Non-crossing partitions and monodromy of Milnor fibres.

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Plan

For each finite real reflection group W,

describe a finite simplicial complex F with the homotopy type of the Milnor fibre of the corresponding complexified arrangement,

exhibit a simplicial automorphism ϕ of F which gives the monodromy action on the fibre,

show that the mapping torus of ϕ is a $K(\pi, 1)$ for the corresponding pure braid group.

Non-crossing partitions

W finite real reflection group of rank n

$|w| =$ total reflection length of w

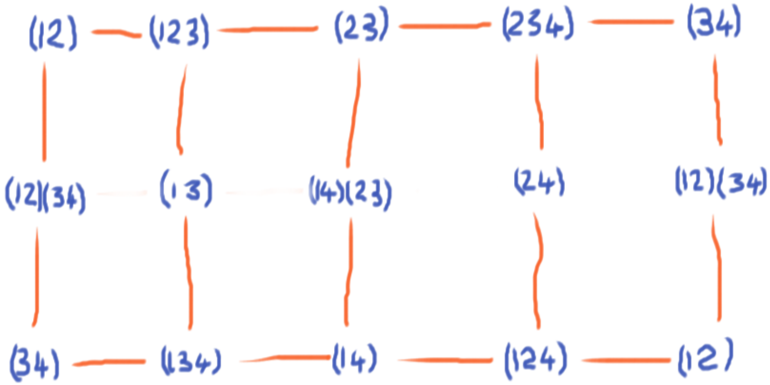
partial order: $w_1 \leq w_2 \Leftrightarrow |w_2| = |w_1| + |w_1^{-1} w_2|$.

$(w_1 \leq w_2 \Rightarrow w_1^{-1} w_2 \leq w_2)$

γ is a fixed Coxeter element.

NCP$=$ elements in $[e, \gamma]$ (lattice under $\leq$)

Proper part of Σ_4 NCP lattice



$K(\pi, 1)$ for pure braid group of W

$PK(W)$ is a trisp with vertex set W

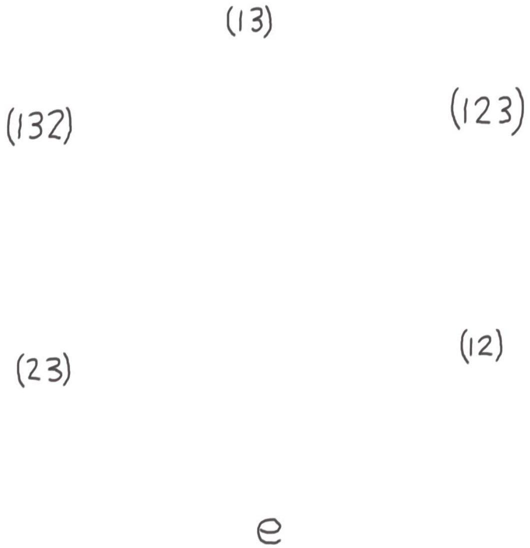
k-cell on $(w, ww_1, ww_2, \ldots, ww_k)$ for each

$w \in W$ and $e < w_1 < w_2 < \cdots < w_k$ in NCP

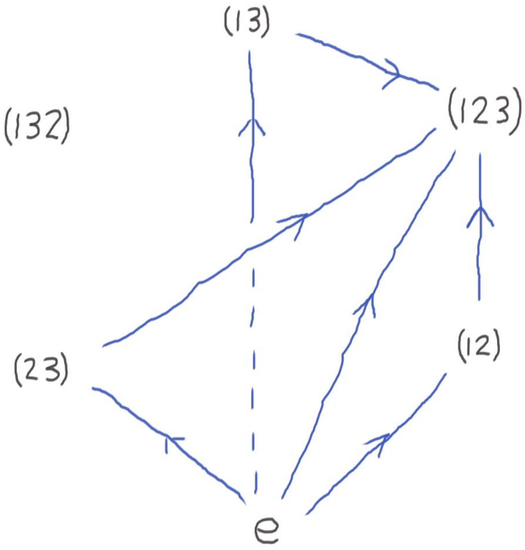
Proposition: $PK(W)$ is a $K(\pi, 1)$ for $PB(W)$.

Proof: $W \backslash PK(W)$ is a $K(\pi, 1)$ for $B(W)$.

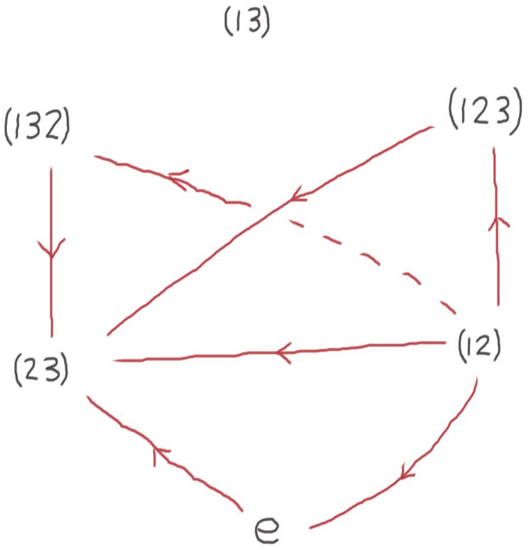
$PK(W)$ for $W = \Sigma_3$, Figure 1



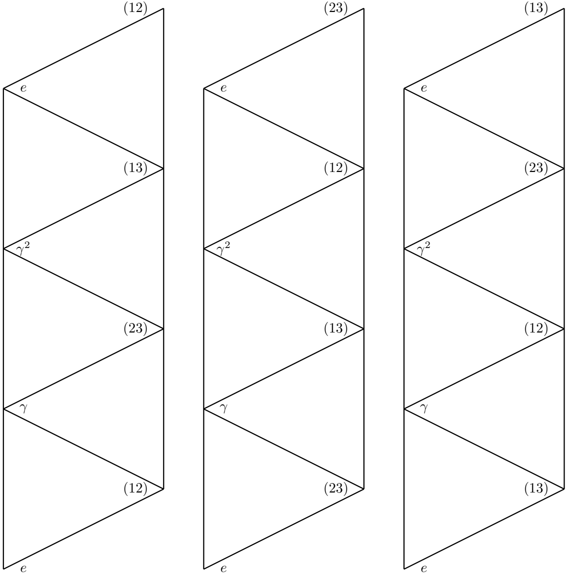
$PK(W)$ for $W = \Sigma_3$, Figure 2



$PK(W)$ for $W = \Sigma_3$, Figure 3

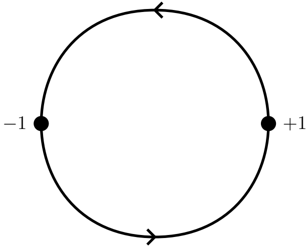


$PK(W)$ for $W = \Sigma_3$, Figure 4



$\mathbb{Z}$ cover of $PK(W)$

Identify S^1 with this cell complex ($PK(\mathbb{Z}_2)$) in $\mathbb{C}^*$.



Construct a map $f : PK(W) \to S^1$ taking W_+ to $+1$, W_- to -1, and an edge (w, ww_i) to a counterclockwise path of length $|w_i|$ from $f(w)$ to $f(ww_i)$. Extend across higher-dimensional cells.

Define $Y(W)$ to be the cover of $PK(W)$ corresponding to the kernel of

$$f_* : \pi_1(PK(W), e) \to \mathbb{Z} = \pi_1(S^1, +1)$$

Structure of $Y(W)$

$Y(W)$ is a simplicial complex with vertex set

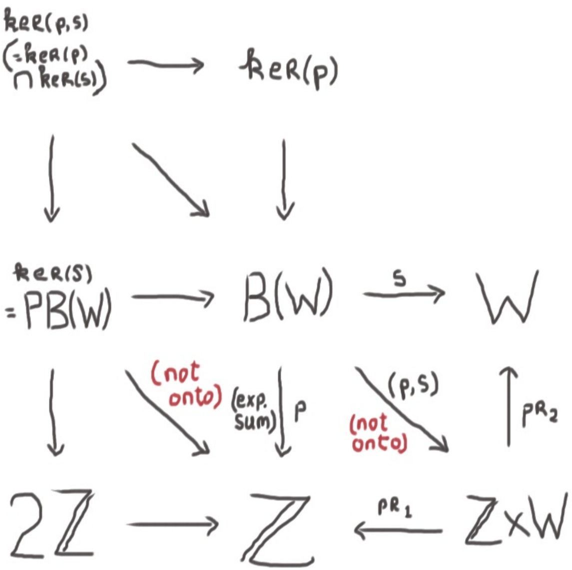
$$\{(m, w) \mid m \in \mathbb{Z}, w \in W \text{ and parity}(w) = \text{parity}(m)\}$$

k-cell on
$((m, w), (m + |w_1|, ww_1), (m + |w_2|, ww_2), \ldots, (m + |w_k|, ww_k))$
for each

$w \in W$ and $e < w_1 < w_2 < \cdots < w_k$ in NCP.

Covering map is projection onto second factor.

Diagram of groups



The finite subcomplex F

Define F to be the finite subcomplex of Y consisting of those simplices whose vertices (m, w) satisfy $0 \leq m \leq n-1$.

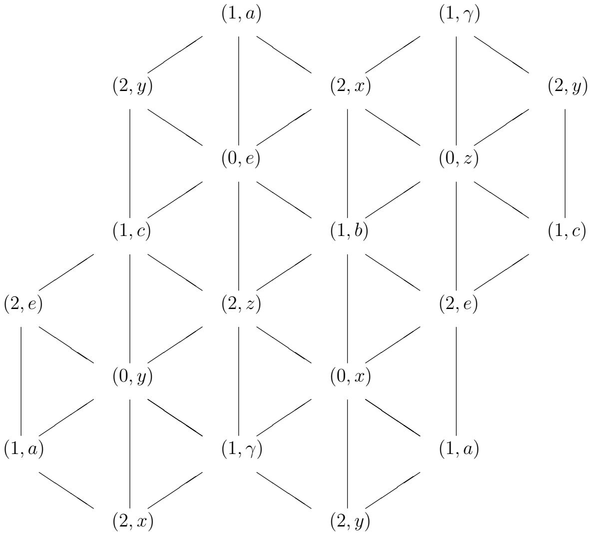
Explicitly, F consists of the faces of the $(n-1)$-simplices

$$((0, w), (1, ww_1), (2, ww_2), \ldots, (n-1, ww_{n-1}))$$

for $w \in W_+$ and $e \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1} \lessdot \gamma$ in NCP.

F for $W = \mathbb{Z}_2^3$

Generators a, b, c. Set $x = ab$, $y = ac$, $z = bc$, $\gamma = abc$.



The basic Lemma

Lemma: If $w_k < \gamma$ then the k-simplex in Y
$((m, w), (m + |w_1|, ww_1), \ldots, (m + |w_k|, ww_k))$
is incident on precisely two $(k+1)$-simplices in Y of the form
$((\ell, u), (\ell + |u_1|, uu_1), \ldots, (\ell + |u_k|, uu_k), (\ell + n, u\gamma))$.

Proof: Can only decrease the height difference between the first and last entries of 'u' cell by deleting either the first entry or the last entry.

Theorem: F is a strong deformation retract of Y.

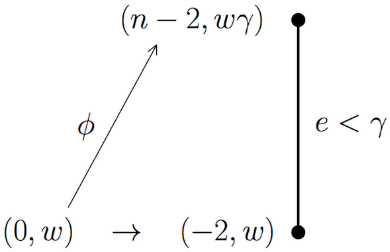
Proof of Theorem: Use lemma to collapse cells above F down to F and cells below F up to F. Match each 'w' cell from the Lemma with the 'u' cell above or below according as the 'w' cell is below or above F.

The monodromy action on vertices.

$\mathbb{Z}$ acts on Y by $(m, w) \to (m-2, w)$, so shift cells of F down by 2 and, when they go below level 0, 'tuck' them back into F using the retraction.

Explicitly, we define ϕ on vertices by

$$\phi(m,w) = \begin{cases} (m-2, w) & 2 \leq m \leq n-1 \\ (n-1, w\gamma) & m = 1 \\ (n-2, w\gamma) & m = 0. \end{cases}$$



The order of ϕ is $nh/2$, where h is the order of γ.

The monodromy action on simplices.

Consider a top-dimensional simplex

$$((0, w), (1, ww_1), (2, ww_2), \ldots, (n-1, ww_{n-1}))$$

Under ϕ its vertices transform to

$$((n-2, w\gamma), (n-1, ww_1\gamma), (0, ww_2), \ldots, (n-3, ww_{n-1}))$$

We know $e \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1} \lessdot \gamma$ and hence
$e \lessdot w_1^{-1}w_2 \lessdot w_1^{-1}w_3 \lessdot \cdots \lessdot w_1^{-1}w_{n-1} \lessdot w_1^{-1}\gamma \lessdot \gamma$ and
$e \lessdot w_2^{-1}w_3 \lessdot w_2^{-1}w_4 \lessdot \cdots \lessdot w_2^{-1}w_{n-1} \lessdot w_2^{-1}\gamma \lessdot w_2^{-1}w_1\gamma \lessdot \gamma$.

Use this last chain to construct a top-dimensional simplex starting at $(0, ww_2)$. This simplex will have the same set of vertices as the ϕ translates above.

The mapping torus of ϕ

Define $MT(\phi) = (F \times I)/\sim$, where $(f, 0) \sim (\phi(f), 1)$.

Proposition: $MT(\phi)$ has the homotopy type of $PK(W)$.

Proof: For each facet σ of F given by

$$((0, w), (1, ww_1), (2, ww_2), \ldots, (n-1, ww_{n-1}))$$

the $MT(\phi)$ cell $\sigma \times I$ has top identified with

$$((n-2, w\gamma), (n-1, ww_1\gamma), (0, ww_2), \ldots, (n-3, ww_{n-1})).$$

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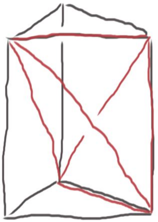
Triangulate $\sigma \times I$ cell in the usual way.



Identify vertices with the same second component.

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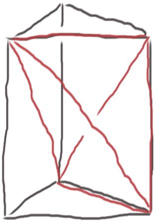

Identify vertices with the same second component. Result is a union of two $PK(W)$ facets:

$(w, ww_1, ww_2, \ldots, ww_{n-1}, w\gamma)$ and $(ww_1, ww_2, \ldots, ww_{n-1}, w\gamma, ww_1\gamma)$