The structure of euclidean Artin groups

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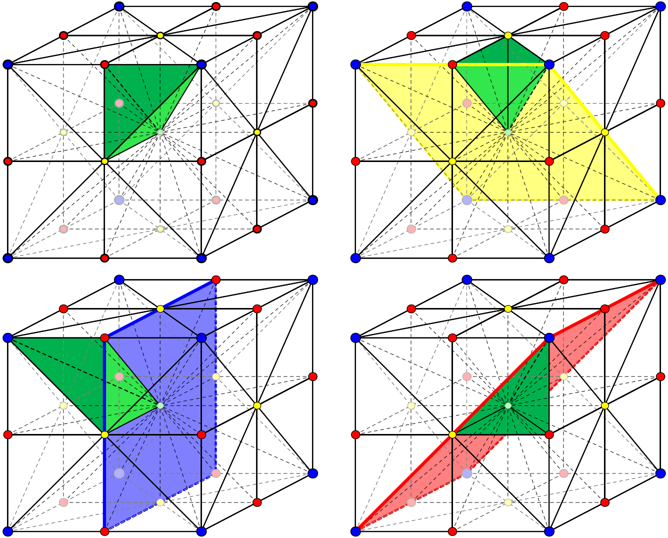
Coxeter groups

The spherical and euclidean Coxeter groups are reflection groups that act geometrically on spheres and euclidean space. They arise in the study of regular polytopes and Lie theory.

Their classification is classical and their presentations are encoded in the well-known Dynkin diagrams and extended Dynkin diagrams, respectively, using conventions sufficient for these groups, but not for general Coxeter groups.

The spherical Coxeter group Cox(B_3)



General Coxeter groups

Spherical and euclidean Coxeter groups are key examples that motivate the general theory introduced by Jacques Tits in the early 1960s. All Coxeter groups are defined by simple presentations encoded in diagrams.

In that first (unpublished) paper, Tits proved that every Coxeter group has a faithful linear representation preserving a symmetric bilinear form and thus has a solvable word problem.

Coxeter groups can be coarsely classified by the signature of the symmetric bilinear forms they preserve. The spherical and euclidean groups are those which have positive definite and positive semi-definite forms.

General Artin groups

Artin groups first appear in print in 1972 (Brieskorn and Saito, Deligne). General Artin groups are defined by simple presentations that can be encoded in the same diagrams as Coxeter groups and then coarsely classified in the same way.

Those early papers connected spherical Artin groups to the fundamental groups of spaces derived from complexified hyperplane complements and successfully analyzed their structure.

Given the centrality of euclidean Coxeter groups and the elegance of their structure, one might have expected euclidean Artin groups to be well understood shortly thereafter. It is now 40 years later and these groups are still revealing their secrets.

Basic Questions

In a recent article Eddy Godelle and Luis Paris highlight four basic conjectures about irreducible Artin groups:

Conjectures

A) *Every Artin group is torsion-free*

B) *Every non-spherical Artin group has trivial center*

C) *Every Artin group has a solvable word problem*

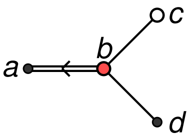
D) *Artin groups satisfy the* $K(\pi, 1)$ *conjecture*

They also remark:

> *"A challenging question in the domain is to prove Conjectures A, B, C, and D for the so-called Artin-Tits groups of affine type, that is, those Artin-Tits groups for which the associated Coxeter group is affine."*

Example: $\mathrm{Art}(\widetilde{B}_3)$

The group $\mathrm{Art}(\widetilde{B}_3)$ has diagram



and presentation

$$\left\langle a, b, c, d \;\middle|\; \begin{matrix} abab = baba & cd = dc \\ bcb = cbc & ad = da \\ bdb = dbd & ac = ca \end{matrix} \right\rangle$$

The basic questions were open for this group until very recently. Callegaro, Moroni and Salvetti answered some of them.

Known: planar Artin groups

The few previously known results about euclidean Artin groups are easy to review.

In 1987 Craig Squier successfully analyzed the structure of the three irreducible euclidean Artin groups $\text{ART}(\widetilde{A}_2)$, $\text{ART}(\widetilde{C}_2)$ and $\text{ART}(\widetilde{G}_2)$ that correspond to the three irreducible euclidean Coxeter groups acting on the euclidean plane.

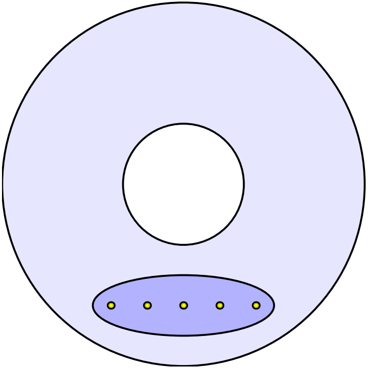
He worked directly with the presentations and analyzed them as amalgamated products and HNN extensions of well-known groups.

His techniques do not appear to generalize to higher dimensions.

Known: euclidean braid groups

The euclidean braid group $\mathrm{ART}(\widetilde{A}_n)$ embeds into the annular braid group $\mathrm{ART}(B_{n+1})$, and this makes its structure clear. In fact, there is a short exact sequence

$$\mathrm{ART}(\widetilde{A}_n) \hookrightarrow \mathrm{ART}(B_{n+1}) \twoheadrightarrow \mathbb{Z}$$



Known: types A and C

Finally, there are recent results due to François Digne.

Theorem (Digne)

The groups $\textsc{Art}(\widetilde{A}_n)$ *and* $\textsc{Art}(\widetilde{C}_n)$ *have Garside structures.*

Digne uses the embedding $\textsc{Art}(\widetilde{A}_n) \hookrightarrow \textsc{Art}(B_{n+1})$ to show that type A has a Garside structure and then an embedding of type C into type A to show the same for type C.

To my knowledge, these are the only euclidean Artin groups that were previously fully understood, and they did not include simple examples such as $\textsc{Art}(\widetilde{B}_3)$.

New: all euclidean Artin groups

Robert Sulway and I provide positive solutions to Conjectures A, B and C for all euclidean Artin groups and we also make progress on Conjecture D. In particular, we prove the following:

Theorem (M-Sulway)

Every irreducible euclidean Artin group $\text{ART}(\widetilde{X}_n)$ is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.

The proof uses the structure of intervals in euclidean Coxeter groups and other euclidean groups generated by reflections.

Coxeter elements

Definition (Coxeter element)

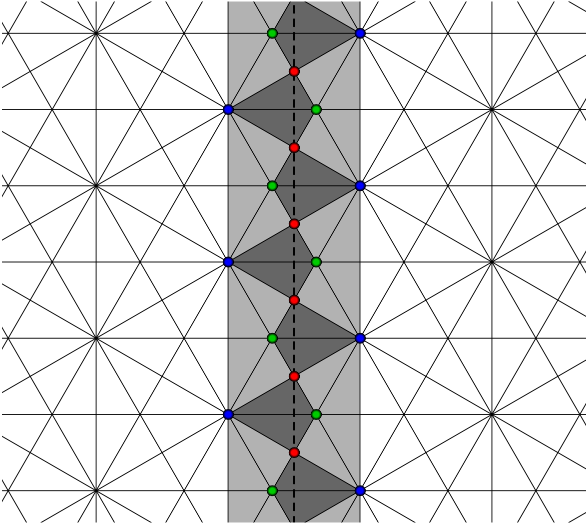
Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with Coxeter generating set S. A *Coxeter element* $w \in W$ is obtained by multiplying the elements of S in some order.

Definition (Axis)

Coxeter elements are hyperbolic isometries of maximal reflection length and the line $\textsc{Min}(w)$ is called its axis. The top-dimensional simplices whose interior nontrivially intersects the axis are called axial simplices and the vertices of these simplices are axial vertices.

The interval $[1, w]^W$ is the portion of the Cayley graph of W w.r.t. all reflections between 1 and w. Not every reflection in W labels an edge in this interval.

The euclidean Coxeter Group $\mathrm{Cox}(\widetilde{G}_2)$



Maximal hyperbolic isometries

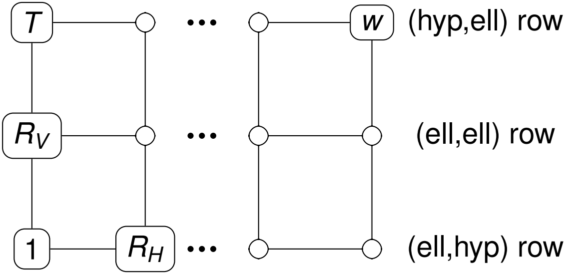
When w is a hyperbolic euclidean isometry of maximal reflection length, its min-set is a line and its move-set is a nonlinear affine hyperplane.

We call the direction of its min-set *vertical* and the orthogonal directions *horizontal*. More generally we call a motion *vertical* if a portion of its motion is in the vertical direction.

For every $u \in [1, w]$ there is a v such that $uv = w$. We split $[1, w]$ into 3 rows based on the types of u and v. When one is hyperbolic, the other is a purely horizontal elliptic. When both are elliptic, both motions have vertical components. Within each row we grade based on the dimensions of the basic invariants.

Coarse structure

Let $L = \textsc{Isom}(E)$ be generated by its reflections. When w is a hyperbolic isometry of maximal reflection length its min-set is a line and $[1, w]^L$ has the following coarse structure:



There is exactly one elliptic in $[1, w]^L$ for each affine subspace $M \subset E$ and exactly one hyperbolic for each affine subspace of $\textsc{Mov}(w) \subset V$. It is NOT a lattice.

Reflection generators

Theorem (M)

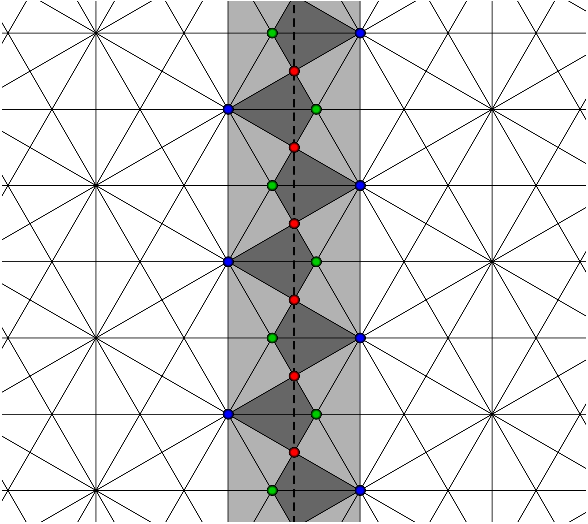
Let w be a Coxeter element of an irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$. A reflection labels an edge in the interval $[1, w]^W$ iff its fixed hyperplane contains an axial vertex.

Definition (Vertical and horizontal)

The set of reflections labeling edges in $[1, w]^W$ consists of every reflection whose hyperplane crosses the Coxeter axis and those reflections which move points horizontally and bound the convex hull of the axial simplices. We call these the vertical and horizontal reflections below w.

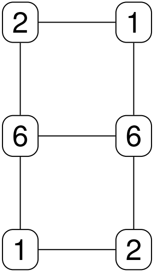
$\textsc{Cox}(\widetilde{G}_2)$ has 2 horizontal reflections.

The euclidean Coxeter Group $\mathrm{Cox}(\widetilde{G}_2)$



Coarse structure of the $\widetilde{G}_2$ interval

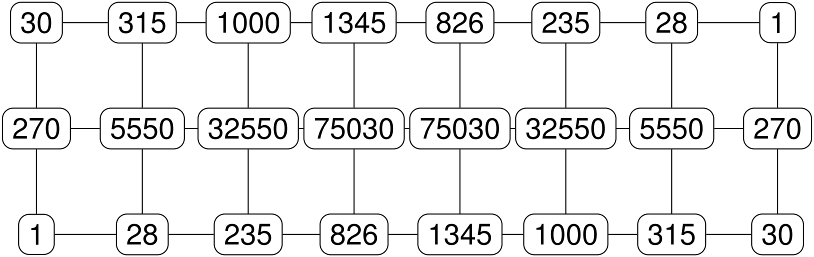
The interval $[1, w]^W$ inside $W = \text{Cox}(\widetilde{G}_2)$ has the following coarse structure:



It has 2 horizontal reflections, 2 translations, 6 infinite families of vertical reflections and 6 infinite families of rotations. Is this a lattice? Yes.

Coarse structure of the $\widetilde{E}_8$ interval

The interval $[1, w]^W$ inside $W = \text{Cox}(\widetilde{E}_8)$ has the following coarse structure:



It has 28 horizontal reflections, 30 translations, 270 infinite families of vertical reflections and 5550 infinite families of vertical rotations about an $\mathbb{R}^6$, etc. Is this a lattice? No.

Intervals and Artin groups

It is now time for two basic questions:

1. Why look at intervals in euclidean Coxeter groups?
2. Why do we care whether or not they are lattices?

And here are the answers:

1. Intervals give alternative presentations of Artin groups.
2. Lattice $\Rightarrow$ Garside $\Rightarrow$ Nice.

We briefly describe how to get presentations from intervals and the consequences of having a Garside structure.

Interval groups and dual presentations

Intervals lead to presentations for new groups.

Definition (Interval groups)

Let $[1, g]^G$ be an interval in a marked group G. The *interval group* G_g is the group generated by the labels of edges in the interval subject to the relations that are visible in the interval.

Intervals in Coxeter groups lead to interesting groups.

Definition (dual Artin groups)

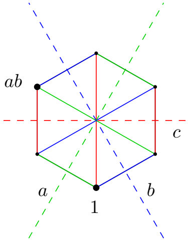
Let w be a Coxter element in a Coxeter group $W = \textsc{Cox}(\Gamma)$ generated by the set of all reflections. The *dual Artin group* $\textsc{Art}^*(\Gamma, w)$ is the interval group W_w, the one defined by the interval $[1, w]^W$.

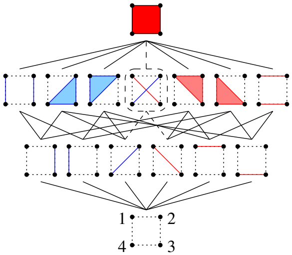
Spherical Artin groups

Theorem (Bessis,Brady-Watt)

If $W = \text{Cox}(X_n)$ is a spherical Coxeter group generated by its reflections, and w is a Coxeter element, then $[1, w]^W$ is a W-noncrossing partition lattice and $W_w = \text{Art}^*(X_n, w)$ is naturally isomorphic to $\text{Art}(X_n)$.

If $W = \text{Sym}_3$, then $W_w = \langle a, b, c \mid ab = bc = ca \rangle \cong \text{Braid}_3$.





Euclidean Artin groups

It is not known in general whether Artin groups and dual Artin groups are isomorphic, hence the distinct names. Fortunately, a key result from quiver representation theory allows us to simplify the dual presentations in the euclidean case and prove the following:

Theorem (M)

If $W = \text{Cox}(\widetilde{X}_n)$ is an irreducible euclidean Coxeter group generated by its reflections, and w is a Coxeter element, then the dual Artin group $W_w = \text{Art}^*(\widetilde{X}_n, w)$ is naturally isomorphic to $\text{Art}(\widetilde{X}_n)$.

In other words, the interval $[1, w]^W$ give a new infinite presentation of the corresponding Artin group.

Garside structures

For this talk we treat Garside structures as a black box. We are only interested in sufficient conditions and consequences.

Theorem (Sufficient conditions)

Let G be a group with a symmetric generating set closed under conjugation. For each $g \in G$, if the interval $[1, g]^G$ is a lattice, then G_g is a Garside group in the expanded sense of Digne.

Theorem (Garside consequences)

If G_g is a Garside group in the expanded sense of Digne, then G_g is a torsion-free group with a solvable word problem and a finite dimensional classifying space.

Artin groups as Garside groups?

Many dual Artin groups are known to be Garside.

Theorem (Artin/Garside)

A dual Artin group W_w is Garside when W_w (1) is spherical (2) is free, (3) is type $\widetilde{A}_n$ or $\widetilde{C}_n$ (4) has rank 3, or (5) has all $m_{ij} \geq 6$.

(1) is due to Bessis and Brady-Watt, (2) is Bessis, (3) is Digne, (4) and (5) are unpublished results with John Crisp.

Conjecture

All dual Artin groups are Garside groups.

This conjecture is too optimistic and false.

Horizontal Roots

It turns out that for euclidean groups, the lattice property is closely related to the structure of its horizontal root system.

Definition (Horizontal root system)

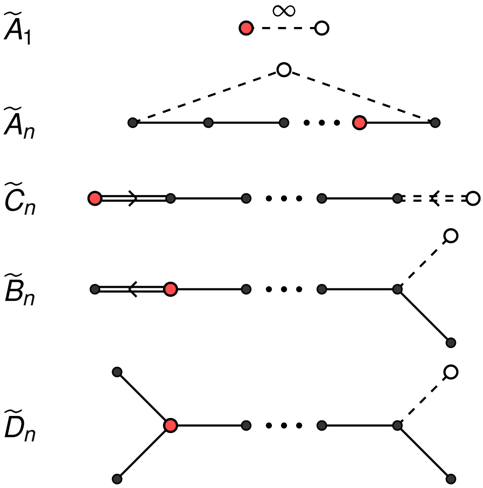
The horizontal reflections have roots orthogonal to the Coxeter axis. These roots form a subroot system that we call the horizontal root system.

Remark (Finding horizontal roots)

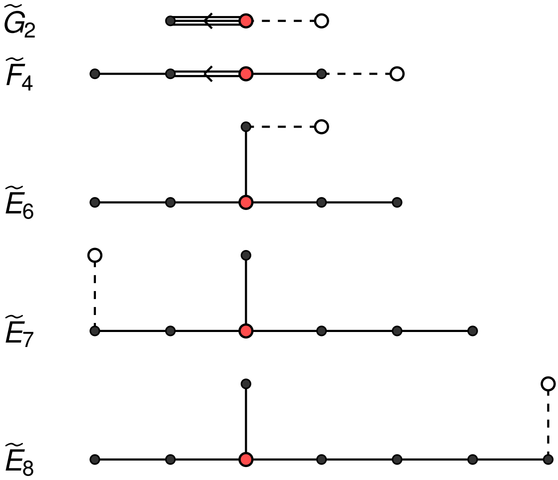
The horizontal root system is described by the subdiagram obtained by removing both the white and the red dots. The red dot is the long end of a multiple bond or the branch point if either exists. For type $\widetilde{A}_n$ there are many choices.

The key property is connectivity of the remaining graph.

Four infinite families



Five sporadic examples



Horizontal Root Systems

Type	Horizontal root system
A_n	$\Phi_{A_{p-1}} \cup \Phi_{A_{q-1}}$
C_n	$\Phi_{A_{n-1}}$
B_n	$\Phi_{A_1} \cup \Phi_{A_{n-2}}$
D_n	$\Phi_{A_1} \cup \Phi_{A_1} \cup \Phi_{A_{n-3}}$
G_2	Φ_{A_1}
F_4	$\Phi_{A_1} \cup \Phi_{A_2}$
E_6	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_2}$
E_7	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_3}$
E_8	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_4}$

Notice that types C and G are irreducible, types B, D, E and F are reducible and for type A it depends.

Failure of the lattice property

Theorem (M)

The interval $[1, w]^W$ is a lattice iff the horizontal root system is irreducible. In particular, types C and G are lattices, types B, D, E and F are not, and for type A it depends on the choice of Coxeter element.

Corollary (M)

The dual Artin group $\textsc{Art}^*(\widetilde{X}_n, w)$ is Garside when X is C or G and it is not Garside when X is B, D, E or F. When the group has type A there are distinct dual presentations and the one investigated by Digne is the only one that is Garside.

These infinite intervals are just barely not lattices and we make further progress by filling in the gaps.

Middle groups

The way we fill the gaps relies of the properties of an elementary group.

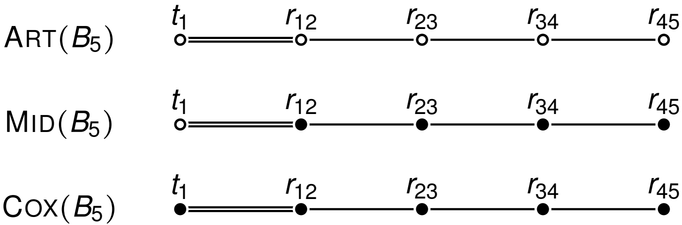
Definition (Middle groups)

We call the symmetries of $\mathbb{Z}^n$ generated by coordinate permutations and integral translations the middle group $\text{MID}(B_n)$. It is generated by the reflections r_{ij} that switch coordinates i and j and the translations t_i that adds 1 to the i-th coordinate.

This is a semidirect product $\mathbb{Z}^n \rtimes \text{SYM}_n$ with the translations generating the normal free abelian subgroup.

Middle groups and presentations

$\text{MID}(B_n)$ is minimally generated by $\{t_1\} \cup \{r_{12}, r_{23}, \ldots, r_{n-1\,n}\}$ and it has a presentation similar to $\text{ART}(B_n)$ and $\text{COX}(B_n)$.



Solid means order 2 and empty means infinite order. Factorizations of $t_1 r_{12} r_{23} \cdots r_{n-1\,n}$ form a type B noncrossing partition lattice. This explains the B_n in the notation.

Relatives of middle groups

The middle group is closely related to several Coxeter groups and Artin groups, hence its name.

$$\begin{array}{ccccc}
\textsc{Art}(\widetilde{A}_{n-1}) & \hookrightarrow & \textsc{Art}(B_n) & \twoheadrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\textsc{Cox}(\widetilde{A}_{n-1}) & \hookrightarrow & \textsc{Mid}(B_n) & \twoheadrightarrow & \mathbb{Z} \\
 & & \downarrow & & \\
 & & \textsc{Cox}(B_n) & &
\end{array}$$

The top row is the short exact sequence that is often used to understand $\textsc{Art}(\widetilde{A}_{n-1})$. Geometrically middle groups are easy to recognize as a symmetric group generated by reflections and a translation with a component out of this subspace.

Diagonal subgroup

The places where the lattice property fails only involve elements from the top and bottom rows of the coarse structure. Thus it makes sense to focus on the corresponding subgroup.

Definition (Diagonal subgroup)

Let R_H and T be horizontal reflections and translations in the interval $[1, w]^W$ and let D denote the subgroup of W generated by $R_H \cup T$. The interval $[1, w]^D$ is a subposet of $[1, w]^W$ consisting of only the top and bottom rows and interval group D_w is the group defined by this restricted interval.

We introduce middle groups because $[1, w]^D$ is almost a poset product and the group D is almost a product of middle groups.

Factored translations

The poset $[1, w]^D$ is almost a product of type B noncrossing partitions lattices and the missing elements are added if we factor the translations.

Definition (Factored translations)

Each pure translation t in $[1, w]^D$ projects nontrivially to the Coxeter axis and to each of the k components of the horizontal root system. Let t_i be the translation which agrees with t on the i-th component and contains $1/k$ of the translation in the Coxeter direction. Let T_F denote the set of all factored translations.

The factorable group F is the crystallographic group generated by $R_H \cup T_F$. The product of the t_i's is t, the i-th horizontal roots and t_i generate a middle group and $[1, w]^F$ is a product of type B noncrossing partition lattices.

New groups

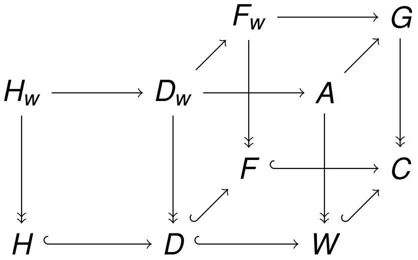
Finally let H be the subgroup of W generated by R_H alone and let C be the crystallographic group generated by $R_H \cup R_V \cup T_F$. This gives five groups so far:

Name	Symbol	Generating set
Horizontal	H	R_H
Diagonal	D	$R_H \cup T$
Coxeter	W	$R_H \cup R_V \; (\cup \; T)$
Factorable	F	$R_H \cup T_F \; (\cup \; T)$
Crystallographic	C	$R_H \cup R_V \cup T_F \; (\cup \; T)$

Let D_w, F_w, W_w and C_w be based on the interval $[1, w]$ in each, and let H_w be the horizontal portion of D_w.

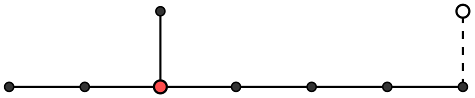
Ten groups

We define ten groups for each choice of $w \in W = \text{Cox}(\widetilde{X}_n)$.
Here are some of the maps between them.



H and W are Coxeter, D, F and C are crystallographic, and the groups on the top are derived from the ones below. We write $A = W_w$ and $G = C_w$ since these are an Artin group and a previously unstudied Garside group.

Example: $\widetilde{E}_8$ groups



Example

Since the horizontal E_8 root system decomposes as $\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_4}$, the group F is a central product of $\textsc{Mid}(B_2)$, $\textsc{Mid}(B_3)$ and $\textsc{Mid}(B_5)$. In addition,

- $[1, w]^F \cong NC_{B_2} \times NC_{B_3} \times NC_{B_5}$,
- $F_w \cong \textsc{Art}(B_2) \times \textsc{Art}(B_3) \times \textsc{Art}(B_5)$,
- $H_w \cong \textsc{Art}(\widetilde{A}_1) \times \textsc{Art}(\widetilde{A}_2) \times \textsc{Art}(\widetilde{A}_4)$, and
- $H \cong \textsc{Cox}(\widetilde{A}_1) \times \textsc{Cox}(\widetilde{A}_2) \times \textsc{Cox}(\widetilde{A}_4)$.

Thm A: crystallographic Garside groups

The addition of the factored translations as generators solves the lattice problem.

Theorem (Crystallographic Garside groups)

If $C = \textsc{Cryst}(\widetilde{X}_n, w)$ is the crystallographic group obtained by adding the factored translations to the Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$, then the interval $[1, w]^C$ is a lattice. As a consequence, this interval defines a group $G = C_w$ with a Garside structure.

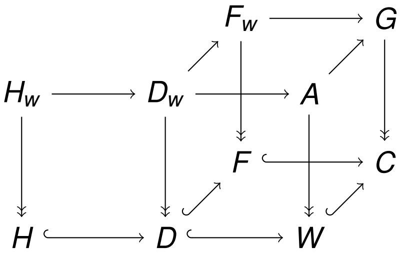
I wrote `GAP/Sage` code to compute the intervals and check the lattice property. We prove the theorem for the infinite families and then rely on the program for the sporadic cases.

Thm B: Artin groups as subgroups

Euclidean Artin groups are understandable because they are subgroups of Garside groups.

Theorem (Subgroup)

For each Coxeter element $w \in W = \mathrm{Cox}(\widetilde{X}_n)$, the Garside group G is an amalgamated product of F_w and A over D_w. As a consequence, the euclidean Artin group $A \hookrightarrow G$.



Thm C: structure of euclidean Artin groups

Once we know that euclidean Artin groups are subgroups of Garside groups, we get many structural results for free.

Theorem (Structure)

Every irreducible euclidean Artin group is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.

The only aspect that requires a bit more work is the center. The Garside structure on G, the product structure on F_w, and the fact that we are amalgamating over D_w are all used in the proof that shows the center of A is trivial.

References

This talk is based on three papers and a survey:

• Noel Brady and Jon McCammond, "Factoring euclidean isometries". arxiv:1312.7780

• (John Crisp and) Jon McCammond, "Dual euclidean Artin groups and the failure of the lattice property". arxiv:1312.7777

• Jon McCammond and Robert Sulway, "Artin groups of euclidean type". arxiv:1312.7770

• Jon McCammond "The structure of euclidean Artin groups" arxiv:1312.7781 (survey)