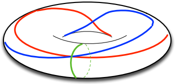
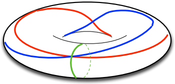
Recent developments in Toric arrangements



Emanuele Delucchi
(SNSF / Université de Fribourg)

Configuration spaces, Cortona, Italy
September 2., 2014

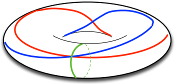
Toric arrangements



– Connection with partition functions and splines [De Concini–Procesi–Vergne]

– New challenges for combinatorial topology.

TORIC ARRANGEMENTS



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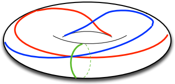

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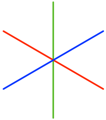
HYPERPLANE ARRANGEMENTS



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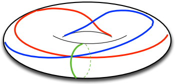

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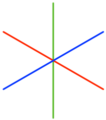

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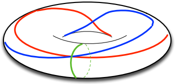

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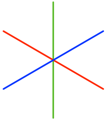
Greater generality

??

Hyperplane arrangements



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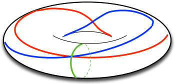

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Hyperplane arrangements

A (central) hyperplane arrangement in a $\mathbb{K}$-vectorspace V is a set

$$\mathscr{A} := \{H_1, \ldots, H_n\}$$

of hyperplanes $H_i = \ker \alpha_i$, with $\{\alpha_1, \ldots, \alpha_n\} \subseteq V^*$

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Topology

$M(\mathscr{A}) := V \setminus \cup\mathscr{A}$ is the arrangement's COMPLEMENT.

Combinatorics

$\mathcal{L}(\mathscr{A})$: poset of intersections (order: reverse inclusion)

$\mathcal{D}(\mathscr{A})$: linearly dependent subsets of $\{\alpha_1, \ldots, \alpha_n\}$

Both encode the associated (simple) MATROID.

The power of matroids (e.g., when $V=\mathbb{C}^d$)

[Arnol'd, Brieskorn ∼'71, Zaslavsky '79, Orlik-Solomon '80]

- $$P(M(\mathscr{A}),t) = \sum_{X\in\mathcal{L}(\mathscr{A})} \underbrace{\mu_{\mathcal{L}(\mathscr{A})}(\hat{0},X)}_{\substack{\text{Möbius}\\ \text{function}\\ \text{of } \mathcal{L}(\mathscr{A})}} (-t)^{\operatorname{rk} X}$$

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- "Orlik-Solomon algebra":

$$H^*(M(\mathscr{A}), \mathbb{Z}) \simeq E/\mathcal{J}(\mathscr{A}), \text{ where}$$

E: exterior $\mathbb{Z}$-algebra with degree-1 generators $e_1, \ldots, e_n$ (one for each H_i);

$\mathcal{J}(\mathscr{A})$: the ideal $\langle\, \sum_{l=1}^{k} (-1)^l e_{j_1} \cdots \widehat{e_{j_l}} \cdots e_{j_k} \mid \{j_1, \ldots, j_k\} \in \min \mathcal{D}(\mathscr{A}) \,\rangle$

The limits of matroids (again, when $V = \mathbb{C}^d$)

[Rybnikov 1995 / 2011]

There are two arrangements $\mathscr{A}_1$, $\mathscr{A}_2$ with

$$\mathcal{L}(\mathscr{A}_1) \simeq \mathcal{L}(\mathscr{A}_2)$$

(i.e., with isomorphic associated matroids), but

$$\pi_1(M(\mathscr{A}_1)) \not\simeq \pi_1(M(\mathscr{A}_2)).$$

Thus the homotopy type is not determined by the matroid alone.

Complexified arrangements

Let $\mathscr{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^d$.

$\mathscr{A}$ is called *complexified* if $\alpha_i \in (\mathbb{R}^d)^*$ for all $i = 1, \ldots, n$.

Consider the arrangement $\mathscr{A}^{\mathbb{R}} := \mathscr{A} \cap \mathbb{R}^d = \{H_1^{\mathbb{R}}, \ldots, H_n^{\mathbb{R}}\}$ in $\mathbb{R}^d$.

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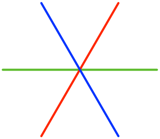
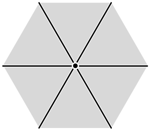
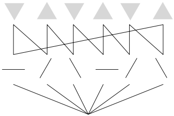
- $\mathscr{A}^{\mathbb{R}}$ has the same defining forms, hence same matroid, as $\mathscr{A}$.
- The hyperplanes of $\mathscr{A}^{\mathbb{R}}$ define a polyhedral fan in $\mathbb{R}^d$.



The arrangement



The fan, with...



...its poset of faces.

Beyond matroids

$\mathscr{A}$: complexified arrangement with defining forms $\{\alpha_1, \ldots, \alpha_n\} \subset (\mathbb{R}^d)^*$.

$\mathcal{F}(\mathscr{A})$: the face poset of the associated polyhedral fan

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($X \in \mathcal{V}(\mathscr{A})$ if and only if $X(i) = \mathrm{sign}(\lambda_i)$ for some

real numbers $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1\alpha_1 + \ldots + \lambda_n\alpha_n = 0$)

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These encode equivalent data and define ("up to reorientation") the

ORIENTED MATROID

associated to $\mathscr{A}^{\mathbb{R}}$.

Salvetti's complex

[Salvetti '87]

Let $\mathscr{A}$ be a complexified arrangement.

The oriented matroid data of $\mathscr{A}$ (e.g., $\mathcal{F}(\mathscr{A})$) determines a poset $\mathrm{Sal}(\mathscr{A})$ such that

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Order complex - the simplicial complex of all chains.

Beyond oriented matroids?

Two approaches towards developing a framework corresponding to oriented matroids for complex vectorspaces.

- Complex matroids [Björner – Ziegler '92, Ziegler '93]
  (Discrete models, topological representation theorem;
  no cryptomorphisms, no characterization of the complex structure)

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- Phased matroids [Anderson – D. '10] (→ Elia's poster)
  (Cryptomorphisms, duality, natural S^1-action;
  not discrete, no topological representation theorem - as yet)

Toric arrangements

A toric arrangement in the complex torus $T := (\mathbb{C}^*)^d$ is a set

$$\mathscr{A} := \{K_1, \dots, K_n\}$$

of 'hypertori' $K_i = \chi_i^{-1}(b_i)$ with $\chi_i \in \mathrm{Hom}(T, \mathbb{C}^*)$ and $b_i \in \mathbb{C}^* / = 1 / \in S^1$

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For simplicity assume that the matrix $[a_1, \dots, a_n]$ has rank d.

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Topology

$M(\mathscr{A}) := T \setminus \cup\mathscr{A}$,
the *complement* of $\mathscr{A}$.

Combinatorics

$\mathcal{C}(\mathscr{A})$: poset of *layers* (connected components of intersections)

The arithmetic matroid of the a_i

The matroid over $\mathbb{Z}$ of the a_i

As yet no overarching theory

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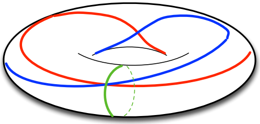
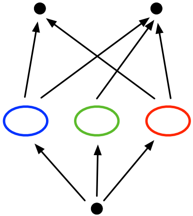


$\mathscr{A}$



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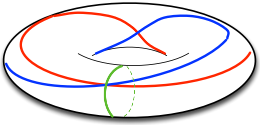

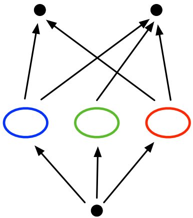

[Looijenga '93, De Concini – Procesi '05]

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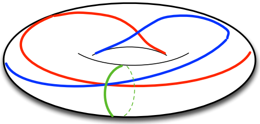

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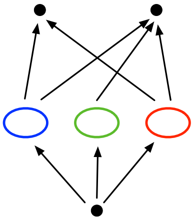

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Homotopy type

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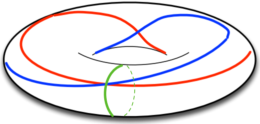
[d'Antonio – D. '12]

Let $\mathscr{A}$ be a complexified toric arrangement (i.e., $K_i = \chi_i^{-1}(b_i)$ for $b_i \in S^1$).

$\mathscr{A}$ defines an arrangement $\mathscr{A}^c$ on the 'compact torus' $T^c = (S^1)^d$.

$\mathcal{F}(\mathscr{A})$ the *face category* of the polyhedral complex induced on T^c.



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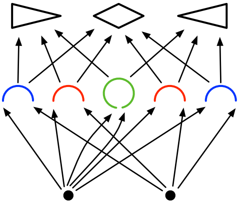

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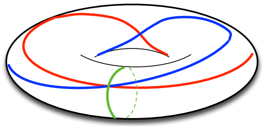

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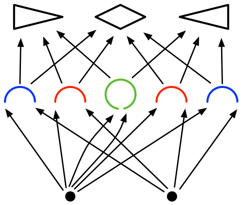


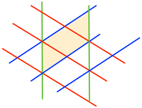
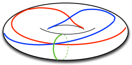
The data of $\mathcal{F}(\mathscr{A})$ determines an acyclic category $\mathrm{Sal}(\mathscr{A})$ such that

$$\Delta(\mathrm{Sal}(\mathscr{A})) \simeq M(\mathscr{A})$$

Complexified toric arrangements

$\mathscr{A}^{\upharpoonright}$

Any complexified toric arrangement $\mathscr{A}$ lifts to a complexified arrangement of affine hyperplanes $\mathscr{A}^{\upharpoonright}$ under the universal cover

$\mathbb{C}^d \to T$, $\quad \mathscr{A}^{\upharpoonright}$:  $\longrightarrow \mathscr{A}$: 

$\mathscr{A}^{\upharpoonleft}$ and the fundamental group

[d'Antonio – D. '12]

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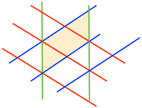
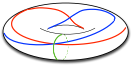
$\mathbb{C}^d \to T, \quad \mathscr{A}^{\upharpoonleft}$:  $\longrightarrow \mathscr{A}$: 

There is a split exact sequence

$$0 \longrightarrow \pi_1(M(\mathscr{A}^{\upharpoonleft})) \longrightarrow \pi_1(M(\mathscr{A})) \overset{\longleftarrow}{\longrightarrow} \mathbb{Z}^d \simeq \pi_1(T^c) \longrightarrow 0.$$

Moreover, via $\mathrm{Sal}(\mathscr{A})$ we obtain a finite presentation for $\pi_1(M(\mathscr{A}))$.

Cohomology

Cohomology I

[De Concini – Procesi '05] compute the cup product in $H^*(M(\mathscr{A}), \mathbb{C})$ when the matrix $[a_1, \dots, a_n]$ is totally unimodular.

[Bibby '14] Studies the rational cohomology algebra of unimodular abelian arrangements and, e.g., describes the deletion-contraction behaviour.

We strive for a description of the integer cohomology algebra. For starters:

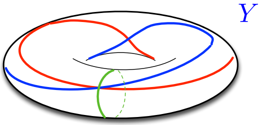
[d'Antonio – D. '13] For any complexified toric arrangement $\mathscr{A}$, the space $M(\mathscr{A})$ is minimal, thus $H^j(M(\mathscr{A}), \mathbb{Z})$ is torsion-free for all j.

Proof. Discrete Morse Theory on $\mathrm{Sal}(\mathscr{A})$.

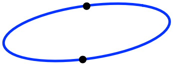
Cohomology II.0

Let $\mathscr{A}$ be a complexified toric arrangement.

For $Y \in \mathcal{C}(\mathscr{A})$ define $\mathscr{A}^Y = \mathscr{A} \cap Y$, the arrangement induced on Y.



$\mathscr{A}$



$\mathscr{A}^Y$

Cohomology II.0

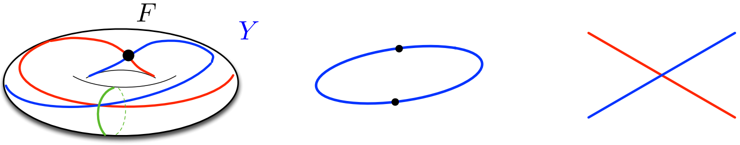
Let $\mathscr{A}$ be a complexified toric arrangement.

For $Y \in \mathcal{C}(\mathscr{A})$ define $\mathscr{A}^Y = \mathscr{A} \cap Y$, the arrangement induced on Y.

For $F \in \mathcal{F}(\mathscr{A})$ choose a lift $F^{\upharpoonleft} \in \mathcal{F}(\mathscr{A}^{\upharpoonleft})$ and let

$$\mathscr{A}[F] = \{H \in \mathscr{A}^{\upharpoonleft} \mid F^{\upharpoonleft} \subseteq H\}$$

be the 'local' hyperplane arrangement at the face F.



$\mathscr{A}$ $\qquad$ $\mathscr{A}^Y$ $\qquad$ $\mathscr{A}[F]$

Cohomology II.1

[Callegaro – D. '14]

(1) $\Delta(\mathrm{Sal}(\mathscr{A})) \simeq \operatorname{hocolim} \mathscr{D}$, where

$$\begin{array}{rccc} \mathscr{D}: & \mathcal{F}(\mathscr{A}) & \to & \mathrm{Top} \\ & F & \mapsto & \Delta(\mathrm{Sal}(\mathscr{A}[F])) \end{array}$$

Call ${}_{\mathscr{D}}E_*^{p,q}$ the associated cohomology spectral sequence [Segal '68].

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Call ${}_{\mathscr{D}}E_*^{p,q}$ the associated cohomology spectral sequence [Segal '68].

(2) Given $Y \in \mathcal{C}(\mathscr{A})$ choose a maximal $F \subseteq Y$ and write $\mathscr{A}[Y] := \mathscr{A}[F]$.
Then in $\Delta(\mathrm{Sal}(\mathscr{A}))$ there is a subcomplex

$$\mathcal{S}_Y \simeq \Delta(\mathcal{F}(\mathscr{A}^Y)) \times \Delta(\mathrm{Sal}(\mathscr{A}[Y])) \simeq Y \times M(\mathscr{A}[Y])$$

Cohomology II.2.1

For every $Y \in \mathcal{C}(\mathscr{A})$, the following commutative square

$$\begin{array}{ccc} M(\mathscr{A}) \simeq \Delta(\mathrm{Sal}(\mathscr{A})) & \overset{\supseteq}{\longleftarrow} & \mathcal{S}_Y \\ \downarrow \pi & & \downarrow \pi_Y \\ \Delta(\mathcal{F}(\mathscr{A})) & \overset{\supseteq}{\longleftarrow} & \Delta(\mathcal{F}(\mathscr{A}^Y)) \end{array}$$

induces a morphism of spectral sequences ${}_{\mathscr{D}}E_*^{p,q} \to {}_Y E_*^{p,q}$.

Next, we examine the morphism of spectral sequences associated to the corresponding map from $\uplus_{Y \in \mathcal{C}(\mathscr{A})} \mathcal{S}_Y$ to $\Delta(\mathrm{Sal}(\mathscr{A}))$.

Cohomology II.2.2

[Callegaro – D., '14 - ongoing] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{array}{ccc}
H^*(M(\mathscr{A})) & \longrightarrow & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \\
\downarrow & & \downarrow \\
{}_{\mathscr{D}}E_2^{p,q} = & & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} {}_Y E_2^{p,q} = \\
\bigoplus_{\substack{Y\in\mathcal{C}(\mathscr{A})\\ \operatorname{rk} Y=q}} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) & \longrightarrow & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^p(Y)\otimes H^q(M(\mathscr{A}[Y]))
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\end{array}$$

On Y_0-summand: $\omega\otimes\lambda \longmapsto \begin{pmatrix} i^*(\omega)\otimes b(\lambda) & \text{if } Y_0\leq Y \\ 0 & \text{else.} \end{pmatrix}_Y$

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[Callegaro – D., '14 - ongoing] (all cohomologies with $\mathbb{Z}$-coefficients)

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$i : Y \hookrightarrow Y_0$

"Brieskorn" inclusion

Cohomology II.2.2

[Callegaro – D., '14 - ongoing] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{array}{ccc}
H^*(M(\mathscr{A})) & \xrightarrow{\text{Hom. of rings}} & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \\
\downarrow \text{bij.} & & \downarrow \text{bij.} \\
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Cohomology II.2.2

[Callegaro – D., '14 - ongoing] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{array}{ccc} H^*(M(\mathscr{A})) & \xrightarrow[\text{Injective}]{\text{Hom. of rings}} & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \\ \downarrow \text{bij.} & & \downarrow \text{bij.} \\ {}_{\mathscr{D}}E_2^{p,q} = & & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} {}_Y E_2^{p,q} = \\ \bigoplus_{\substack{Y\in\mathcal{C}(\mathscr{A})\\ \operatorname{rk} Y=q}} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) & \xrightarrow{\text{Hom. of rings}} & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) \end{array}$$

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"Brieskorn" inclusion

Cohomology III

[Callegaro – D., '14]

$\mathcal{C}(\mathscr{A})$ determines the cohomology ring *if* $\mathscr{A}$ *has a unimodular basis*.

"Proof": A realizable arithmetic matroid with an unimodular basis has a unique realization.

In general, we do not know whether $\mathcal{C}(\mathscr{A})$ determines the ring structure.

Cohomology III

[Callegaro – D., '14]

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"Proof": A realizable arithmetic matroid with an unimodular basis has a unique realization.

In general, we do not know whether $\mathcal{C}(\mathscr{A})$ determines the ring structure.

(...is it even the 'right' combinatorial invariant to look at?)

Other related combinatorial structures

Let $\mathscr{A}$ be a toric arrangement, recall the vectors $a_1, \ldots, a_n \in \mathbb{Z}^d$.

For every $I \subset \{1, \ldots, n\}$ define $\mathscr{M}_{\mathscr{A}}(I) := \mathbb{Z}^d / \langle a_i \mid i \in I \rangle$.

A matroid over a ring R [Fink – Moci, '13] is a family of R-modules satisfying two abstract axioms (→ Luca's talk).

Example: the $\mathbb{Z}$-modules $\mathscr{M}_{\mathscr{A}}(\cdot)$.

Other related combinatorial structures

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For every $I \subset \{1, \ldots, n\}$ define $\mathscr{M}_{\mathscr{A}}(I) := \mathbb{Z}^d / \langle a_i \mid i \in I \rangle$.

Let $m(I)$ denote the cardinality of the torsion part of $\mathscr{M}_{\mathscr{A}}(I)$.

A matroid over a ring R [Fink – Moci, '13] is a family of R-modules satisfying two abstract axioms (→ Luca's talk).

Example: the $\mathbb{Z}$-modules $\mathscr{M}_{\mathscr{A}}(\cdot)$.

An arithmetic matroid [d'Adderio, Brändén, Moci, '10 – '12] is a matroid with a multiplicity function defined on its ground set, satisfying (→ Luca).

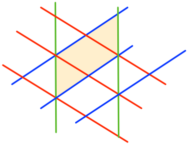
Example: our $m : 2^{[n]} \to \mathbb{N}$ on the matroid of $\mathbb{Q}$-dependencies of the a_i.

Every arithmetic matroid has an arithmetic Tutte polynomial.

Combinatorial framework

Ansatz [D. – Riedel, current]:

$\mathcal{L}(\mathscr{A}^{\upharpoonright})$
Infinite (finitary)
Geometric semilattice



$\mathscr{A}^{\upharpoonright}$

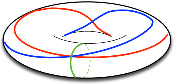
$\mathcal{F}(\mathscr{A}^{\upharpoonright})$
Infinite (finitary)
affine oriented matroid
(→ Sonja's poster)

$\downarrow$ $/\mathbb{Z}^d$(as posets)

$\downarrow$ $/\mathbb{Z}^d$

$\downarrow$ $/\mathbb{Z}^d$ (as acyclic categories)

$\mathcal{C}(\mathscr{A})$



$\mathscr{A}$

$\mathcal{F}(\mathscr{A})$

Characterize axiomatically the involved posets and the group actions.

Finitary semimatroids

[Riedel '13] The following notions are equivalent.

A finitary semimatroid [cp. Ardila '06] is a triple $(S, \mathcal{K}, \mathrm{rk})$, where

- $\mathcal{K}$ is a finite-dimensional simplicial complex on a (possibly infinite) set S
- $\mathrm{rk} : \mathcal{K} \to \mathbb{N}$ satisfies some axioms (generalizing matroid-rank axioms)

Example/Intuition: $S = \mathscr{A}^{\upharpoonright}$, $\mathcal{K} = \{$'central sets' $\}$, $\mathrm{rk}(K) = \mathrm{codim} \cap K$.

A finitary geometric semilattice [cp. Walker – Wachs '86] is a ranked meet-semilattice $\mathcal{L}$ such that

- $\mathcal{L}$ has finite rank and every interval is a geometric lattice
- $\mathcal{L}$ satisfies a global condition about existence of joins

Example: $\mathcal{L}(\mathscr{A}^{\upharpoonright})$.

Group actions on finitary semimatroids?

Let a f. g. abelian group G act on a finitary geometric semilattice $\mathcal{L}$.

(Equivalently, let G act on a finitary semimatroid $(S, \mathcal{K}, \mathrm{rk})$).

For every $X \subseteq S$ define:

$$\underline{X} := \{Gx \mid x \in X\}, \text{ the set of orbits meeting } X$$

Assume that

- there is $X \in \mathcal{K}$ with $\underline{X} = \underline{S}$, and
- for all $g \in G$, $x \in S$, if $\{x, g.x\} \in \mathcal{K}$ then $x = g.x$.

Sample fact 1: Under this assumptions, the rank function rk induces a matroid rank function $\underline{\mathrm{rk}}$ on $\underline{S}$.

Group actions on finitary semimatroids?

Moreover, for every $X \subseteq S$ define

$$\Gamma(X) := G/\operatorname{stab}(X)$$

and suppose that

for every $X \in \mathcal{K}$, $\Gamma(X)$ is free of rank $\operatorname{rk}(X)$.

For $A \subseteq \underline{S}$, define

$$m(A) := |\{X \in \mathcal{K} \mid \underline{X} = A\}/G| = |\{\text{ min. u. b. of } A \text{ in } \mathcal{L}/G\ \}|$$

Sample fact 2: Under this additional hypothesis

- $(\underline{S}, \underline{\operatorname{rk}}, m)$ is a – often nonrealizable – quasiarithmetic matroid
- whose arithmetic Tutte polynomial satisfies Crapo's formula and
- evaluates as the rank-generating function of $\mathcal{L}/G$.

Summary