Matroids over a ring: motivations, examples, perspectives

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Matroids and their realizations

The notion of a matroid axiomatizes the relations of linear dependence of a *vector configuration* (i.e. of a list of elements in a vector space).
If a matroid actually comes from such a list, one says that it is realizable.

Given a commutative ring R, we are going to introduce the notion of a matroid over R, that axiomatizes "relations of dependence" of a list of elements in an R-module.
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Example 1: $R = \mathbb{Z}$

Let X be a list of vectors with integer coordinates.
As we have seen in Emanuele's talk, the toric arrangement defined by X do not depends only on the linear algebra of X, but also on its "arithmetics". The same is true for other objects associated to X, such as the *Dahmen-Micchelli space* $DM(X)$.

Then it is desirable to have a structure keeping track of the linear algebra *and of the arithmetics* of X.
This is precisely what matroids over $\mathbb{Z}$ (and previously defined arithmetic matroids) do.

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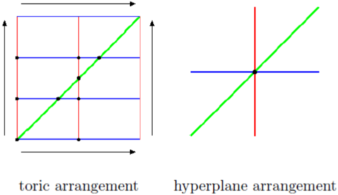

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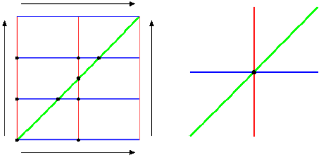

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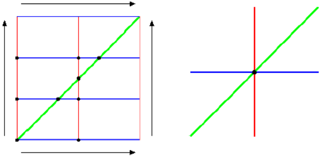

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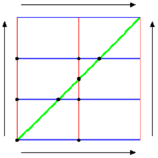

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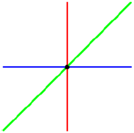

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Example 2: R is a valuation ring

Let F be a field with valuation (for instance the p-adic numbers $\mathbb{Q}_p$, or the Puiseux series $\{\sum_{i=k}^{\infty} a_i t^{i/n}\}$).

Let X be an "integer vector configuration", e.g. a list of elements of F^d with entries in $R = \mathcal{O}_F$. Then we may want to remember not only the linear dependencies, but also the valuations involved.

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Classical matroids: definition and example

A structure that retains the linear algebraic information of a list of vector already exists since the 30s: matroids [Whitney, Maclane].

It has many appeerently unrelated definitions. (Rota: “cryptomorphism”.)

Definition

A matroid M on the finite *ground set* E assigns to each subset $A \subseteq E$ a rank $\mathrm{rk}(A) \in \mathbb{Z}_{\geq 0}$, such that: [. . .]

Main example: realizable matroids

Let $v_1, \ldots, v_n$ be vectors in a vector space V.

$$\mathrm{rk}(A) := \dim \mathrm{span}\{v_i : i \in A\}$$

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(0) $\mathrm{rk}(\emptyset) = 0$

(1) $\mathrm{rk}(A) \leq \mathrm{rk}(A \cup \{b\}) \leq \mathrm{rk}(A) + 1 \qquad \forall A \not\ni b$

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Example

$X = \{(2, 0), (0, 3), (1, -1)\}$

A	$\emptyset$	1	2	3	13	12	23	123
$\mathrm{rk}(A)$	0	1	1	1	2	2	2	2

Matroids over $\mathbb{Z}$: an example

Let $v_1, \ldots, v_n$ be a configuration of vectors in an R-module N.
Already in the case $R = \mathbb{Z}$ we see that it is convenient to take a system of axioms for the *quotients* $N/\langle v_i | i \in A\rangle$:

Realizable example

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Enriched variants of matroids

In general, as we have seen, one might want to capture more than just the linear dependences of a list of vectors:

- Arithmetic matroids come from configurations over $\mathbb{Z}$, and remember indices of sublattices. [D'Adderio-M.]
- Valuated matroids come from configurations over a *field with valuation*, and remember valuations. [Dress-Wenzel]

Matroids over rings encompass these constructions, by taking a new approach: not a matroid decorated with extra data, but a theory with only *one* simple, algebraic axiom.

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Definition

Let R be a commutative ring and E be a finite set.
A matroid over R on the *ground set* E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R-module $M(A)$ satisfying the following axiom:

for all $A \subseteq E$ and $b \neq c \notin A$, there exists a pushout square where all four morphisms are surjections with cyclic kernel:

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{b\}) \\ \downarrow & & \downarrow \\ M(A \cup \{c\}) & \longrightarrow & M(A \cup \{b, c\}) \end{array}$$

Polymatroids are defined similarly, by discarding the “cyclic kernel” condition.

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$$x = x(b, c), \quad y = y(b, c) \in M(A)$$

such that there is a diagram

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Realizability

Fundamental example: "vector configurations" in an R-module.

Given a f.g. R-module N and a list $X = x_1, \ldots, x_n$ of elements of N, we have a matroid M_X associating to $A \subseteq X$ the quotient

$$M_X(A) = N \Big/ \left(\sum_{x \in A} Rx \right).$$

For each $x_i \in X$ there is a quotient map

$$M_X(A) \xrightarrow{/\overline{x_i}} M_X(A \cup \{x_i\})$$

and this system of maps obviously satisfies the axiom.

We say that a matroid M over R is realizable if it actually comes from such a list.

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Classical matroids are matroids over fields

We can, and will, assume that the module $M(E)$ has no nontrivial projective summands, since this makes many results simpler to state.

Proposition

Matroids over a field $\mathbb{K}$ are equivalent to matroids.

A f.g. $\mathbb{K}$-module is determined by its *dimension* $\in \mathbb{Z}$.

If $v_1, \ldots, v_n$ are vectors in $\mathbb{K}^r$,
the dimension of $\mathbb{K}^r / \langle v_i : i \in N \rangle$ is $r - \mathrm{rk}(A)$, the corank of A.

Example

$X = \{(2,0), (0,3), (1,-1)\}$

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$M(A)$	$\mathbb{R}^2$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	0	0	0

Note: The definition of matroids over $\mathbb{K}$ is blind to which field $\mathbb{K}$ is, but for *realizability* the choice of $\mathbb{K}$ matters.

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Sum, deletion, contraction, duality

Let M and M' be matroids over R on E and E'.
We define their direct sum $M \oplus M'$ on $E \amalg E'$ by

$$(M \oplus M')(A \amalg A') = M(A) \oplus M'(A').$$

For $i \in E$, we define two matroids over R on the ground set $E \setminus \{i\}$:
the deletion of i in M, denoted $M \setminus i$, by

$$(M \setminus i)(A) = M(A)$$

and the contraction of i in M, denoted $M \setminus i$, by

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When R is a Dedekind domain, we can also define a dual matroid M^* having the expected properties (omitted).

If M is realizable, $M \setminus i$ and M/i can be realized in the usual way, while M^* can be realized by a generalization of Gale duality.

Sum, deletion, contraction, duality

Let M and M' be matroids over R on E and E'.
We define their direct sum $M \oplus M'$ on $E \amalg E'$ by

$$(M \oplus M')(A \amalg A') = M(A) \oplus M'(A').$$

For $i \in E$, we define two matroids over R on the ground set $E \setminus \{i\}$:
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Tensor product. Localizations and generic matroid

Let $R \to S$ be a map of rings. Then the tensor product $- \otimes_R S$ is a functor $R\text{-Mod} \to S\text{-Mod}$. If M is a matroid over R, then

$$(M \otimes_R S)(A) \doteq M(A) \otimes_R S.$$

defines a matroid over S.

Two special cases will be fundamental for us:

1. For every prime ideal $\mathfrak{m}$ of R, let $R_\mathfrak{m}$ be the localization of R at $\mathfrak{m}$. We call $M \otimes_R R_\mathfrak{m}$ the localization of M at $\mathfrak{m}$.
2. If R is a domain, let $\mathrm{Frac}(R)$ be the fraction field of R. Then we call $M \otimes_R \mathrm{Frac}(R)$ the generic matroid of M.

Notice that every matroid over $R_\mathfrak{m}$ induces a matroid over the residue field $R_\mathfrak{m}/(\mathfrak{m})$.

We can study the matroid M via all these "classical" matroids.

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Dedekind rings and DVR

From now on, we will always assume R to be a Dedekind domain (i.e., an integral domain in which every nonzero proper ideal factors into a product of prime ideals).

The localization of a Dedekind domain at a prime ideal is a DVR (i.e. a Dedekind domain that is not a field and has a unique maximal ideal $\mathfrak{m}$).

(Actually, the theory works in a more general framework:

R is a *Prüfer domain*, i.e. its localizations are *valuation rings*).

Any indecomposible f.g. module over a DVR R is isomorphic to either R or $R/\mathfrak{m}^n$ for some integer $n \geq 1$.

So a f.g. R-module are parametrized by "partitions" that may have some infinitely long lines.

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Local theory: matroids over a DVR

We denote by r_i the cardinality of the i-th row of such a "partition", and by $s_i \doteq \sum_{j \geq i} r_j$.
Let $r_i(Ab)$ be stenography for $r_i(M(A \cup \{b\}))$ and so on.
Our first result is a combinatorial characterization of matroids over a DVR:

Theorem (Fink, M.)

$M : 2^E \to \{$f. g. R-modules$\}$ is a matroid over R if and only if:

- *for every 1-element minor $M(A) \to M(A \cup \{b\})$ the difference of the two "partitions" is a (Pieri-like) stripe (i.e. $r_i(A) \geq r_i(Ab) \geq r_{i+1}(A)$);*
- *for every 2-element minor, the miminum of the three quantities*

$$s_i(Ab) + s_{i+1}(Ac),\ s_i(Ac) + s_{i+1}(Ab),\ s_i(Abc) + s_{i+1}(A)$$

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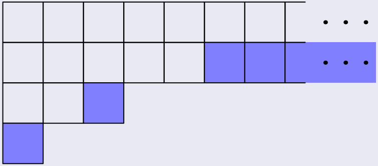

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Furthermore, by looking at the 3-element minors of the matroid M, we get that the minimum of

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These are tropicalizations of the Plücker relations for the Grassmanian! Then we get:

Proposition (Fink, M.)

The vector $(s_i(M(A)), |A| = k)$ *defines a point on the Dressian* $Dr(k, |E|)$*

In fact, we conjecture that in this way we get a point on the Dressian analogue of the full flag variety*. (* polyhedral fans parametrizing tropical linear spaces, and full flags of t.l.s., respectively).

Connections with tropical geometry

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Valuated matroids

A valuated matroid is defined as a matroid decorated with an integer valued function $\mathcal{V}$ on the set of the bases $\mathcal{B}$, satisfying a certain axiom [Dress and Wenzel]. There is a bijection

$$\{\text{tropical linear spaces}\} \longleftrightarrow \{\text{valuated matroids}\}$$

Then, as consequence of the Proposition above, we get:

Corollary (Fink, M.)

Let M be a matroid over a DVR $(R, \mathfrak{m})$.
Then the function $\mathcal{V}(A) \doteq \dim_{R/\mathfrak{m}} M(A)$ makes the generic matroid of M into a valuated matroid.

Then a matroid over a DVR contains richer information than the valuated matroid.

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Matroid polytope - work in progress!

We can also define an (unbounded) polytope in $\mathbb{R}^{|E|+2}$ as follows:

$$P(M) \doteq Conv\Big\{\big(e_A, i, s_i(A)\big), A \subseteq E, i \in \mathbb{N}\Big\} + \mathbb{R}_{\geq 0}(\underline{0}, 1, 0)$$

It is easy to see that $P(M)$ has indeed a finite number of vertices, and that it is possible to recover $P(M)$ from M. Furthermore:

Proposition (Fink, M.)

If we disregard the last coordinate, the direction of each edge of $P(M)$ has the shape $e_i - e_j$ for some i, j.

This generalizes a known fact for classical matroids. Consequences:

- by adding a few simple conditions, one gets a characterization of the polytopes that are $P(M)$ for some M, and hence a cryptomorphic axiomatization for matroids over a valuation ring!
- By replacing A_n by other root systems, Coxeter matroids over a valuation rings can be defined!
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Modules over Dedekind domains

We can now pass to the global theory. Let R be a Dedekind domain.

In order to have a matroid over R, is it sufficient that every localization $M_{\mathfrak{m}}$ is a matroid over the DVR $R_{\mathfrak{m}}$?

NO! In general there is an extra "global" condition.
This will be simple to state, once we will have recalled some facts.

Given an R-module N, let $N_{\text{tors}} \subseteq N$ denote the submodule of its torsion elements, and N_{proj} denote the projective module N/N_{tors}.
Then $N \simeq N_{\text{tors}} \oplus N_{\text{proj}}$.

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NO! In general there is an extra "global" condition.
This will be simple to state, once we will have recalled some facts.

Given an R-module N, let $N_{\mathrm{tors}} \subseteq N$ denote the submodule of its torsion elements, and N_{proj} denote the projective module N/N_{tors}.
Then $N \simeq N_{\mathrm{tors}} \oplus N_{\mathrm{proj}}$.

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Determinant of a module

The Picard group of R, $\mathrm{Pic}(R)$, is the set of the isomorphism classes of f.g. projective modules of rank 1, with product induced by the tensor product.
If P is a projective module of rank n, then $\bigwedge^n P$ is a f.g. projective module of rank $\binom{n}{n} = 1$. We call determinant, and denote by $\det(P)$, its class in $\mathrm{Pic}(R)$.
The algebraic K-theory group $K_0(R)$ of f.g. R-modules is the abelian group generated by iso classes $[N]$ of f.g. R-modules, modulo the relations $[N] = [N'] + [N'']$ for any exact sequence $0 \to N' \to N \to N'' \to 0$.
Fact: there is an isomorphism of groups

$$\Phi : K_0(R) \longrightarrow \mathbb{Z} \oplus \mathrm{Pic}(R).$$

In fact, when P is a projective module, the map above is simply given by $\Phi([P]) = (\mathrm{rk}(P), \det(P))$.
Then for any f.g. R-module N we will still denote by $\mathrm{rk}(N)$ the first summand of $\Phi([N])$, and by $\det(N)$ the second summand of $\Phi([N])$. The former coincides with the rank of N_{proj}.

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Global theory: matroids over a Dedekind domain

By this det function we can characterize matroids over a Dedekind domain R:

Theorem (Fink, M.)

$M : 2^E \to \{$f. g. R-modules$\}$ is a matroid over R if and only if every localization at a prime ideal $\mathfrak{m}$ is a matroid over $R_\mathfrak{m}$, and for every 1-element minor $N \to N'$ we have:

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Arithmetic matroids

If M is a matroid over $\mathbb{Z}$, we define the two functions

$$\operatorname{cork}(A) = \operatorname{rk}(M(A)_{\mathrm{proj}}) \text{ and } m(A) \doteq |M(A)_{\mathrm{tors}}|.$$

As a corollary of the previous theorem, we can prove that $(E, \operatorname{cork}, m)$ is (essentially) an arithmetic matroid, i.e. that the function m satisfies the axioms introduced by [D'Adderio-M].

Notice that matroids over $\mathbb{Z}$ and arithmetic matroids and are *not* truly equivalent, since the information contained in the former is richer, since there are many groups with the same cardinality.

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Definition of the Tutte-Grothendieck ring

Several invariants can be associated to a classical matroid; the universal deletion-contraction invariant is the well-known *Tutte polynomial*.
We will now define and compute the universal deletion-contraction invariant of matroids over any Dedekind domain R.

Essentially following Brylawski, define the Tutte-Grothendieck ring of matroids over R, $K(R\text{-Mat})$, to be the abelian group generated by a symbol $\mathbf{T}_M$ for each matroid M over R, modulo the relations

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Define $\mathbb{Z}[R\text{-Mod}]$ to be the ring of $\mathbb{Z}$-linear combinations of symbols X^N, one for each f.g. R-module N up to isomorphism, and product $X^N X^{N'} = X^{N \oplus N'}$.

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When R is a field, $\mathrm{Pic}(R)$ is trivial and there is no torsion, thus $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}] \simeq \mathbb{Z}[X, Y]$.

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Another invariant that we can obtain from the Grothendieck-Tutte invariant $\mathbf{T}_M$ in the case $R = \mathbb{Z}$ is the Tutte quasi-polynomial

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where $q = (x-1)(y-1)$.

This is a quasi-polynomial in q, interpolating between the classical and the arithmetic Tutte polynomials.

This polynomial was introduced in [Brändén- M.], and has application to generalized colorings and flows on graphs with labeled edges.

Notice that $\mathbf{Q}_M(x, y)$ is not an invariant of the arithmetic matroid, (as it depends on the groups $M(A)_{\mathrm{tors}}$ and not just on their cardinalities), but it is an invariant of the matroid over $\mathbb{Z}$.

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Developments and applications

Future developments:

- study other examples, such as R coordinate ring of an algebraic curve (e.g. the affine line or an elliptic curve);
- provide more cryptomorphic definitions;
- study Coxeter matroids over a valuation ring;

Possible applications:

- combinatorial topology: [Bajo-Burdick-Chmutov], [Duval-Klivans-Martin], ...;
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Developments and applications

Future developments:

- study other examples, such as R coordinate ring of an algebraic curve (e.g. the affine line or an elliptic curve);
- provide more cryptomorphic definitions;
- study Coxeter matroids over a valuation ring;

Possible applications:

- combinatorial topology: [Bajo-Burdick-Chmutov], [Duval-Klivans-Martin], ...;
- tropical geometry;
- intersection theory for arrangements of subtori, toric varieties, ...;
- error-correcting codes over rings.

THANK YOU!