Configuration Spaces
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Matrix and Lie algebras invariants

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Abstract:

"The classical theory of cohomology of compact Lie groups, started by Hopf and E. Cartan, was analized by Chevalley, Weil, H. Cartan, Koszul through the study of invariants in the exterior algebra of simple Lie algebras and the idea of transgression, the entire structure as representation of this exterior algebra poses several challenging problems and I will show some recent results in this area" I will start from matrix theory

1. *Joint work with Matej Brešar and Špela Špenko*
2. *Joint work with Paolo Papi and Corrado De Concini*
3. *Joint work with Papi, De Concini, Moseneder*
4. *work of Salvatore Dolce*

the Cayley–Hamilton identity

A basic Theorem for matrices is the *Cayley–Hamilton identity.* Take an $n \times n$ matrix $X = (x_{i,j})$, one starts from the characteristic polynomial $\chi_X(t) : \det(t1_n - X)$ and the CH theorem is $\chi_X(X) = 0$. if X is a matrix of variables denote this by $\chi_n(X)$

$$e.g.\ n = 2 \quad \chi_2(X) = X^2 - (x_{1,1} + x_{2,2})X + x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$$

$$= X^2 - tr(X)X + 1/2(tr(X)^2 - tr(X^2))$$

quasi–identities

The Cayley–Hamilton identity is a special kind of identity involving matrix variables, in this case just one variable X and the entries $x_{i,j}$ of this matrix variable.

One gets from this, by substituting to X some expression in matrix variables and entries variables, infinitely many identities.

the polarized Cayley–Hamilton identity

$$e.g.\ n = 2 \quad \chi_2(X+Y) - \chi_2(X) - \chi_2(Y) = 0$$

$$= XY + YX - tr(X)Y - tr(Y)X + tr(X)tr(Y) - tr(XY) =$$

$$XY+YX-(x_{1,1}+x_{2,2})Y-(y_{1,1}+y_{2,2})X+x_{1,1}y_{2,2}-x_{1,2}y_{2,1}+y_{1,1}x_{2,2}-y_{1,2}x_{2,1}$$

Amitsur–Levitzki

In non–commutative algebra of particular importance are *polynomial identities* which depend only upon the non–commutative variables (and no entries). In particular recall the

Amitsur–Levitzki theorem

For any integer h one may define the standard polynomial

$$St_h(X_1, \dots, X_h) := \sum_{\sigma \in S_h} \epsilon_\sigma X_{\sigma(1)} \dots X_{\sigma(h)}$$

it is a *non commutative polynomial* which for $h = 2$ gives the *commutative law*.

Amitsur–Levitzki

The theorem of Amitsur–Levitzki states that the standard polynomial

$$St_{2n}(X_1, \ldots, X_{2n})$$

vanishes on the algebra of $n \times n$ matrices over a commutative ring, we speak of the standard identity for matrices.

Why $2n$?, how this 2 arises?, is a rather interesting fact.

The identities of matrices

Denote by $M_n(F)$ the algebra of $n \times n$ matrices with entries in a field F, I will assume F of characteristic 0

- The study of the non commutative polynomials vanishing on matrices is quite a difficult issue.
- In between quasi–identities and polynomial identities there are some quite remarkable type of formal identities, between matrices and their traces.
- These are called *Trace identities of matrices* and arise from invariant theory.

Trace identities of matrices

One should start from the algebra of polynomial maps

$$\boxed{M_n(A) := \{f : M_n(F)^m \to M_n(F)\}}$$

where $A = S[(M_n(F)^*] = F[\xi_{i,j}^h],\ i,j = 1,\dots n,\ h = 1,\dots,m$ is the usual polynomial functions on the mn^2 dimensional space $M_n(F)^m$ of which the $\xi_{i,j}^h$ are coordinates.

The identities of matrices

The linear group $GL(n, F)$ acts on maps by conjugation

$$(gf)(X_1, \dots, X_m) := gf(g^{-1}X_1g, \dots, g^{-1}X_mg)g^{-1}$$

so that an invariant map $f : M_n(F)^m \to M_n(F)$ is an equivariant polynomial map that is

$$f(gX_1g^{-1}, \dots, X_mg^{-1})g^{-1} = gf(X_1, \dots, X_m)g^{-1}$$

in particular we have the coordinate maps

$$X_i : (X_1, \dots, X_m) \mapsto X_i$$

The identities of matrices

Inside the matrix algebra $M_n(F[\xi_{i,j}^h])$ there are three remarkable algebras.

- The algebra of *quasi polynomial maps* generated by the coordinates X_i and by the polynomials in the coordinates $F[\xi_{i,j}^h]$.
- The algebra of equivariant maps $M_n(F[\xi_{i,j}^h])^{GL(n,F)}$.
- The algebra of generic matrices generated by the coordinates X_i.

Encoding quasi, trace and polynomial identities of matrices. The center of the first is $F[\xi_{i,j}^h]$, of the second $F[\xi_{i,j}^h]^{GL(n,F)}$ and the third has a rather mysterious center the *central polynomials*.

The algebra of equivariant maps

The algebra of equivariant maps $f : M_n(F)^m \to M_n(F)$ is a remarkable non–commutative algebra (for $m > 1$) which in a way is the basis of invariant theory for matrices.
Let us denote it by

$$R_{n,m} = (S[(M_n(F)^*)^m] \otimes M_n(F))^{GL(n,F)}.$$

As in classical invariant theory we have a first and a second fundamental theorem (FFT and SFT).

The algebra of equivariant maps

For $m = 1$ the ring of invariants is generated by the coefficients of the characteristic polynomial of X.
$R_{n,1}$ is generated over the invariants by X which satisfies the Cayley–Hamilton identity.
In particular $R_{n,1}$ is a free module, with basis $X^i, \; i = 0, \ldots, n-1$ over the invariants, a polynomial ring in n variables.

The identities of matrices

In general

FFT

$R_{n,m}$ is generated by the variables X_i and the traces $tr(X_{i_1}X_{i_2}\dots X_{i_k})$ of the monomials in the X_i.

SFT

The relations between the variables X_i and the traces $tr(X_{i_1}X_{i_2}\dots X_{i_k})$ of the monomials in the X_i are all consequences of the Cayley–Hamilton identity.

$$e.g.\ n = 2 \quad X^2 - tr(X)X + 1/2(tr(X)^2 - tr(X^2))$$

Representations

For $m > 1$ the algebra $R_{n,m}$ of equivariant maps does not admit an explicit description, except for $n = 1, 2$.

About a year ago I looked at the question of Brešar and Špenko, is it also true that quasi–identities are all consequences of Cayley–Hamilton?
The answer is NO but not easy,

quasi identities and CH

We have a multiplication map

$$0 \to K \to F[\xi_{i,j}^h] \otimes_{F[\xi_{i,j}^h]^{GL(n,F)}} R_{n,m} \to M_n(F[\xi_{i,j}^h])$$

The kernel K measures the quasi identities not consequence of CH. The fact that this may be non zero is difficult, for this we needed to look at a related algebra:

what is the structure of the algebra (under exterior mutiplication)

of antisymmetric equivariant functions from matrices to matrices?

Representations

This is just the invariant algebra

$$[\bigwedge M_n(F)^* \otimes M_n(F)]^{GL(n,F)}$$

under conjugation action.

As a set antisymmetric equivariant functions from matrices to matrices belong to the algebra $R_{n,n^2} = (S[(M_n(F)^*)^{n^2}] \otimes M_n(F))^{GL(n,F)}$ so are describable by the FFT, but!

exterior multiplication is a different structure!

What is exterior multiplication?

of two alternating maps $f(v_1, \ldots, v_h)$, $g(v_1, \ldots, v_k)$ from any vector space V to any algebra L is

$$(f \wedge g)((v_1, \ldots, v_{h+k})$$

$$:= \frac{1}{h!k!} \sum_{\sigma \in S_{h+k}} \epsilon_\sigma f(v_{\sigma(1)}, \ldots, v_{\sigma(h)}) g(v_{\sigma(h+1)}, \ldots, v_{\sigma(h+k)})$$

the product is in L (not necessarily associative) the algebra of alternating maps then is

$$\bigwedge V^* \otimes L.$$

with the tensor product algebra structure.

The structure of $[\bigwedge M_n(F)^* \otimes M_n(F)]^{GL(n,F)}$

Since $F \subset M_n(F)$ as scalar matrices, this (associative) algebra contains the algebra of invariants, $[\bigwedge M_n(F)^*]^{GL(n,F)}$.

This has been long studied since it describes cohomology, it is then classical that $[\bigwedge M_n(F)^*]^{GL(n,F)}$ is a Hopf algebra and by Hopf theorem an exterior algebra in its primitive generators:

$$tr(St_{2k+1}(X_1, \dots, X_{2k+1})), \;\; k = 0, \dots, n-1.$$

The structure of $[\bigwedge M_n(F)^* \otimes M_n(F)]^{GL(n,F)}$

The algebra $[\bigwedge M_n(F)]^{GL(n,F)}$ also contains

all the standard polynomials

$St_k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \epsilon_\sigma X_{\sigma(1)} \ldots X_{\sigma(k)}$

which are clearly equivariant multilinear and antisymmetric functions from k–tuples of matrices to matrices.

The structure of $[\bigwedge M_n(F)^* \otimes M_n(F)]^{GL(n,F)}$

In particular it contains $X = St_1(X_1)$, the identity map!

As matrix, with entries in the exterior algebra $\bigwedge M_n(F)^*$, the map X is the

generic matrix with entries $x_{i,j}$, i.e. n^2 Grassmann variables!.

Exterior multiplication

One easily sees, be the definition of exterior multiplication, that

$$St_k(X_1, \ldots, X_k) = X^k$$

By invariant theory it follows that

$[\bigwedge M_n(F)^* \otimes M_n(F)]^{GL(n,F)}$ is generated by X and by the elements $tr(X^{2i+1})$ which are just the primitive generators of cohomology. In fact it is $\mathbb{Z}/(2)$ graded commutative!

Exercise: Compare it with $[S[M_n(F)^*] \otimes M_n(F)]^{GL(n,F)}$

Theorem

- *The algebra* $[\bigwedge M_n(F)^* \otimes M_n(F)]^{GL(n,F)}$ *is generated by* X *and the elements* $tr(X^{2i-1}),\ i = 1, \dots, n$.
- *All these elements anti commute.*
- $[\bigwedge M_n(F) \otimes M_n(F)]^{GL(n,F)}$ *is a free module with basis* $X^i,\ i = 0, \dots, 2n-1$ *over the Grassman algebra in the elements* $tr(X^{2i-1}),\ i = 1, \dots, n-1$ *and*
- *we have the two defining identities*

$$\boxed{X^{2n} = 0, \quad tr(X^{2n-1}) = -\sum_{i=1}^{n-1} X^{2i} \wedge tr(X^{2(n-i)-1}) + nX^{2n-1}}.$$

How do you prove this theorem?

You have to know a priori that

$$\dim[\bigwedge M_n(F) \otimes M_n(F)]^{GL(n,F)} = n2^n$$

this can be proved in several different ways.

The relation $X^{2n} = 0$ is Amitsur–Levitzki.

Finally you deduce
$tr(X^{2n-1}) = -\sum_{i=1}^{n-1} X^{2i} \wedge tr(X^{2(n-i)-1}) + nX^{2n-1}$ from the Cayley–Hamilton identity.

Proof of Amitsur–Levitzki

We have seen that this Theorem is $X^{2n} = 0$, with X the matrix in Grassmann variables.

Now X^2 is an $n \times n$ matrix with entries in the *even Grassmann algebra*, a commutative algebra.

So $X^{2n} = (X^2)^n = 0$ follows immediately from the fact that

$$tr(X^2)^i = tr(X^{2i}) = tr(St_{2i}(X_1, \dots, X_{2i})) = 0, \ \forall i$$

a statement easy to verify by symmetry.

Other examples

Salvatore Dolce has studied many variations of this case where, using classical invariant theory, one takes symmetric or skew symmetric matrices for orthogonal or symplectic groups.

Simple Lie algebras

The trace zero matrices are a simple Lie algebra, in fact the simplest class and then it was natural to ask if the previous theorem has a counterpart for simple Lie algebras or for other representations.

This is related to cohomology of groups.

Cohomology of compact Lie groups

Due to the work of Hopf, one knows that the cohomology of a group is a special Hopf algebra, that is an exterior algebra in certain odd generators, called primitive of degrees $2m_i + 1$ where the numbers m_i are the *exponents* of the group.
The theory can be quickly reduced to the case of simple groups.

Transgression

If G is a simple compact Lie group one considers the complexified Lie algebra $\mathfrak{g}$ of G and obtains that

$$H^*(G) = \wedge(\mathfrak{g}^*)^G$$

The algebra of Invariant forms, is an exterior algebra

$$\bigwedge(\mathfrak{g}^*)^G = \bigwedge[P_1, \ldots, P_r]$$

where the P_i are the *primitive generators* of the Hopf algebra and r is the *rank* of G.

How to compute the P_i?

Example

For $SU(n, \mathbb{C})$ the complexified Lie algebra is the Lie algebra of $n \times n$ matrices with trace 0.

$$P_i = tr(St_{2i+1}(X_1, \ldots, X_{2i+1})), \quad i = 1, \ldots, n-1$$

$$St_k(X_1, \ldots, X_k) = \sum_{\sigma \in S_k} \epsilon_\sigma X_{\sigma(1)} \ldots X_{\sigma(k)}, \ \forall k$$

Similar formulas involving standard polynomials hold for symplectic and odd orthogonal groups, for even orthogonal groups there is an extra generator, related to the Pfaffian.

Lie algebras

We perform the same construction when $\mathfrak{g}$ is a simple Lie algebra

- Define A as the space of multilinear alternating functions from $\mathfrak{g}$ to $\mathfrak{g}^*$ which are $\mathfrak{g}$-equivariant.

$$A := \left(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*\right)^{\mathfrak{g}} = \hom(\bigwedge \mathfrak{g}, \mathfrak{g}^*)^{\mathfrak{g}} = \hom(\mathfrak{g}, \bigwedge \mathfrak{g}^*)^{\mathfrak{g}}. \quad (1)$$

- By the work of Kostant, it is known that $\dim(A) = 2^r r$.
- The Poincaré polynomial $GM(q)$ describing the dimension of A in each degree is given by a formula conjectured by Joseph and proved by Bazlov (m_i the exponents):

$$GM(q) = (1 + q^{-1}) \prod_{i=1}^{r-1} (1 + q^{2m_i+1}) \sum_{i=1}^{r} q^{2m_i}. \quad (2)$$

Lie algebras

The dimension formulas of Kostant and Bazlov suggested to extend the theory of alternating maps, from matrices to all simple Lie algebras (although in the general case we loose the associative structure).

The space $A := (\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*)^G$ is still a module over the invariant algebra $(\bigwedge \mathfrak{g}^*)^G = \bigwedge(P_1, \dots, P_r)$ cohomology of $\mathfrak{g}$, we have

Theorem

There exist elements $f_i, u_i \in (\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*)^G = A$ of degrees $2m_i, 2m_i - 1$ so that A is a free module, with basis the elements $f_i, u_i, i = 1, \dots, r$, over the exterior algebra $\bigwedge(P_1, \dots, P_{r-1})$.

Lie algebras

The proof of this Theorem requires the formalism of transgression and the theory of Koszul differential, the elements f_i, u_i are constructed this way.
Then we need a non degeneracy property of the scalar product of the differentials of the invariants of the symmetric algebra. This property appears also in a work of Givental, Slodowy on deformations of surface singularities (the swallowtails of Arnhold).
We complete the Theorem with explicit formula of the multiplication for the missing generator P_r.

IN GENERAL

Constructing the primitive generators

- The construction of the elements P_i and even the computation of their degrees has been a challenging project.
- For classical groups we have computations by Richard Brauer,
- for exceptional groups one has to understand first a simpler problem, the study of invariants of the symmetric algebra $S[\mathfrak{g}^*]$ that is the algebra of invariant polynomials on $\mathfrak{g}$.

Cartan subalgebras and Weyl groups

The root system is the set of eigenvalues Φ of a maximal subalgebra of semisimple commuting elements, a Cartan or toral subalgebra $\mathfrak{h}$.
The Weyl group W acts on $\mathfrak{h}$ as reflection group.
A great amount of structure theory for simple Lie algebras is extracted from $\mathfrak{h}, \Phi, W$.

Symmetric algebra

Invariants

When we study the invariants of $S[\mathfrak{g}^*]$ the main result is Chevalley's restriction theorem

Theorem

Under restriction to a Cartan's subalgebra the invariants $S[\mathfrak{g}^*]^G$ are isomorphic to the invariants $S[\mathfrak{h}^*]^W$ under the Weyl group of the reflection representation on the root system.

exponents of exceptional groups

By Chevalley

- the invariants of a reflection representation form a polynomial ring in r generators,
- r is the rank, the degrees of the generators are $m_i + 1$ where the numbers m_i, are the exponents
- The exponents can be computed directly from the root system or from the Coxeter element in the Weyl group.

The exponents (and Coxeter number)

Type	Exponents	h
E_6	1, 4, 5, 7, 8, 11	12
E_7	1, 5, 7, 9, 11, 13, 17	18
E_8	1, 7, 11, 13, 17, 19, 23, 29	30
F_4	1, 5, 7 ,11	12
G_2	1, 5	6

Table 1: exponents for the exceptional types.

IN GENERAL

The P_i are constructed in three steps

Step 1 The dual of the Lie multiplication gives a map

$$\mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$$

this map extends to a homomorphism

$$s : S[\mathfrak{g}^*] \to \bigwedge^{even} \mathfrak{g}^*$$

Transgression

Step 2 there is another map T called transgression from the symmetric algebra of $\mathfrak{g}^*$ to the exterior algebra of $\mathfrak{g}^*$ defined by

$$T(f) = m(s \otimes 1)(df)$$

df is the usual differential.

$$df \in S[\mathfrak{g}^*] \otimes \mathfrak{g}^*, \quad s \otimes 1(df) \in \bigwedge[\mathfrak{g}^*] \otimes \mathfrak{g}^* \ {}^{m} \xrightarrow{\text{multiplication}} \bigwedge[\mathfrak{g}^*]$$

Transgression

Step 3 the generators P_i are obtained by transgression from generators of invariants of the symmetric algebra of $\mathfrak{g}^*$
thus a generator $f \in S[\mathfrak{g}^*]^G$ of degree $e+1$ where e is an exponent gives rise by transgression to a primitive generator $T(f) \in \bigwedge[\mathfrak{g}^*]^G$ of degree $2e+1$

Example

The generator $tr(St_{2i+1}(X_1, \ldots, X_{2i+1}))$ is the transgression of $tr(X^{i+1})$.

The elements f_i, u_i

we have for the generators a_i of $S[\mathfrak{g}^*]^G$ the formulas

$$P_i = T(a_i), \quad f_i = (s \otimes 1)da_i, \quad u_i = (T \otimes 1)da_i.$$

So the main theorem is that $(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*)^G$ is a free module, with basis the elements $f_i, u_i, i = 1, \ldots, r$, over the exterior algebra $\bigwedge(P_1, \ldots, P_{r-1})$. There is an explicit formula for the action of P_r.

In principle this should be a first step in representation theory of the exterior algebra.

Representations

The full representation

It is then interesting to study both the symmetric $S[\mathfrak{g}^*]$ and the exterior algebra $\bigwedge[\mathfrak{g}^*]$ as representations of the adjoint group.

For the symmetric algebra there is a nice theorem of Kostant which exhibits

$$S[\mathfrak{g}^*] = S[\mathfrak{g}^*]^G \otimes H$$

where H are *harmonic polynomials* isomorphic to the coordinate ring of the nilpotent cone.

In particular when we study the isotypic component of some irreducible representation N we have to analyze

$$\boxed{(S[\mathfrak{g}^*] \otimes N^*)^G = S[\mathfrak{g}^*]^G \otimes (H \otimes N^*)^G}$$

We see that

Theorem

Each $(S[\mathfrak{g}^*] \otimes N^*)^G$ *is a a free module over the invariants* $S[\mathfrak{g}^*]^G$.

This Theorem is called separation of variables.

Representations

In particular

it follows that $(S[\mathfrak{g}^*] \otimes \mathfrak{g}^*)^G$ is a free rank r module over the ring of invariants $S[\mathfrak{g}^*]^G$.

For the exterior algebra we have a more elusive picture, there is no separation of variables and different isotypic components behave in different ways. Nevertheless Kostant proved a

Clifford algebra

separation of variables for the Clifford algebra of $\mathfrak{g}$ with the Killing form which as representation is the same as the exterior algebra.

But this gives us no information on the graded structure.

Representations

The Theorem I explained

Theorem

There exist elements $f_i, u_i \in (\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*)^G$ of degrees $2m_i, 2m_i - 1$ so that $(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*)^G$ is a free module, with basis the elements $f_i, u_i, i = 1, \dots, r$, over the exterior algebra $\bigwedge(P_1, \dots, P_{r-1})$.

gives a precise description of the isotypic component of type $\mathfrak{g}$ in $\bigwedge \mathfrak{g}$.
For the other isotypic components at the moment we have no nice description as module over the invariants.

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Invariant theory of symplectic and orthogonal groups
Salvatore Dolce

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