Large Random Spaces and Groups

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Further facts about M_l and N_l:

- If for some permutation $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ one has

$$l' = \left(l_{\sigma(1)},\dots,l_{\sigma(n)}\right)$$

then $M_l \cong M_{l'}$ and $N_l \cong N_{l'}$ (are diffeomorphic).

- $M_{\lambda l}$=M_l

for $\lambda > 0$ (normalization). Thus we may always assume that

$$\sum_{i=1}^{n} l_i = 1.$$

Plan:

Part I:

Configuration spaces of linkages;
The Walker conjecture (classification of configuration spaces of linkages);
Random manifolds arising as configuration spaces of linkages;

Part II

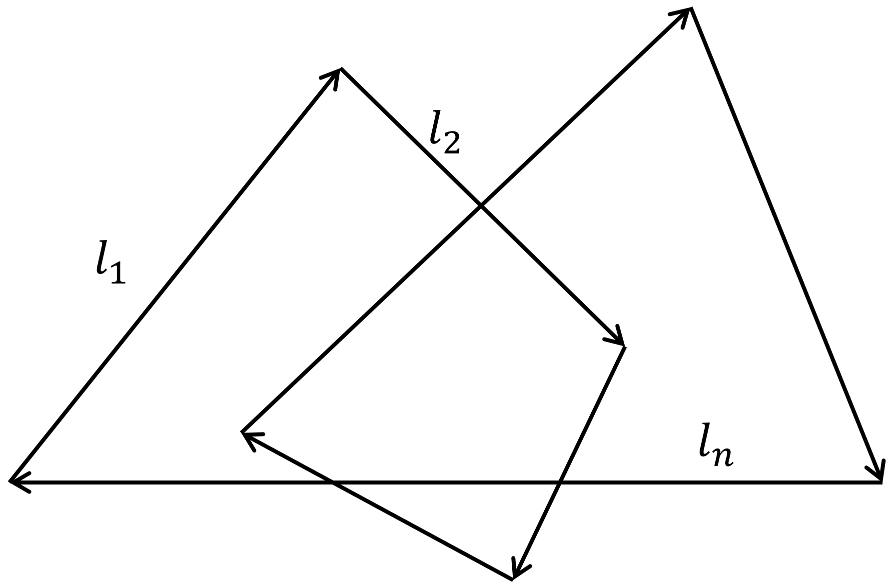
Large random simplicial complexes and their fundamental groups;
Eilenberg – Ganea conjecture for random groups;
Whitehead conjecture for random aspherical 2-complexes.

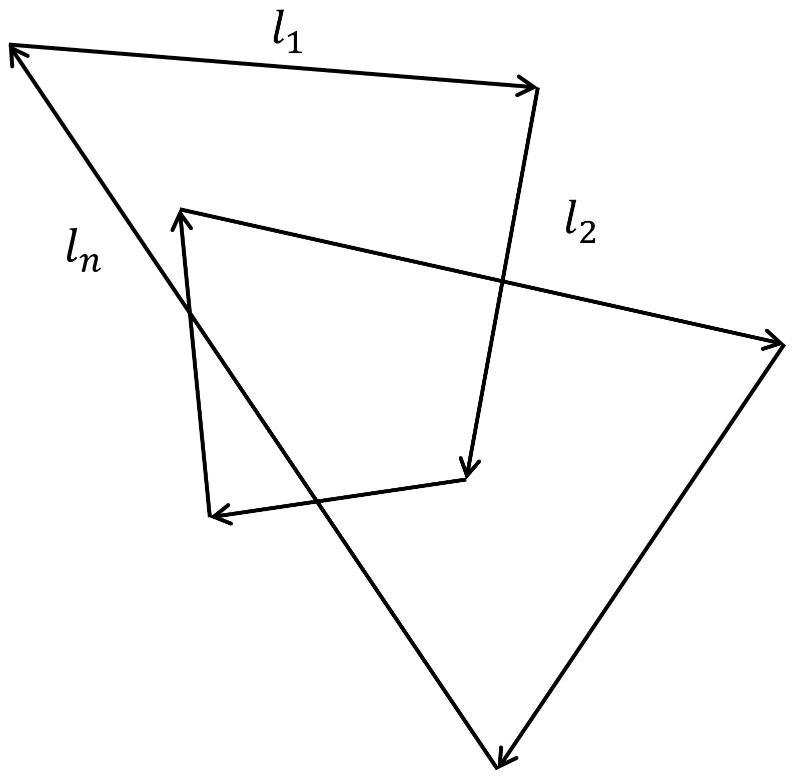
Models:

a. Configuration spaces of linkages.
b. Erdös – Rényi random graphs.
c. Linial – Meshulam model of random simplicial complexes.
d. Random clique (flag) complexes.
e. Random triangulated surfaces of N. Pippenger and K. Schleich.

Part I

Shapes of planar n-gons with given sides



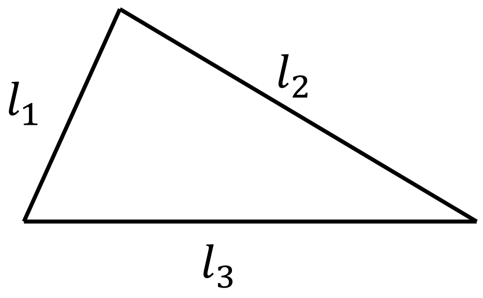


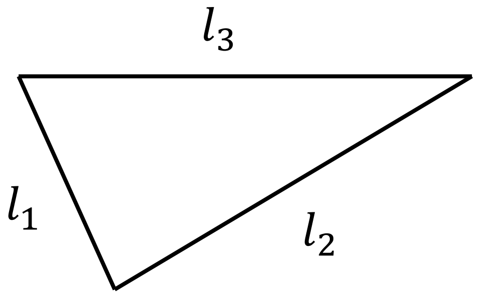
$l = (l_1, l_2, \ldots, l_n)$ is called the *length vector*.
The space of shapes of n-gons with length vector l is defined as

$$M_l = \{(u_1, \ldots, u_n) \in S^1 \times \cdots \times S^1;\ \sum_{i=1}^{n} l_i u_i = 0\}/SO(2).$$

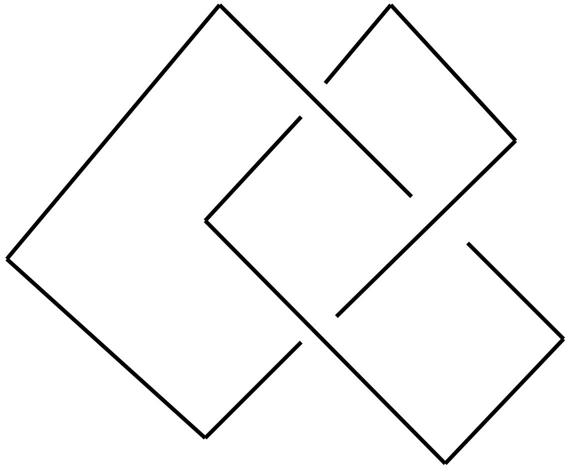
Case n=3, $l = (l_1, l_2, l_3)$

$$M_l = \begin{matrix} \emptyset \\ * \\ ** \end{matrix}$$





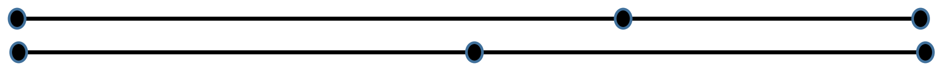
Shapes of spatial n-gons



$$N_l = \{(u_1, \ldots, u_n) \in S^2 \times \cdots \times S^2;\ \sum_{i=1}^{n} l_i u_i = 0\}/SO(3).$$

General facts about M_l and N_l

- For l generic, M_l is a closed smooth manifold of dimension n-3.
- For l generic, N_l is a closed smooth manifold of dimension 2(n-3).
- If l is not generic then M_l and N_l have finitely many singular points which correspond to collinear configurations.



$$l_1+l_2 = l_3+l_4$$

The manifolds N_l and M_l are relevant to applications in

- Topological robotics
- Molecular biology
- Statistical shape theory

(see the book: D. G. Kendall, D. Barden, T. K. Carne and H. Le, *Shape and Shape Theory*, John Wiley & Sons, 1999).

Chambers

$\Delta^{n-1} \subset \mathbb{R}^n$ - unit simplex, $l = (l_1, \dots, l_n)$, $l_i > 0, \sum_{i=1}^{n} l_i = 1$.

$l \in \Delta^{n-1}, M_l$ - field of manifolds.

$J \subset \{1, \dots, n\}$ - subset,

Hyperplane $H_J \subset \mathbb{R}^n$ given by the equation $\sum_{i \in J} l_i = \sum_{i \notin J} l_i$.

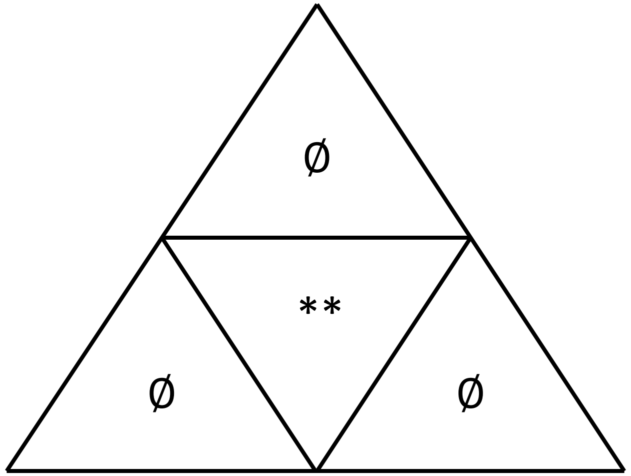
Definition: Connected components of the complement

$$\Delta^{n-1} - \cup_J H_J$$

are called chambers.

Fact: If two generic length vectors l, l' lie in the same chamber then the manifolds M_l and $M_{l'}$ are diffeomorphic; besides, the manifolds N_l and $N_{l'}$ are diffeomorphic.

Example: n=3.



Walker's conjecture: Let $l, l' \in \Delta^{n-1}$ be two generic length vectors. If the corresponding polygon spaces M_l and $M_{l'}$ have isomorphic graded integral cohomology rings then for some permutation σ : {1, . . . , n} → {1, . . ., n} the length vectors l and $\sigma(l')$ lie in the same chamber of Δ^{n-1}.

Kevin Walker, 1985.

A length vector $l = (l_1, \dots, l_n)$ is ordered if

$$l_1 \leq l_2 \leq \cdots \leq l_n.$$

Theorem (J.-C. Hausmann, D. Schuetz, MF):

A. If for two ordered generic length vectors l, l' there exists a graded ring isomorphism of the integral cohomology algebras $H^*(M_l;\ \mathbb{Z}) \to H^*(M_{l'};\ \mathbb{Z})$ then l and l' lie in the same chamber of Δ^{n-1}.

B. For $n \neq 4$, if for two ordered generic length vectors l, l' there exists a graded ring isomorphism of the cohomology algebras $H^*(N_l;\ \mathbb{Z}_2) \to H^*(N_{l'};\ \mathbb{Z}_2)$ then the vectors l and l' lie in the same chamber of Δ^{n-1}.

A crucial role in the solution of the Walker Conjecture plays an important result of J. Gubeladze who solved the Isomorphism Problem for commutative monoidal rings (1998).

Let c_n denote the number of distinct Σ_n - orbits of chambers.

Corollary:

- For any given n, the number of distinct diffeomorphism types of manifolds M_l equals c_n.
- For any given $n \neq 4$, the number of distinct diffeomorphism types of manifolds N_l equals c_n.

J-C. Hausmann, E. Rodriguez:

n	3	4	5	6	7	8	9
c_n	2	3	7	21	135	2470	175428

Corollary:

- Two manifolds M_l and $M_{l'}$ are diffeomorphic iff they are homotopy equivalent or have isomorphic integral cohomology algebras.

- For $n \neq 4$, two manifolds N_l and $N_{l'}$ are diffeomorphic iff they are homotopy equivalent or have isomorphic cohomology algebras with $\mathbb{Z}_2$ coefficients.

Spaces of high dimensional polygons

$E_d(l) = \left\{(u_1, \dots, u_n) \in \left(S^{d-1}\right)^n;\ \sum_{i=1}^{n} l_i u_i = 0\right\}.$

The points of $E_d(l)$ can be understood as closed *n*-gons in $\mathbb{R}^d$ with sides of length $l_1, \dots, l_n$, viewed up to Euclidean translations.

For $d \geq 4$ the quotient $E_d(l)\ /\ SO(d)$ has singularities even for generic vector l. However, for a generic l, the space of polygons $E_d(l)$ is a closed smooth manifold of dimension $(n-1)(d-1)-1$.

Theorem (V. Fromm, MF, 2013):

Let $l, l' \in \mathbb{R}^n$ be two generic length vectors and let d ≥ 3. The following conditions are equivalent:

a) The manifolds $E_d(l)$ and $E_d(l')$ are $SO(d)$-equivariantly diffeomorphic.
b) The cohomology rings $H^*(E_d(l);\ \mathbb{Z}_2)$ and $H^*(E_d(l');\ \mathbb{Z}_2)$ are isomorphic as graded rings.
c) The rings $H^{*(d-1)}(E_d(l);\ \mathbb{Z}_2)$ and $H^{*(d-1)}(E_d(l');\ \mathbb{Z}_2)$ are isomorphic.
d) For some permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, the length vectors l and $\sigma(l')$ lie in the same chamber.

1. What happens when n is large?

The simplex Δ^{n-1} is subdivided on a large number of tiny chambers, and each orbit of chambers represents a different $(n-3)$- dimensional closed smooth manifold.

2. Suppose that the length vector $l \in \Delta^{n-1}$ is chosen randomly. What are the expected topological properties of M_l?

C. Dombri, C. Mazza, T. Kappeler and MF:

Theorem: For a large class of probability measures μ_n on Δ^{n-1}, the expectation of the random variable $b_p(M_l)$ satisfies

$$\mathbb{E}\big(b_p(M_l)\big) \sim \binom{n-1}{p},$$

and besides,

$$\mathbb{E}\big(b_{2p}(N_l)\big) \sim \sum_{i=0}^{n} \binom{n-1}{i}.$$

In both cases, the error is exponentially small in n.

Part II

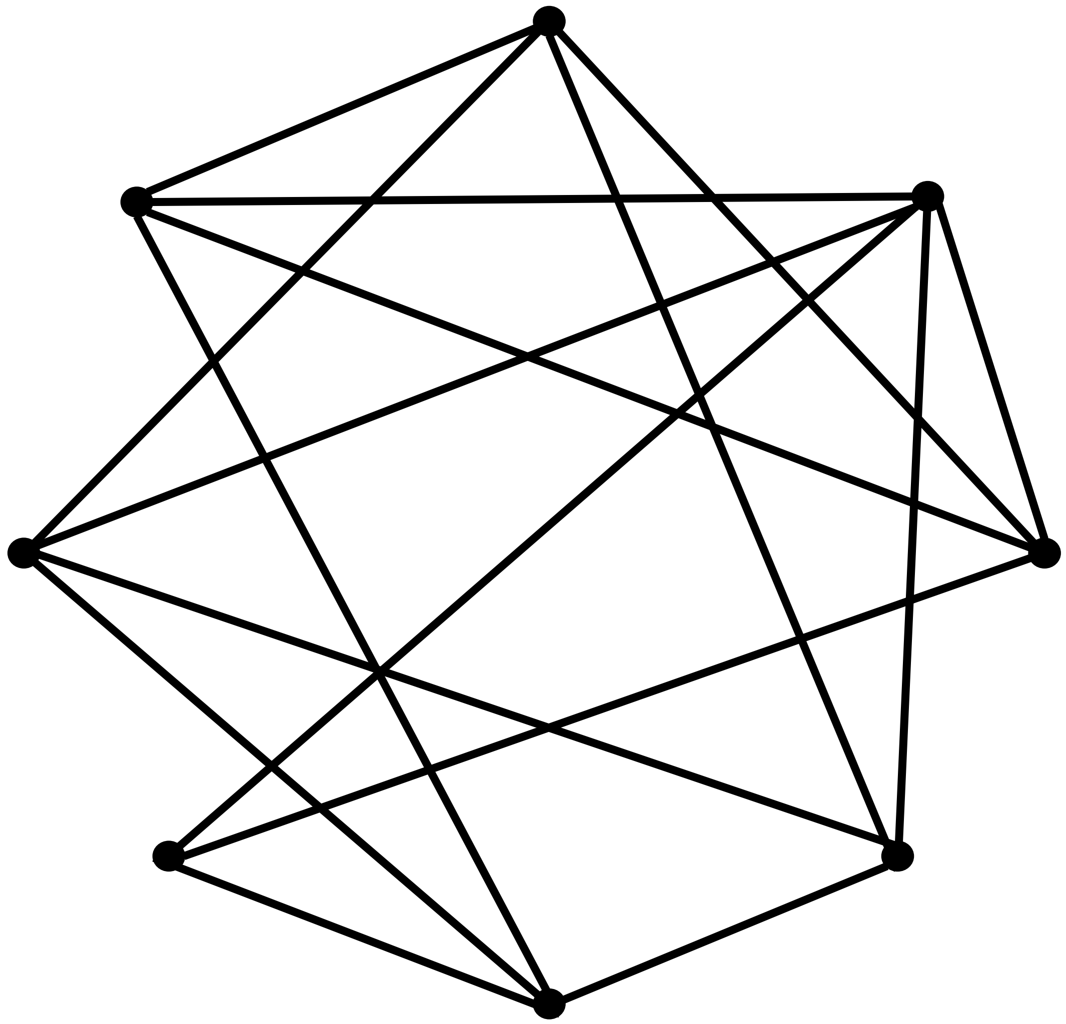
Erdös – Rényi random graphs

$\Gamma \in G(n, p)$

n vertices $\{1,...,n\}$, $\quad n \to \infty$

each edge (ij) is included with probability p

independently of the other edges.



A random graph with n=8 and p=1/2.

Formally one considers the probability space

$G(n, p)$ consisting of all graphs on n vertices

$\Delta_n^{(0)} \subset \Gamma \subset \Delta_n^{(1)}$,

where Δ_n is the simplex on n vertices,

with the probability function $P : G(n, p) \to \mathbb{R}$

$$P(\Gamma) = p^{E(\Gamma)} \cdot (1 - p)^{\binom{n}{2} - E(\Gamma)} .$$

We are interested in properties of large random graphs,

i.e. in properties which hold with probability tending to 1 as $n \to \infty$.

We say that a graph property Q holds a.a.s. (asymptotically almost surely)

if the probability of the set

$P\left(\Gamma \in G(n, p); \Gamma \in Q\right) \to 1$ as $n \to \infty$.

Theorem (Erdos - Renyi, 1959) :

Let $\omega \to \infty$. If

$$p \geq \frac{\log n + \omega}{n}$$

then a random graph $\Gamma \in G(n, p)$ is connected, a.a.s.

If

$$p \leq \frac{\log n - \omega}{n}$$

then a random graph $\Gamma \in G(n, p)$ is disconnected, a.a.s.

Linial - Meshulam model for random simplicial complexes

In 2006 N. Linial and R. Meshulam initiated the topological study of random simplical complexes.

One starts with a complete graph on n vertices $\{\mathbf{1},\mathbf{...},n\}$ and adds each triangle (ijk) with probability $p \in (\mathbf{0},\mathbf{1})$ independently of each other.

Formally, one denotes by $Y(n, p)$ the set of subcomplexes

$$\Delta_n^{(1)} \subset Y \subset \Delta_n^{(2)}$$

and defines the probability function

$$P : Y(n, p) \to \mathbb{R}$$

by the formula

$$P(Y) = p^{f_2(Y)} \cdot (1 - p)^{\binom{n}{3} - f_2(Y)}, \quad Y \in Y(n, p),$$

where $f_2(Y)$ denotes the number of faces in Y.

Theorem (Linial and Meshulam, 2006) :

Let $\omega \to \infty$. If

$$p \geq \frac{2\boldsymbol{log}\, n + \omega}{n}$$

then a random 2-complex $\mathrm{Y} \in \mathrm{Y}(n, p)$ satisfies $H_{\mathbf{1}}(Y; \mathbb{Z}_{\mathbf{2}}) = \mathbf{0}$, a.a.s.

If

$$p \leq \frac{2\boldsymbol{log}\, n - \omega}{n}$$

then a random 2-complex $Y \in \mathrm{Y}(n, p)$ satisfies $H_{\mathbf{1}}(Y; \mathbb{Z}_{\mathbf{2}}) \neq \mathbf{0}$, a.a.s.

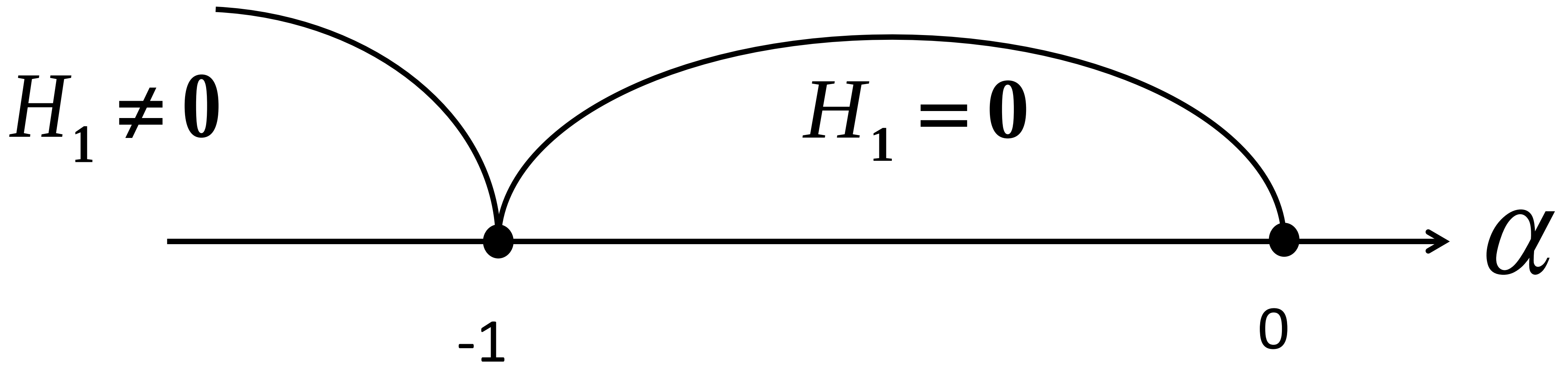
In deterministic topology one studies the topological properties of specific spaces and manifolds.

In stochastic topology (dealing with large random spaces) one may predict (with high probability) the topological properties of a space knowing how many simplexes of various dimensions it has.

Simplifying assumption: $p = n^{\alpha}, \quad \alpha < 0.$

Then the above Theorem of Linial and Meshulam

can be expressed as follows:



A recent Theorem of Hoffman, Kahle and Paquette (2013) states that the same phase transition happens with the integral homology at $\alpha = -\mathbf{1}$:

If $\alpha > -\mathbf{1}$ then $H_1(Y;\mathbb{Z}) = \mathbf{0}, \quad a.a.s.$

If $\alpha < -\mathbf{1}$ then $H_1(Y;\mathbb{Z}) \neq \mathbf{0}, \quad a.a.s.$

$f_2(Y)$ - the number of 2-simplexes in Y.

$$E(f_2) = p\binom{n}{3} \sim n^{\alpha+3}.$$

Intuitively, when α increases a random 2-complex Y has more faces.

The fundamental group of Y

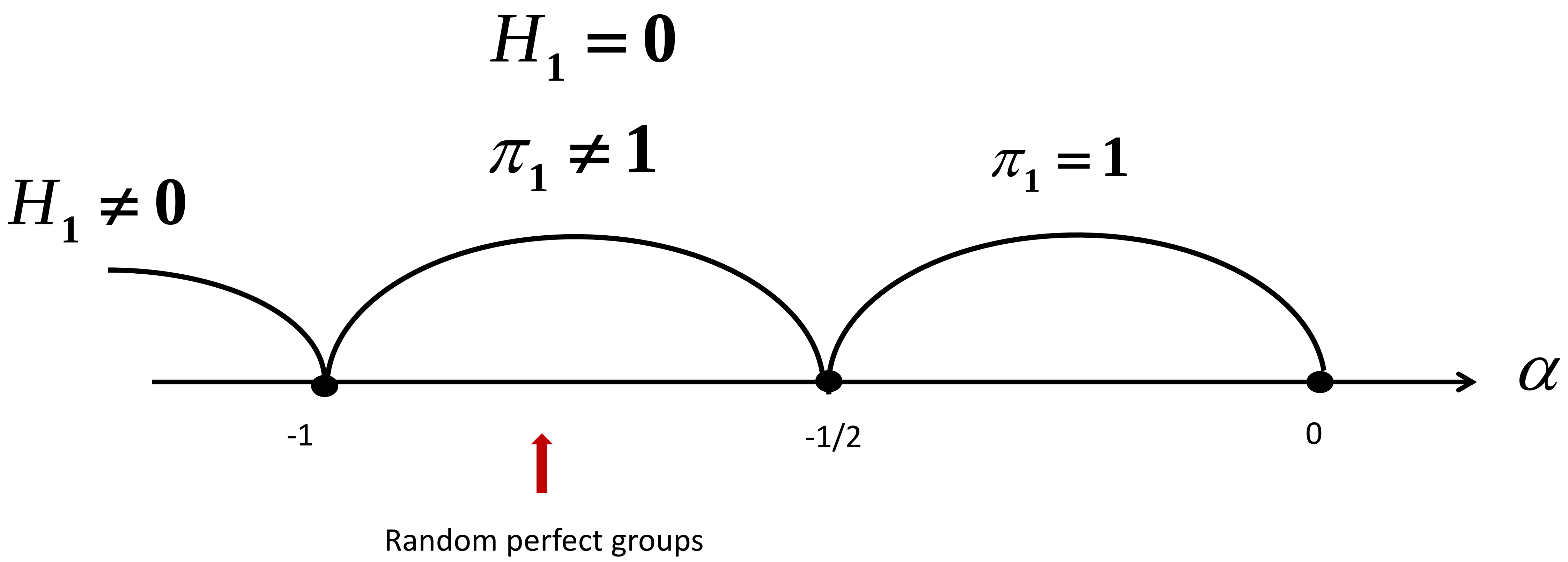
If $\alpha > -\frac{1}{2}$ then $\pi_1(Y) = 1,$ a.a.s.

If $\alpha < -\frac{1}{2}$ then $\pi_1(Y) \neq 1,$ a.a.s.

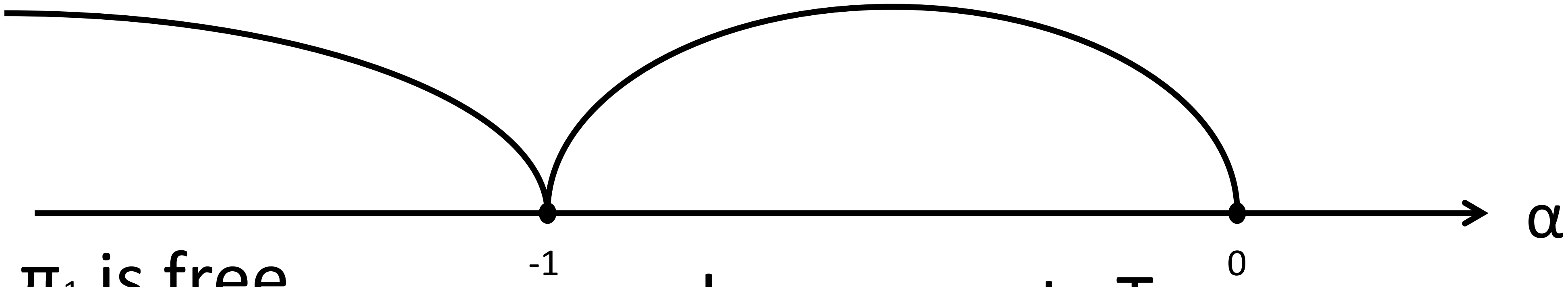
Moreover, for $\alpha < -\frac{1}{2}$ the fundamental group $\pi_1(Y)$

is hyperbolic in the sense of Gromov.

E. Babson, C. Hoffman, M. Kahle (2011).



$H_1 = 0$
$\pi_1 \neq 1$
$\pi_1 = 1$
$H_1 \neq 0$
α
-1
-1/2
0
Random perfect groups



π_1 is free

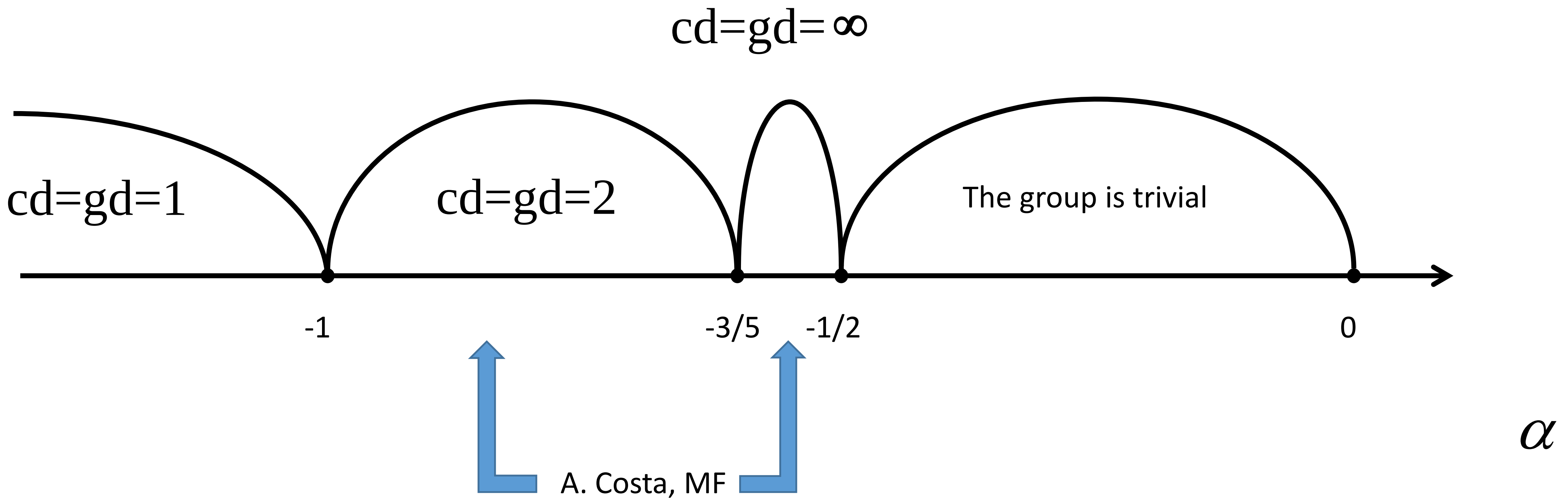
CCFK

π_1 has property T

M. Kahle

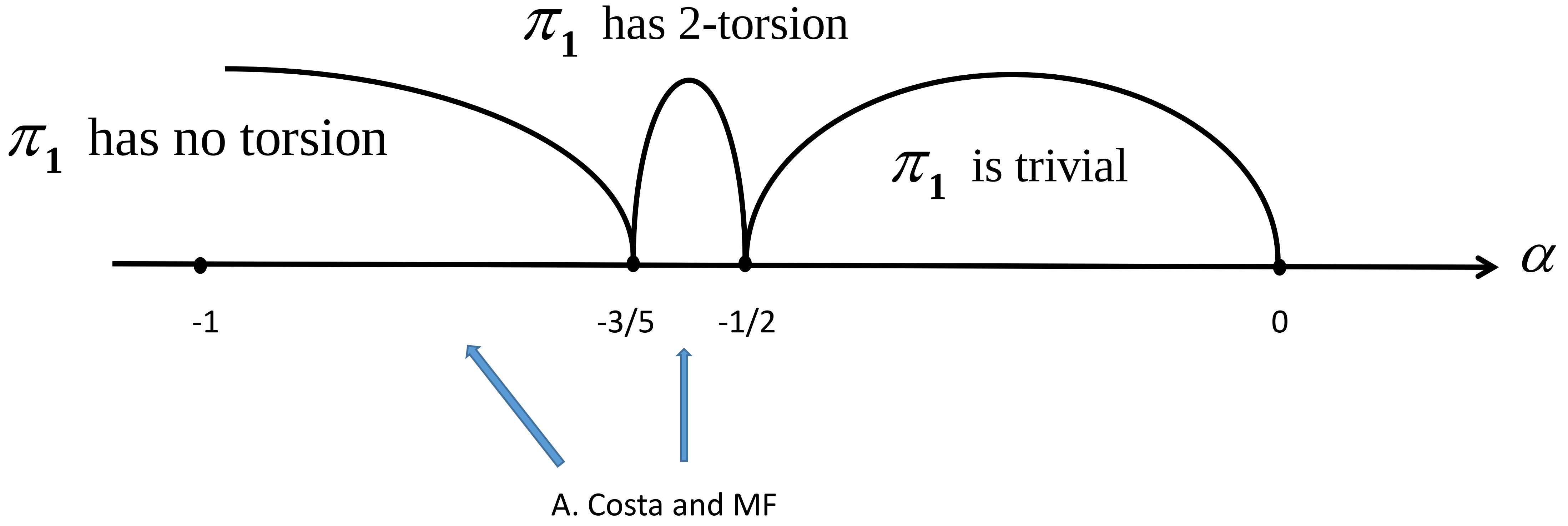
In fact, for α<-1 the complex $Y \in Y\left(n,\, p = n^{\alpha}\right)$ collapses simplicially to a graph, a.a.s.

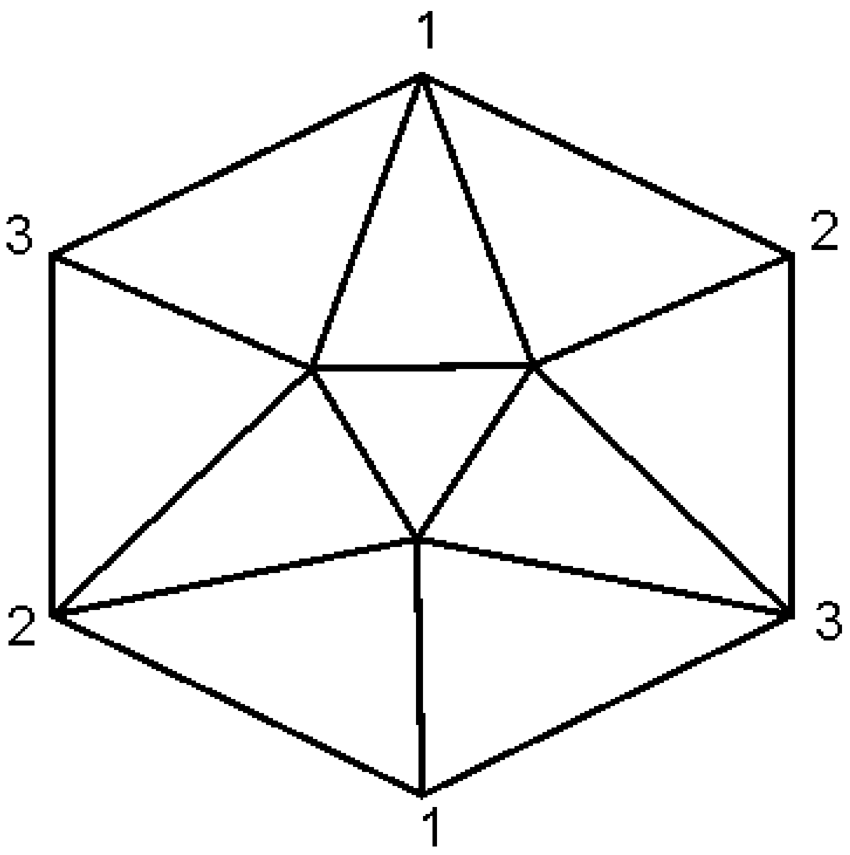
Cohomological and Geometric dimension



We see that probabilistically the Eilenberg-Ganea conjecture is satisfied, for any given $\alpha \neq -1, -\frac{3}{5}, -\frac{1}{2}$, i.e. probability that it holds tends to 1 as $n \to \infty$.

Torsion in the fundamental group of random 2-compelxes





Triangulation S of the real projective plane with 6 vertices and 10 faces

$3/5 = 6/10 = v/f$

If α>-3/5 then a random 2-complex $Y \in Y\left(n, n^{\alpha}\right)$ contains

S as an essential subcomplex, i.e.

$\pi_1\left(S\right) = \mathbb{Z}_2 \to \pi_1\left(Y\right)$

is injective.

Hence cd$\left(\pi_1\left(Y\right)\right)$=∞.

Let $m > \mathbf{2}$ be an odd prime.

Then for any $\alpha \neq$ -1/2 the fundamental group $\pi_{\mathbf{1}}(Y)$,

where $Y \in Y(n, n^{\alpha})$, has no $m-$torsion, a.a.s.

A.Costa and MF

The Whitehead Conjecture

Let X be a 2-dimensional finite simplicial complex.

X is called *aspherical* if $\pi_2(X) = 0$.

Equivalently, X is aspherical if the universal cover $\tilde{X}$ is contractible.

Examples of aspherical 2-complexes: Σ_g with $g > 0$;

N_g with $g > 1$.

Non-aspherical are S^2 and P^2 (the real projective plane).

In 1941, J.H.C. Whitehead suggested the following question:

Is every subcomplex of an aspherical 2-complex also aspherical?

This question is known as the Whitehead conjecture.

Theorem: If $p = n^{\alpha}$, where $\alpha < -1/2$, then a random 2-complex $Y \in Y(n, p)$ with probability tending to one as $n \to \infty$ has the following property: any aspherical subcomplex $Y' \subset Y$ satisfies the Whitehead Conjecture, i.e. all subcomplexes $Y'' \subset Y'$ are also aspherical.

A. Costa, MF