ALTA
Institute for Algebra, Geometry, Topology and their applications

What is a toric pseudoarrangement?

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Objective

Characterization of toric arrangements via matroid theory
(cf. relationship between oriented matroids and pseudosphere arrangement).
Or: "What is a toric pseudoarrangement?"

Toric Arrangements

Let $X = S^1$ or $\mathbb{C}^*$, so $T = X^d$ is either the compact or complex torus. A **toric arrangement** is a finite collection $\mathcal{A}$ of toric hyperplanes in the torus. A toric hyperplane is given as a level set of character.

A character of T is a map $\chi : (\mathbb{C}^*)^d \to \mathbb{C}^*$ given by

$$x \mapsto \chi(x) = x_1^{a_1} \dots x_d^{a_d} \text{ with } a_\chi = (a_1, \dots, a_d) \in \mathbb{Z}^d.$$

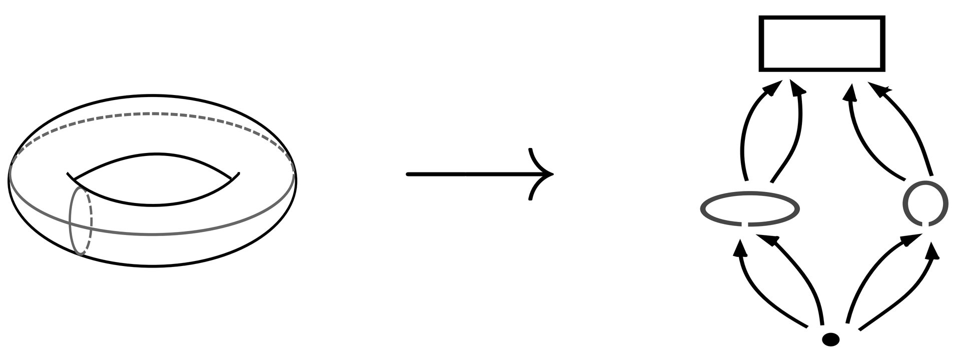
A toric arrangement can be regarded as the set $\mathcal{A} = \{(\chi_i, b_i)\}_i \subset \mathbb{Z}^d \times X$, each pair (χ_i, b_i) defining a toric hyperplane $K_i = \{x \in T \mid \chi_i(x) = b_i\}$. Its complement is denoted by

$$M(\mathcal{A}) = (\mathbb{C}^*)^d - \cup\mathcal{A}.$$

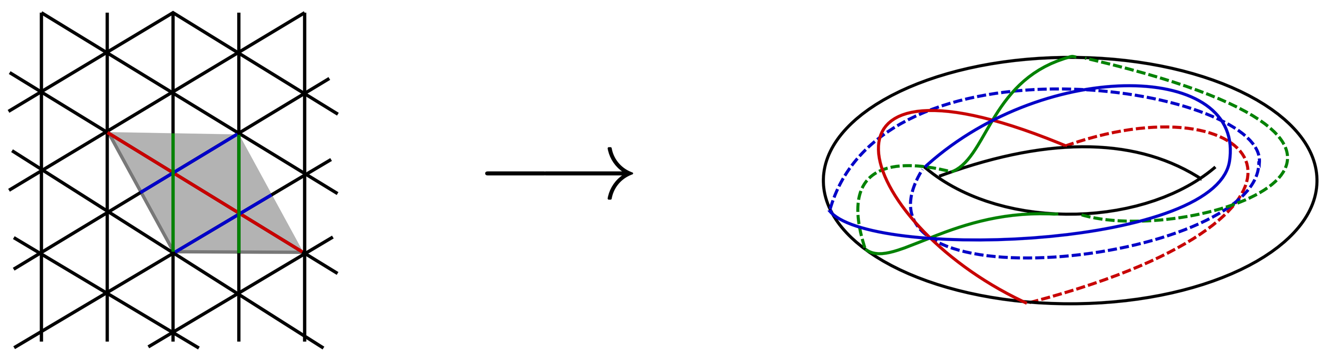
The arrangement $\mathcal{A}$ is called *complexified* if $b_i \in S^1$ for all $i = 1 \dots n$. We will suppose from now that $\mathcal{A}$ is complexified.

Motivation and Background

A complexified toric arrangement gives a polytopal decomposition of the compact torus $(S^1)^d$ into faces. We denote the face category of $\mathcal{A}$ by $F(\mathcal{A})$. Notice that the combinatorial data of $F(\mathcal{A})$ determines the homotopy type of $M(\mathcal{A})$ (see [1]).



The lift of a toric arrangement through the universal cover $\mathbb{C}^d \to (\mathbb{C}^*)^d$ resp. $\mathbb{R}^d \to (S^1)^d$ is a periodic hyperplane arrangement $\mathcal{B}_\mathcal{A}$.

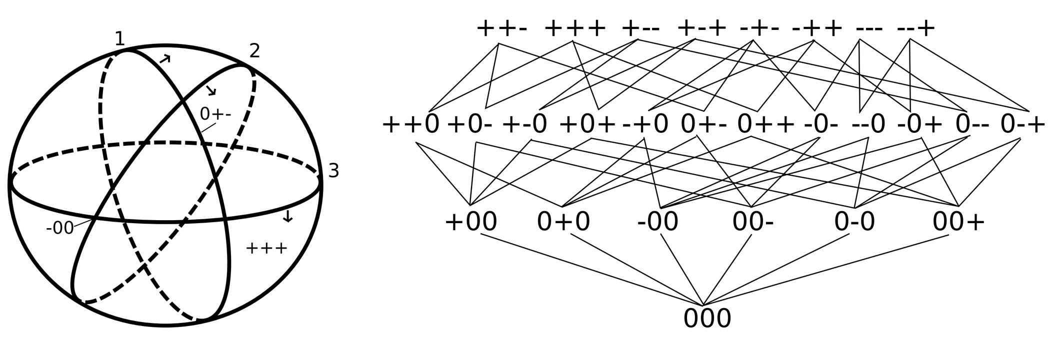


Furthermore, the character lattice $\Lambda \simeq \mathbb{Z}^d$ acts on the face poset $F(\mathcal{B}_\mathcal{A})$ of the affine arrangement $\mathcal{B}_\mathcal{A}$ (see [1]), s.t. regarded as categories we obtain the relation

$$F(\mathcal{A}) = F(\mathcal{B}_\mathcal{A})/\Lambda. \qquad (1)$$

Oriented Matroids

For inspiration we look at central hyperplane arrangements in $\mathbb{R}^d$, which correspond to arrangements of $(d-2)$-subspheres on S^{d-1}. Each face of the decomposition of the sphere can be characterized by a tuple which denotes the position corresponding to each hyperplane (equipped with an orientation). Abstractly this can be given as follows.



Let E be finite, $\mathcal{L} \subseteq \{+, -, 0\}^E$ is a set of covectors of an **oriented matroid** if

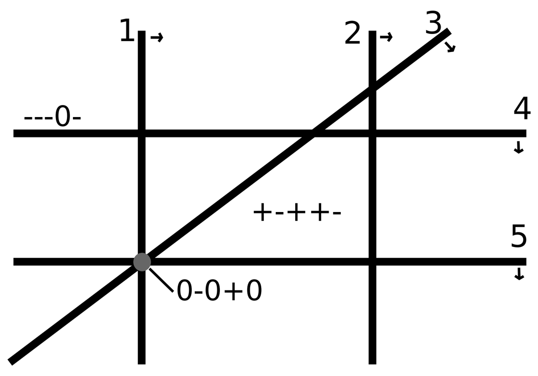
(L0) $0 \in \mathcal{L}$,
(L1) $X \in \mathcal{L}$ implies $-X \in \mathcal{L}$,
(L2) $X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$,
(L3) if $X, Y \in \mathcal{L}$ and $e \in S(X,Y)$ then there exists a $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X,Y)$.

Composition: $(X \circ Y)_e = \begin{cases} X_e, & \text{if } X_e \neq 0, \\ Y_e, & \text{otherwise.} \end{cases}$ Separation set: $S(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}$.

Topological Representation Theorem (Lawrence '78, see e.g. [2]). Oriented matroids correspond bijectively to the cell decompositions obtained by arrangements of *pseudospheres* (centrally symmetric codimension 1 subspheres with "good" intersection properties, for the formal definition see [2]).

Approach

- Abstract characterization of the intersection poset and the face poset of non-central arrangements.
- Toric version via an abstractly characterized action of the integer lattice (e.g., as (1)).



(Sanity check: The finite case should correspond to an affine oriented matroid, i.e., a pseudoarrangement restricted to a halfsphere.)

Semimatroids and Geometric Semilattices

Semimatroids are a generalization of matroids which aims at describing non-central arrangements as well. We extend the definition from Ardila [3] to an infinite ground set.

A *semimatroid* is given by

- a (possibly infinite) set S,
- a m-dim. simpl. complex $\mathcal{C}$ on S (the "central sets"),
- a rank function r on $\mathcal{C}$.

A *geometric semilattice* (see [4]) is a ranked meet semilattice $\mathcal{L}$ satisfying:

(G3) Every interval is isomorphic to the poset of flats of a matroid.
(G4) Let A be an independent set of atoms and $x \in \mathcal{L}$ with $r(x) < r(\vee A)$, then there exists an $a \in A$ such that $a \not\leq x$ and $x \vee a \in \mathcal{L}$.

Theorem. (R. '14) A poset is an (infinite) geometric semilattice if and only if it is isomorphic to the poset of flats of an (infinite) semimatroid.

Oriented Semimatroids

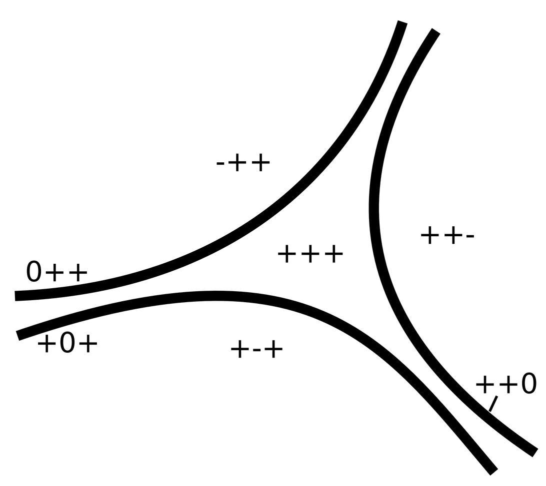
Let E be an arbitrary set, $\mathcal{F} \subseteq \{+, -, 0\}^E$ is the set of covectors of an **oriented semimatroid** if

(F1) $X, Y \in \mathcal{F}$ implies $X \circ Y \in \mathcal{F}$,
(F2) if $X, Y \in \mathcal{F}$ and $e \in S(X,Y)$ then there exists a $Z \in \mathcal{F}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f = (Y \circ X)_f$ for all $f \notin S(X,Y)$,
(F3) $S(X,Y)$ is finite for all $X, Y \in \mathcal{F}$,
(F4) all zero sets X^0 are finite and bounded by $m \in \mathbb{N}$,
(F5) the deletion $\mathcal{F} \backslash \underline{X} = \{Y|_{X^0} : Y \in \mathcal{F}\}$ is an oriented matroid for all $X \in \mathcal{F}$,
(F6) if $A \subseteq E$ and $X \in \{V \in \mathcal{F} | A \subseteq V^0, \nexists\, W \in \mathcal{F} : A \subseteq W^0 \subsetneq V^0\}$ then holds for all $Y \in \mathcal{F}$ with $\mathrm{rk}(\mathcal{F}\backslash\underline{X}) > \mathrm{rk}(\mathcal{F}\backslash\underline{Y})$ that there exist an $a \in A - Y^0$ and a $Z \in \mathcal{F}$ such that $a \cup Y^0 \subseteq Z^0$.

Proposition. (R. '14) The collection of zero sets of $\mathcal{F}$ forms a geometric semilattice. Hence, $\mathcal{F}$ determines an underlying semimatroid.

Conjecture. The order complex of the poset of covectors of an oriented semimatroid is homeomorphic to $\mathbb{R}^{\mathrm{rk}\,\mathcal{F}}$.

Observation. An finite oriented semimatroid is more general as an affine oriented matroid (as the figure shows). We need an additional restriction.



Application. Understanding which affine topological arrangements correspond to oriented matroids (Open problem, see e.g. Forge and Zaslavsky [5]).

References

[1] Giacomo d'Antonio and Emanuele Delucchi.
A Salvetti complex for toric arrangements and its fundamental group.
Int. Math. Res. Not. IMRN, 6:Art. ID rnr161, 32, 2011.

[2] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler.
Oriented matroids, volume 46 of *Encyclopedia of Mathematics and its Applications*.
Cambridge University Press, Cambridge, second edition, 1999.

[3] Federico Ardila.
Semimatroids and their Tutte polynomials.
Rev. Colombiana Mat., 41(1):39–66, 2007.

[4] Michelle Wachs and James Walker.
On geometric semilattices.
Order **2**, pages 367–385, 1986.

[5] David Forge and Thomas Zaslavsky.
On the division of space by topological hyperplanes.
European J. Combin., 30(8):1835–1845, 2009.