Cohomology of abelian arrangements

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Definitions/Preliminaries

1. Let X be a complex abelian variety. An **abelian arrangement** in X is a set $\mathcal{A} = \{Y_1, \ldots, Y_\ell\}$ of codimension-one abelian subvarieties. Denote the complement of an abelian arrangement by
$$M(\mathcal{A}) := X \setminus \bigcup_{Y \in \mathcal{A}} Y.$$
2. A **component** of the arrangement is a connected component of an intersection $\cap_{Y \in S} Y$ for some $S \subseteq \mathcal{A}$.
3. An arrangement is **unimodular** if all intersections $\cap_{Y \in S} Y$ are connected.
4. The **rank** of a component F is its complex codimension in X, $\mathrm{rk}(F) := \mathrm{codim}(F)$.
5. Let $S \subseteq \mathcal{A}$, and F a connected component of $\cap_{Y \in S} Y$. If $\mathrm{rk}(F) = |S|$, we say that S is **independent**. Otherwise, $\mathrm{rk}(F) < |S|$ and S is dependent.
6. Let F be a component of $\mathcal{A}$ and $x \in F$. Define a central hyperplane arrangement in $T_x X$ by
$$\mathcal{A}_F^{(x)} = \{T_x Y \ : \ Y \in \mathcal{A}, Y \supseteq F\}$$
called the **localization** of $\mathcal{A}$ at F. Denote its complement by
$$M\left(\mathcal{A}_F^{(x)}\right) := T_x X \setminus \bigcup_{T_x Y \in \mathcal{A}_F^{(x)}} T_x Y$$
   - If x is not contained in a smaller component, then there is a neighborhood $U \ni x$ such that $U \cap M(\mathcal{A}) \cong M\left(\mathcal{A}_F^{(x)}\right)$.
   - The combinatorics of $\mathcal{A}_F^{(x)}$ does not depend on the choice of $x \in F$, and hence $H^*\left(M\left(\mathcal{A}_F^{(x)}\right);\mathbb{Q}\right)$ does not depend on the choice of $x \in F$.
7. (Deligne) For a smooth complex variety, there is a canonical increasing filtration on $H^*(X;\mathbb{Q})$, called the weight filtration,
$$0 \subseteq W_k \subseteq \cdots \subseteq W_{2k} \subseteq H^k(X;\mathbb{Q})$$
   - If X is projective, then $H^k(X;\mathbb{Q})$ is pure of weight k.
   - If X is the complement of a complex hyperplane arrangement, then $H^k(X;\mathbb{Q})$ is pure of weight $2k$.

Example

Let E be a complex elliptic curve, define $\alpha_{ij} : E^n \to E$ by $\alpha_{ij}(e_1, \ldots, e_n) = e_i - e_j$ for $1 \leq i < j \leq n$, and take the arrangement $\mathcal{A} = \{Y_{ij} = \ker(\alpha_{ij})\}$.

This is an analogue of the braid arrangement, and its complement $M(\mathcal{A})$ is a configuration space of ordered n-tuples of points in E.

Main Theorem

Let X be a complex abelian variety, $\mathcal{A} = \{Y_1, \ldots, Y_\ell\}$ a unimodular arrangement of codimension-one abelian subvarieties, and $M(\mathcal{A}) = X \setminus \cup_{i=1}^{\ell} Y_i$. Then
$$H^*(M(\mathcal{A});\mathbb{Q}) \cong H^*(A^\bullet, d)$$
where $A^\bullet$ is the quotient of the graded-commutative algebra $H^*(X;\mathbb{Q})[g_1, \ldots, g_\ell]$ by the ideal generated by:

1. $\sum_{j=1}^{k} (-1)^{j-1} g_{i_1} \cdots \hat{g}_{i_j} \cdots g_{i_k}$ whenever $\{Y_{i_1}, \ldots, Y_{i_k}\}$ is dependent.
2. $\alpha_i^*(x) g_i$, where Y_i is the kernel of the map $\alpha_i : X \to E_i$ for an elliptic curve E_i, and $x \in H^1(E_i;\mathbb{Q})$.

with differential d defined by $dg_i = [Y_i] \in H^2(X;\mathbb{Q})$ and $dx = 0$ for $x \in H^*(X;\mathbb{Q})$.

Moreover, this algebra comes with a bi-grading which describes the weight filtration on $H^*(M(\mathcal{A});\mathbb{Q})$.

Method/Idea of Proof

B. Totaro and I. Kriz independently showed an analogous result for configuration spaces of projective varieties. The method used here is a generalization of that used by Totaro. Our work overlaps in the case of a configuration space of ordered n-tuples of points on an elliptic curve.

We study the Leray Spectral Sequence for the inclusion $f : M(\mathcal{A}) \hookrightarrow X$, given by
$$H^p(X; R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(M(\mathcal{A});\mathbb{Q})$$
where $R^q f_* \mathbb{Q}$ is the sheaf that takes an open $U \subseteq X$ to $H^q(U \cap M(\mathcal{A});\mathbb{Q})$.

1. $H^p(X; R^q f_* \mathbb{Q}) \cong \bigoplus_{\mathrm{rk}(F)=q} H^p(F;\mathbb{Q}) \otimes H^q(M(\mathcal{A}_F);\mathbb{Q})$
2. $H^p(F;\mathbb{Q})$ is pure of weight p, and $H^q(M(\mathcal{A}_F);\mathbb{Q})$ is pure of weight $2q$. Hence, $E_j^{p,q}$ is pure of weight $p+2q$
3. The differential d_j on the E_j-term is trivial for $j > 2$; that is, the sequence degenerates at the E_3-term. This is because the differential must respect the weight filtration.
4. Since the spectral sequence degenerates, $E_3 \cong \mathrm{gr}\, H^*(M(\mathcal{A});\mathbb{Q})$, the associated graded with respect to the Leray filtration. But in this case, this is the associated graded with respect to the weight filtration, which is isomorphic to $H^*(M(\mathcal{A});\mathbb{Q})$.
5. The E_2-term is a differential graded algebra, isomorphic to the presentation given in the main result. The algebra structure comes from both $H^*(X;\mathbb{Q})$ and $H^*(M(\mathcal{A}_0);\mathbb{Q})$, where $\mathcal{A}_0$ denotes the localization at $\cap_{Y \in \mathcal{A}} Y$.

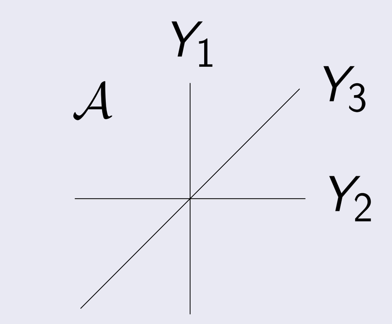
Example

For a more general example, let $M = [\alpha_1 | \cdots | \alpha_\ell]$ be an $n \times \ell$ integer matrix. Each column represents a map $\alpha_i : E^n \to E$, and we can take $Y_i = \ker(\alpha_i)$.

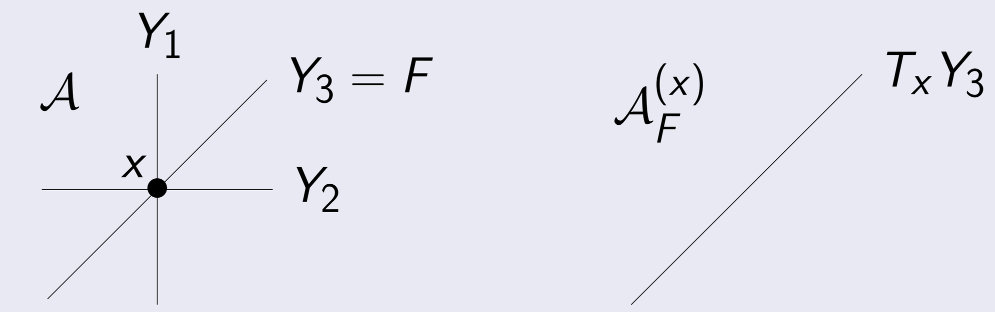
1. For a subset $S \subseteq \mathcal{A}$, the intersection $\cap_{Y \in S} Y$ is the kernel of the corresponding submatrix.
2. The codimension of $\cap_{Y \in S} Y$ is the rank of the corresponding submatrix. In this way, the dependencies of the subvarieties correspond to dependencies of the α_i's in $\mathbb{Z}^n$.
3. If $|S| = n$, and the determinant of the corresponding submatrix is ± 1, then $\cap_{Y \in S} Y$ is connected. If the matrix is unimodular, then so is $\mathcal{A}$.

Example

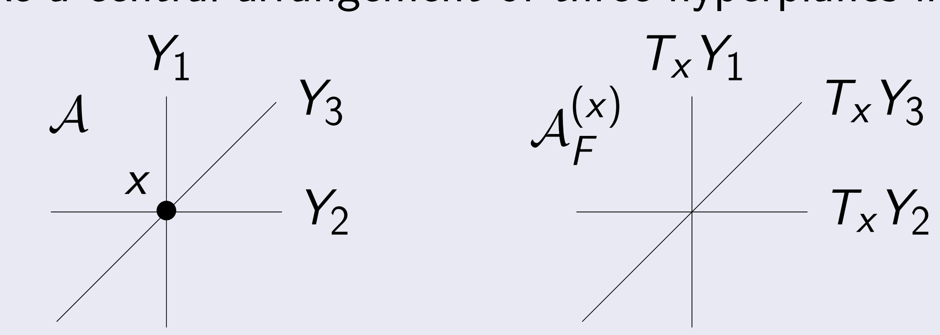
Let $X = E^2$, and $\mathcal{A} = \{Y_1, Y_2, Y_3\}$ with Y_i the kernel of $\alpha_i : E^2 \to E$, where
$$\alpha_1(e_1, e_2) = e_1$$
$$\alpha_2(e_1, e_2) = e_2$$
$$\alpha_3(e_1, e_2) = e_1 - e_2$$
The picture to the right depicts the combinatorics of the arrangement.



1. The components of the arrangement are:
   - E^2 is the only rank-0 component.
   - Y_1, Y_2, and Y_3 are the rank-1 components.
   - $Y_1 \cap Y_2 \cap Y_3$ is the only rank-2 component.
2. This arrangement is unimodular.
3. The set $S = \{Y_1, Y_2, Y_3\}$ is the only dependent subset of $\mathcal{A}$; all other subsets are independent.
4. The localization of $\mathcal{A}$ at the component $F = Y_3$ looks like a single hyperplane in $\mathbb{C}^2$



5. The localization of $\mathcal{A}$ at the component $F = Y_1 \cap Y_2 \cap Y_3$ looks like a central arrangement of three hyperplanes in $\mathbb{C}^2$.



6. The DGA from the main result is the quotient of the exterior algebra $\bigwedge(x_1, y_1, x_2, y_2, g_1, g_2, g_3)$ by the ideal generated by
   1. $g_2 g_3 - g_1 g_3 + g_1 g_2$
   2. $x_1 g_1$, $y_1 g_1$, $x_2 g_2$, $y_2 g_2$, $(x_1 - x_2) g_3$, $(y_1 - y_2) g_3$

   with differential $dx_i = 0 = dy_i$, $dg_1 = x_1 y_1$, $dg_2 = x_2 y_2$, and $dg_3 = (x_1 - x_2)(y_1 - y_2)$.
7. Computing cohomology, we get the Poincare polynomial
$$P(t) = 1 + 4t + 5t^2.$$
Moreover, the weight filtration on $H^*(M(\mathcal{A});\mathbb{Q})$ is nontrivial:
$$\sum_{i,j} \dim \mathrm{gr}_j\, H^i(M(\mathcal{A});\mathbb{Q}) t^i u^j = 1 + 4tu + 3t^2u^2 + 2t^2u^3$$