Braid Groups and Euclidean Simplices

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Introduction

When Krammer and Bigelow independently proved that braid groups are linear, they used the Lawrence-Krammer-Bigelow representation for generic values of its variables q and t [Kra00, Big01, Kra02]. The t variable is closely connected to the traditional Garside structure of the braid groups and it plays a major role in Krammer's proof [Kra02]. The q variable, associated with the dual Garside structure of the braid groups, has received less attention.

In the special case $t = 1$ and q real, we show that there is an elegant geometric interpretation of the LKB representation that highlights the role of the q variable, at least when it is viewed in Krammer's original basis. Concretely, braid group elements can be viewed as acting on and systematically reshaping euclidean simplices (Theorem A). In fact, for each simple element in the dual Garside structure, the reshaping is an elementary operation that we call edge rescaling (Theorem B).

Braids act by reshaping simplices

The specialized LKB representation that we work with is easy to describe and, in light of our first theorem, we call it the *simplicial representation* of the braid group.

Definition 1 (Simplicial Representation)**.** Let q be a nonzero positive real number, let $\mathcal{E}$ be the set $\{e_{i,j}\}$ with $1 \leq i < j \leq n$ and let V be the $\binom{n}{2}$-dimensional real vector space with $\mathcal{E}$ as its basis. When writing explicit matrices we order the basis $\mathcal{E}$ lexicographically (so that for $n = 4$ the order is $\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$). The *simplicial representation* $\rho : \textsc{Braid}_n \to GL(V)$ is defined by explicitly describing the action of the standard minimal generators of the braid group $r_{1,2}, \ldots, r_{n-1,n}$. If we let $r_{i,i+1}$ also denote the matrix that represents $\rho(r_{i,i+1})$ with respect to the basis $\mathcal{E}$, then:

$$(e_{jk})r_{i,i+1} = \begin{cases} q^2 e_{j,k} & i = j = k-1 \\ qe_{j,i} + (1-q)e_{j,k} + (q^2-q)e_{i,k} & i = k-1 \neq j \\ qe_{i,k} + (1-q)e_{j,k} + (q^2-q)e_{i,j} & i = j-1 \\ e_{j+1,k} & i = j \neq k-1 \\ e_{j,k+1} & i = k \\ e_{j,k} & i, i+1 \notin \{j,k\} \end{cases}$$

Note that the t variable does not appear because we have set it equal to 1 and that the matrix $r_{i,i+1}$ as defined above is acting from the *right*. This differs from the literature but we make this choice so that the action we are interested in is an action from the left.

One part of the action merely permutes the subscripts of the e_{ij}'s according to the standard permutation representation of the braid group. We write $R_{i,i+1}$ for the matrix which remains when this permutation has been stripped away. As an illustration, in the simplicial representation of $\textsc{Braid}_4$, the matrix R_{12} acts on column vectors as follows:

$$R_{12} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} q^2 a \\ (q^2-q)a + qb + (1-q)d \\ (q^2-q)a + qc + (1-q)e \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ c' \\ d' \\ e' \\ f' \end{bmatrix} \qquad (1)$$

Our first claim is that if the column vector with entries a through f represents the squared edge lengths of a euclidean tetrahedron then the same is true for the column vector with entries a' through f'. More precisely we prove the following:

Theorem A (Braids act by reshaping simplices)**.** *The simplicial representation ρ as defined above preserves the set of $\binom{n}{2}$-tuples of positive reals that represent the squared edge lengths of a euclidean $(n-1)$-simplex when acting from the left. In particular, if $\mathbf{v}$ is a column vector that records the squared edge lengths of a euclidean simplex and β is a braid, then the column vector $\rho(\beta)\cdot\mathbf{v}$ also records the squared edge lengths of a euclidean simplex.*

The idea behind the proof is to use *Cayley-Menger determinants*, a well-known way to test whether or not a list of real numbers come from squared edge lengths of a euclidean simplex. To come from an actual simplex, it is necessary and sufficient that the Cayley-Menger determinant for the full simplex and for various subsimplices have certain specified signs. For edges and triangles, the determinant inequalities require that the entries are strictly positive, and that their square roots satisfy the triangle inequality. The first non-obvious restriction is for a tetrahedron and we illustrate it with the column vectors shown in (1). Using standard row and column operations it is straightforward to show that the follow equality holds:

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a' & b' & c' \\ 1 & a' & 0 & d' & e' \\ 1 & b' & d' & 0 & f' \\ 1 & c' & e' & f' & 0 \end{bmatrix} = q^2 \cdot \det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & b & c \\ 1 & a & 0 & d & e \\ 1 & b & d & 0 & f \\ 1 & c & e & f & 0 \end{bmatrix} \qquad (2)$$

This shows that the two Cayley-Menger determinants have the same sign. Similar results hold for every standard generator, for any number of strings, and for every subsimplex. This is sufficient to prove Theorem A.

Generators act by edge rescaling

The standard generators of the braid group (in the simplicial representation) reshape simplices in a very elementary way that we call edge rescaling.

Definition 2 (Edge Rescaling)**.** Let Δ and Δ' be two euclidean simplices with labeled vertices in a common vector space. We say that an edge e in Δ is merely *rescaled* if it and the corresponding edge e' in Δ' point in the same direction. More generally, we say that Δ' is an *edge rescaling* of Δ if there exist enough pairs of corresponding edges pointing in the same direction (but with possibly different lengths) to form a vector space basis out of these common direction vectors. An example is shown in Figure 1.

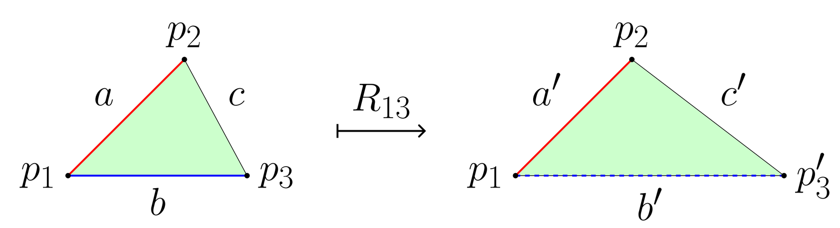

Figure 1: A reshaping that fixes e_{12} and rescales e_{13}.

Proposition 3 (Rescaling an edge of a triangle)**.** *Let Δ be a triangle whose edges have squared lengths a, b and c. If Δ' is the triangle obtained by fixing the a edge and rescaling the b edge by a factor of q, then the squared edge lengths of the new triangle are $a' = a, b' = q^2 b$, and $c' = (1-q)a + (q^2-q)b + qc$.*

The values a' and b' are immediate and c' follows from the law of cosines. Note the similarity with the entries of the simplicial representation. The reshaping described in (1) fixes the edges in the triangle Δ_{234} and rescales the edge e_{12} by a factor of q. See Figure 2.

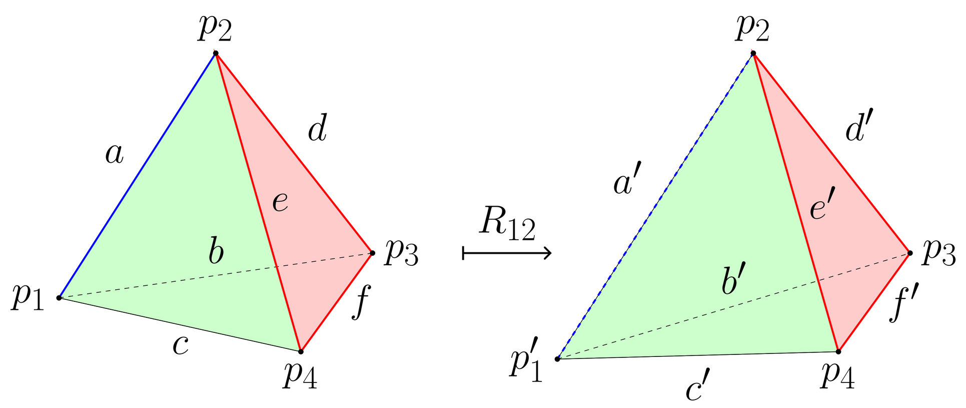

Figure 2: A reshaping that fixes Δ_{234} and rescales e_{12}.

A more precise statement of Theorem A would be that the standard generators act on the set of labeled euclidean simplices by an edge rescaling followed by a permutation of the vertex labels.

Noncrossing partitions and dual simples

To describe the way that dual simple elements reshape simplices, we need to recall noncrossing partitions and the dual Garside structure of the braid group.

Definition 4 (Noncrossing partitions)**.** Let $\mathbb{D}_n$ be a disc in $\mathbb{R}^2$ with n points arranged so that they are the vertices of a convex n-gon labeled 1 through n in the order they occur in its boundary. A partition of these n points is called a *noncrossing partition* if distinct blocks have disjoint convex hulls. These partitions form a bounded graded lattice under the refinement order. See Figure 3.

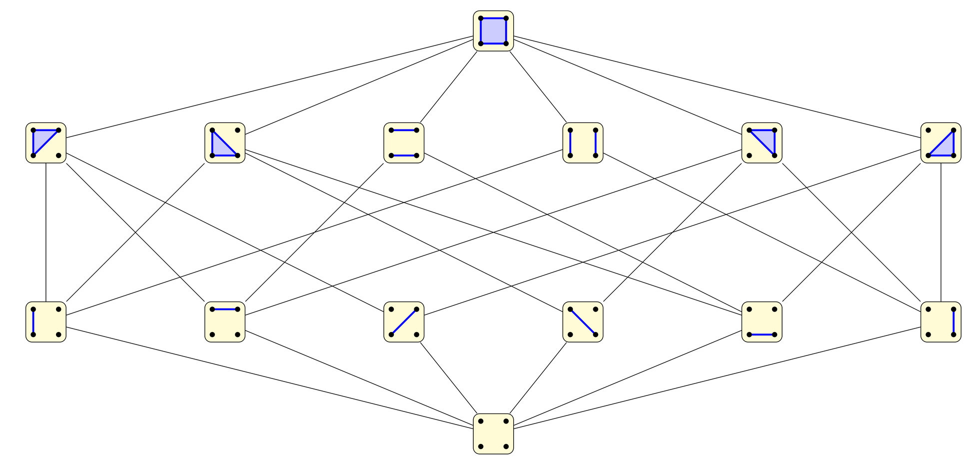

Figure 3: Noncrossing partitions for $n = 4$.

Elements of the braid group can be identified with (equivalence classes of) motions of the labeled points in the disc $\mathbb{D}_n$ and the dual simple elements are a set of braid group elements indexed by the noncrossing partitions as follows.

Definition 5 (Rotations)**.** The *dual Garside element* δ of the n-string braid group is the motion where each labeled point in $\mathbb{D}_n$ moves clockwise along the boundary of the convex hull of all n points to the next vertex. We call this rotating the vertices. More generally, for each set $B \subset \{1, \ldots, n\}$, let P_B be the convex hull of the vertices indexed by B and let $\mathbb{D}_B$ be an ϵ-neighborhood of P_B. The braid group element r_B is a similiar motion restricted to the subdisc $\mathbb{D}_B$, i.e. the vertices in the subdisc move clockwise along one side of the polygon P_B to the next vertex, leaving all other vertices fixed. See Figure 4. In this notation, the dual Garside element δ is the rotation $r_{1,2,\ldots,n}$ and the identity $1 = r_\emptyset$. When B has two elements, the points avoid collisions by passing on the left.

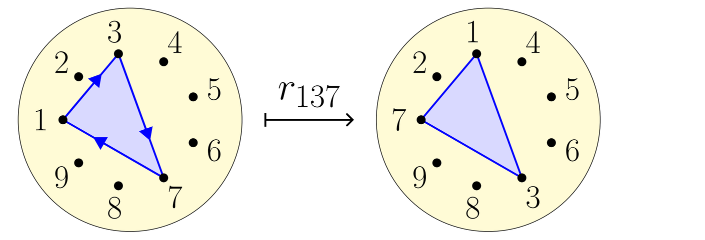

Figure 4: The rotation r_{137}.

Rotations can be used to assign a braid to each noncrossing partition.

Definition 6 (Dual simple elements)**.** The *dual simple elements* of the braid group are in one-to-one correspondence with the set of noncrossing partitions. More precisely, for each noncrossing partition, we associate the product of the rotations corresponding to each of its blocks. Because rotations of noncrossing blocks commute, the resulting element in the braid group is well-defined. The dual simple elements in $\textsc{Braid}_4$ written as products of rotations are shown in Figure 5.

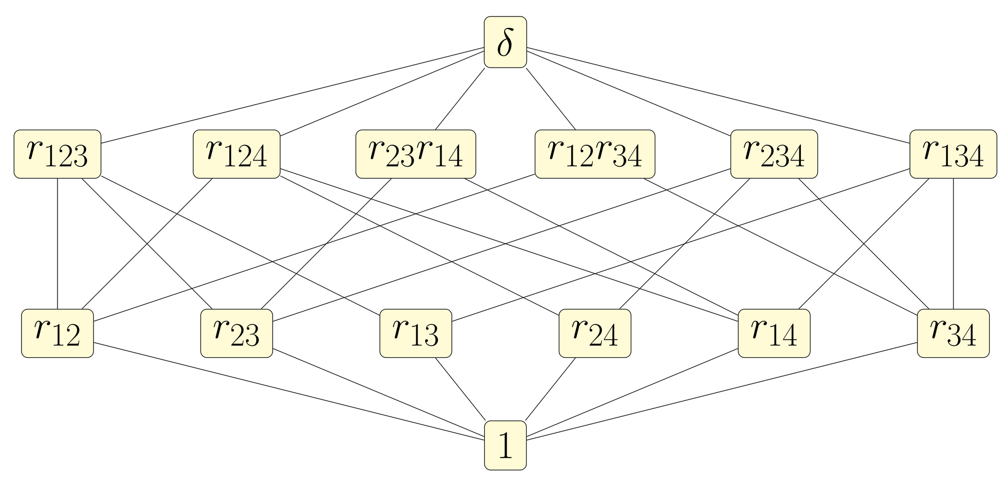

Figure 5: Dual simple elements for $n = 4$.

Dual simples act by edge rescaling

The action of the dual simple elements under the simplicial representation can be described as an edge rescaling based on a noncrossing partition and its left/right complement.

Definition 7 (Left/right complements)**.** Given two group elements s and δ, there are, of course, unique elements s' and s'' such that $s's = \delta$ and $ss'' = \delta$. When δ is the dual Garside element of the braid group and s is one of its dual simple elements, it turns out that the elements s' and s'' are also dual simple elements called the *left* and *right complement* of s, respectively. For example, the left complement of r_{136} in $\textsc{Braid}_9$ is $r_{12}r_{345}r_{6789}$ and its right complement is $r_{23}r_{456}r_{1789}$.

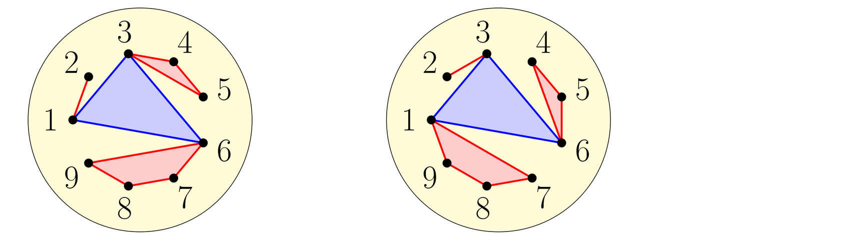

Figure 6: The left and right complements of r_{136}.

Definition 8 (Hypergraphs and hypertrees)**.** A *hypergraph* is a generalization of a graph where its *hyperedges* are allowed span more than two vertices, and a *hypertree* is the natural generalization of a tree. As can be seen in Figure 6, the blocks of the noncrossing partition associated to a dual simple element and the blocks of one of its complements together form the hyperedges of a planar hypertree.

We connect diagrams in the disc $\mathbb{D}_n$ to high-dimensional simplices via their vertex labelings. For example, the three blocks of the left complement of r_{136} shown in Figure 6 correspond to an edge, a triangle and a tetrahedron in any 8-dimensional simplex with 9 labeled vertices.

Theorem B (Dual simples act by edge rescaling)**.** *Under the simplicial representation of the braid group, each dual simple element acts by fixing the length and direction of the edges corresponding to the blocks of its right complement, rescaling the edges corresponding to its own blocks by a factor of q and then permuting the labels on the vertices. If the vertex relabeling is performed first, then the left complement is used instead of the right complement.*

Final remarks

We conclude with a few remarks about the broader context.

- The set of euclidean simplices, with dilated simplices identified, is one of the standard parameterizations of the higher rank symmetric space $SL(V)/SO(V)$ and the simplicial representation induces a braid group action by isometries on this space.
- The simplicial representation is not faithful for large n because it is essentially the same as the symmetric tensor square of the Burau representation (which is known to not be faithful for $n \geq 5$).
- Similar constructions/interpretations should be possible for the other spherical Artin groups, but we have not yet investigated these. The idea would be to rescale root systems rather than euclidean simplices.

References

[Big01] Stephen J. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. **14** (2001), no. 2, 471–486 (electronic). MR 2002a:20043

[Kra00] Daan Krammer, *The braid group B_4 is linear*, Invent. Math. **142** (2000), no. 3, 451–486. MR 2001k:20078

[Kra02] ——, *Braid groups are linear*, Ann. of Math. (2) **155** (2002), no. 1, 131–156. MR 2003c:20040