Big polygon spaces

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Cohomology & Poincaré pairing

X compact oriented manifold, $H^*(X) =$ de Rham cohomology

Poincaré pairing:

$$PP\colon H^*(X) \times H^*(X) \to \mathbb{R}$$
$$([\alpha],[\beta]) \mapsto \int_X \alpha \wedge \beta$$

Classical fact

The Poincaré pairing PP is perfect ($\Leftrightarrow$ non-degenerate).

Equivariant cohomology

G compact connected Lie group of rank r, acting on X

Equivariant cohomology: $H^*_G(X) = H^*(\Omega^*_G(X))$

Cartan model: $\Omega^*_G(X) = \left(\Omega^*(X) \otimes \mathbb{R}[\mathfrak{g}^*]\right)^G$ with differential

$$d_G(\alpha \otimes f) = d\alpha \otimes f + \sum_{k=1}^{r} \iota_{\xi_k}\alpha \otimes x_k f$$

($\xi_1, \ldots, \xi_r$ generating v. f. of basis of $\mathfrak{g}$; $x_1, \ldots, x_r$ dual basis of $\mathfrak{g}^*$)

Definition can be extended to G-stable closed subsets of X.

$A = \mathbb{R}[\mathfrak{g}^*]^G$ polynomial ring in r indeterminates of even degrees

$H^*_G(X)$ is an A-algebra, f. g. as A-module

trivial action $\Rightarrow H^*_G(X) \cong H^*(X) \otimes A$ free A-module

(locally) free action $\Rightarrow H^*_G(X) \cong H^*(X/G)$ torsion module

Equivariant Poincaré pairing

The A-bilinear **equivariant Poincaré pairing**

$$PP_G\colon H^*_G(X) \times H^*_G(X) \to A$$

is induced by the pairing

$$\Omega^*_G(X) \times \Omega^*_G(X) \to A$$
$$(\alpha \otimes f, \beta \otimes g) \mapsto \left(\int_X \alpha \wedge \beta\right) fg$$

Proposition (Ginzburg, Brion)

$H^*_G(X)$ free $/A \Rightarrow PP_G$ perfect

Equivariant Poincaré pairing II

Proposition

PP_G non-degenerate $\Leftrightarrow$ $H_G^*(X)$ torsion-free $/A$

(Special cases: Chang–Skjelbred, Bredon, Guillemin–Ginzburg–Karshon)

Theorem

PP_G perfect

$\Leftrightarrow$ *$H_G^*(X)$ reflexive A-module*

$\Leftrightarrow$ *The following sequence is exact:*

$$0 \to H_G^*(X) \to H_G^*(X_0) \xrightarrow{\delta} H_G^{*+1}(X_1, X_0)$$

(Special cases: Brion, Goertsches–Rollenske, Allday–F–Puppe)

$X_0 = \{x \in X \mid \operatorname{rank} G_x = r\}$, $X_1 = \{x \in X \mid \operatorname{rank} G_x \geq r - 1\}$

$X_0 = X^T$ if $G = T$. Both X_0 and X_1 may be singular.

Syzygies

$A \cong \mathbb{R}[t_1, \dots, t_r]$, M f. g. A-module

M ***k*-th syzygy**: $\exists$ exact sequence

$$0 \to M \to F_1 \to \cdots \to F_{k-1} \to F_k$$

with $F_1, \dots, F_k$ f. g. free $/A$

Syzygies interpolate between torsion-freeness and freeness:

$$\begin{aligned} \text{zeroeth syzygy} &= \text{any } M \\ \text{first syzygy} &= \text{torsion-free} \\ \text{second syzygy} &= \text{reflexive} \\ &\vdots \\ r\text{-th syzygy} &= \text{free} \end{aligned}$$

Syzygies in equivariant cohomology I

Let T be a maximal torus of G.

Restriction: X G-manifold $\Rightarrow$ X T-manifold

Induction: Y T-manifold $\Rightarrow$ $\hat{Y} = G \times_T Y$ G-manifold

Proposition

i) $H^*_G(X)$ k-th syzygy$/A_G \iff H^*_T(X)$ k-th syzygy$/A_T$

ii) $H^*_T(Y)$ k-th syzygy$/A_T \iff H^*_G(\hat{Y})$ k-th syzygy$/A_G$

Syzygies in equivariant cohomology II

For non-compact X, syzygies of any order can appear as $H_G^*(X)$.
However, in the presence of Poincaré duality:

Theorem

If $H_G^*(X)$ is a syzygy of order $\geq r/2$, then it is free over A.

(Torus case: Allday ($r = 2$), Allday–F–Puppe)

For instance: $r \leq 2$: $H_G^*(X)$ torsion-free $\Rightarrow$ free
$r \leq 4$: $H_G^*(X)$ reflexive $\Rightarrow$ free

Proof of the theorem

Set $X_i = \{x \in X \mid \text{rank } G_x \geq r - i\}$, $k \geq 1$

1) $H_G^*(X)$ is k-th syzygy $\iff$ the first line of the sequence

$$0 \to H_G^*(X) \to H_G^*(X_0) \to H_G^{*+1}(X_1, X_0) \to \cdots \to H_G^{*+k-1}(X_{k-1}, X_{k-2}) \\ \to H_G^{*+k}(X_k, X_{k-1}) \to \cdots \to H_G^{*+r}(X_r, X_{r-1}) \to 0$$

is exact

2) For $j \geq 1$, the j-th cohomology of this complex is

$$\operatorname{Ext}_A^j(H_G^*(X), A)$$

(For general X this would involve *equivariant homology*.)

3) M is k-th syzygy $\implies \operatorname{Ext}_A^j(M, A) = 0$ for $j > r - k$

Syzygies in equivariant cohomology II

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Question

Is the bound "$r/2$" sharp?

Enough to look at torus actions, so $G = T$ for the rest of the talk.

Syzygies in equivariant cohomology III

Example (F–Puppe)

There is a 7-dimensional compact orientable **mutant** Z with an action of $T = (S^1)^3$ s. t.

$$H_T^*(Z) \cong A \oplus \mathfrak{m}[1] \oplus A[6] \oplus A[7]$$

is torsion-free, but not reflexive. ($\mathfrak{m} \lhd A$ maximal ideal)

Construction uses Hopf fibration $S^3 \to S^2$ (also possible: $r = 5, 9$).

Non-equivariantly, Z is a connected sum of products of spheres:

$$Z \approx \underset{3}{\#}\, S^3 \times S^4$$

Similar connected sums appear in the work of López de Medrano and Bosio–Meersseman on intersections of real quadrics.

Looking for a generalization

First observation (Gómez–López de Medrano)

The mutant "looks like" the real algebraic variety Y defined by

$$\sum_{j=1}^{3} \lambda_j |z_j|^2 + uv = 0$$

$$\sum_{j=1}^{3} |z_j|^2 + |u|^2 + |v|^2 = 1$$

where $\lambda_j = e^{2\pi i j/3}$ and z_1, z_2, z_3, u, $v \in \mathbb{C}$. T rotates the z_j.

Second observation

There is a change of variables such that Y is given by

$$|z_j|^2 + |u_j|^2 = 1 \quad (j = 1, 2, 3), \qquad u_1 + u_2 + u_3 = 0$$

where z_1, z_2, z_3, u_1, u_2, $u_3 \in \mathbb{C}$.

Polygon spaces

Choose a **length vector** $\ell \in \mathbb{R}^r_{\geq 0}$.

The **polygon space** $E_{2a}(\ell)$ is the real algebraic variety defined by

$$\begin{cases} |u_j|^2 = 1 \quad (j = 1, \ldots, r) \\ \ell_1 u_1 + \cdots + \ell_r u_r = 0 \end{cases}$$

where $a \geq 1$ and $u_1, \ldots u_r \in \mathbb{C}^a$.

Polygon space have been studied by Walker (1985), Hausmann, Klyachko, Knutson, Farber, Schütz, Fromm, ...

$E_{2a}(\ell)$ is a compact orientable manifold if ℓ is **generic**:

$$\forall J \subset \{1, \ldots, r\} \qquad \sum_{j \in J} \ell_j \neq \sum_{j \notin J} \ell_j$$

Depending on which side dominates, J is called ℓ-**long** or ℓ-**short**.

Equivalent length vectors (= same long/short sets) give diffeomorphic polygon spaces.

Big polygon spaces

The **big polygon space** $X_{a,b}(\ell)$ is defined by

$$\begin{cases} |z_j|^2 + |u_j|^2 = 1 \quad (j = 1, \dots, r) \\ \ell_1 u_1 + \cdots + \ell_r u_r = 0 \end{cases}$$

where a, $b \geq 1$ and $u_1, \dots\ u_r \in \mathbb{C}^a$, $z_1, \dots\ z_r \in \mathbb{C}^b$.

$=$ configuration space of chains of vectors of prescribed lengths, starting at $0 \in \mathbb{C}^{a+b}$ and ending on a fixed subspace $\mathbb{C}^b \subset \mathbb{C}^{a+b}$

For generic ℓ, $X_{a,b}(\ell)$ is a compact orientable manifold.
$T = (S^1)^r$ acts by rotating the z_j's. The fixed point set is $E_{2a}(\ell)$.

The mutant Z is T-homeomorphic to $X_{1,1}(1,1,1)$.

Observation

Betti sum of $X_{a,b}(\ell) = 2^r >$ Betti sum of $E_{2a}(\ell)$

$\implies H_T^*(X_{a,b}(\ell))$ not free $/A \implies$ syzord $H_T^*(X_{a,b}(\ell)) < r/2$

Equivariant cohomology of big polygon spaces

Theorem

- syzord $H^*_T(X_{a,b}(\ell)) \le \mu(\ell) - 1$, *where*

$$\sigma_\ell(J) = \#\{\, j \in J \mid J \setminus j \;\; \ell\text{-short} \,\}$$
$$\mu(\ell) = \min\{\, \sigma_\ell(J) \mid J \; \ell\text{-long and } \sigma_\ell(J) > 0 \,\}$$

- *Assume* $0 \le m < r/2$. *Then*

$$\operatorname{syzord} H^*_T(X_{a,b}(0, \ldots, 0, \underbrace{1, \ldots, 1}_{2m+1})) = m$$

- $r = 2m+1$: syzord $H^*_T(X_{a,b}(\ell)) = m \;\Leftrightarrow\; \ell \sim (1, \ldots, 1)$
- $r = 2m+2$: syzord $H^*_T(X_{a,b}(\ell)) = m \;\Leftrightarrow\; \ell \sim (0, 1, \ldots, 1)$

Conjecture: One has equality in the first part.

Open questions

Geometric structures? More examples of (maximal) syzygies?

Let $r = 2m + 1$. Our smallest examples for maximal syzygies are the equilateral big polygon spaces $Y_m = X_{1,1}(1, \ldots, 1)$ with $\dim Y_m = 6m + 1$. Are there lower-dimensional examples?

$m = 0$: $\dim Y_0 = \dim S^1 = 1$. This is minimal.

$m = 1$: $\dim Y_1 = 7$. This is also minimal:

Proposition

If X^T is discrete, then $H_T^*(X)$ torsion-free $\Leftrightarrow$ $H_T^*(X)$ free$/A$

$m = 2$: $\dim Y_2 = 13$. The Proposition gives the lower bound 11.

Question

Assume $\dim X^T \leq 2k - 2$. Is the following true?

$$H_T^*(X) \text{ } k\text{-th syzygy} \quad \Leftrightarrow \quad H_T^*(X) \text{ free}/A$$

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Thank you.