Phasing Classes of Matroids

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Phased matroids $\mathcal{R}_{\mathbb{C}}(M)$

Let $E = \{1, \dots, m\}$ be a finite ground set. A nonzero alternating function $\varphi : E^d \longrightarrow S^1 \cup \{0\}$ is called a *rank d phirotope* [AD12] if

$$0 \in \operatorname{pconv}\left(\left\{(-1)^k \varphi(x_1, x_2, \dots, \hat{x}_k, \dots, x_{d+1})\varphi(x_k, y_1, \dots, y_{d-1})\right\}\right)$$

for any two subsets $\{x_1, \dots, x_{d+1}\}$ and $\{y_1, \dots, y_{d-1}\}$ of E. Two phirotopes φ_1, φ_2 are called *equivalent* if $\varphi_1 = \alpha\varphi_2$ for some $\alpha \in S^1$.

A *phased matroid* is defined in [AD12] to be a pair $(E, \overline{\varphi})$, where $\overline{\varphi}$ is an equivalence class of phirotopes. The support of any representative φ of $\overline{\varphi}$ is the set of bases of a matroid M which is called the *underlying matroid* of $(E, \overline{\varphi})$.

Two phased matroids $(E, \overline{\varphi_1}), (E, \overline{\varphi_2})$ with same underlying matroid M are in the same *phasing class* if there exist $\alpha, \eta(1), \dots, \eta(m) \in S^1$ such that

$$\varphi_1(z_1, \dots, z_d) = \alpha \left(\prod_{j=1}^{d} \eta(z_j)\right) \varphi_2(z_1, \dots, z_d).$$

Definition. $\mathcal{R}_{\mathbb{C}}(M)$ is the set of phasing classes of phased matroids with underlying matroid M.

Problem 1. Characterize the set $\mathcal{R}_{\mathbb{C}}(M)$ of phasing classes of phased matroids with underlying matroid M, as a phased counterpart to the results of [GRS95].

Review of basic definitions

The phase $\operatorname{ph}(x)$ of $x \in \mathbb{C}$ is defined to be 0 if $x = 0$ and $\frac{x}{|x|}$ otherwise.

Let $E = \{1, \dots, m\}$. For $v \in \mathbb{C}^E$, $\operatorname{ph}(v)$ is defined componentwise.

A *phased vector* is any $X \in (S^1 \cup \{0\})^E$.

Let X be a phased vector. The *support* of X is the set

$$\operatorname{supp}(X) = \{e \in E \mid X(e) \neq 0\}.$$

Let $S \subseteq (S^1 \cup \{0\})$ be a finite set. The *phase convex hull* $\operatorname{pconv}(S)$ is the set of all phases of (real) strictly positive linear combinations of S. We set $\operatorname{pconv}(\emptyset) = \{0\}$.

Two phased vectors X and Y are *orthogonal* if

$$0 \in \operatorname{pconv}\left(\left\{\frac{X(e)}{Y(e)} \,\middle|\, e \in \operatorname{supp}(X) \cap \operatorname{supp}(Y)\right\}\right).$$

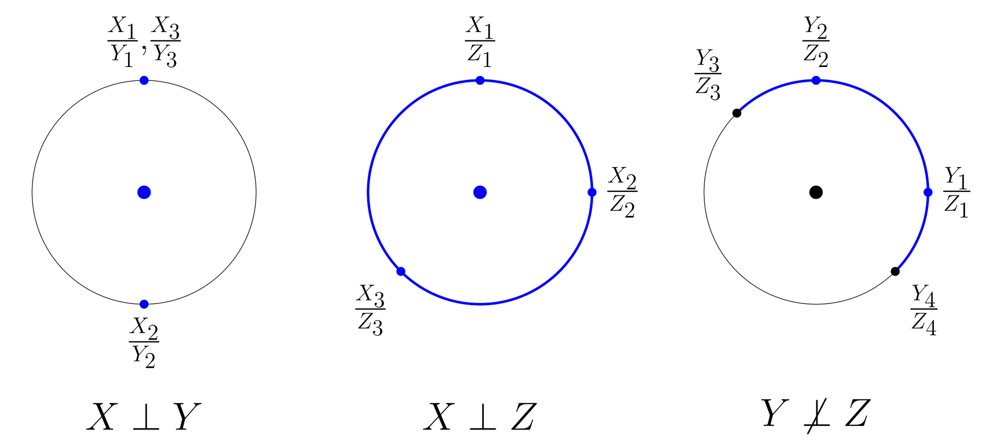


Figure 1: the orthogonality relations between the phased vectors $X = (i, 1, -1, 0)$, $Y = (1, i, i, 1)$ and $Z = (1, 1, e^{-i\pi/4}, e^{i\pi/4})$. The phase convex hulls are coloured in blue.

Projective phasings $\mathcal{P}_{\mathbb{C}}(M)$

Extending and generalizing a construction of [GRS95] in the oriented case, we define a *projective phasing* $\mathbb{P}_{\mathbb{C}}$ of a matroid M with set of circuits $\mathfrak{C}$, set of cocircuits $\mathfrak{C}^*$ and finite ground set E to be a function

$$\mathbb{P}_{\mathbb{C}} : \left\{(C, D, x, y) \in \mathfrak{C} \times \mathfrak{C}^* \times E \times E \,\middle|\, \begin{array}{c} C \cap D \neq \emptyset \\ x, y \in C \cap D \end{array}\right\} \longrightarrow S^1$$

whose values, denoted by $\left(\begin{smallmatrix} C & D \\ x & y \end{smallmatrix}\right)$, satisfy conditions

$$\begin{pmatrix} C & D \\ x & x \end{pmatrix} = 1,$$

$$\begin{pmatrix} C & D \\ x & y \end{pmatrix}\begin{pmatrix} C & D \\ y & z \end{pmatrix}\begin{pmatrix} C & D \\ z & x \end{pmatrix} = 1,$$

$$\begin{pmatrix} C_1 & D_1 \\ x & y \end{pmatrix}\begin{pmatrix} C_2 & D_2 \\ x & y \end{pmatrix} = \begin{pmatrix} C_1 & D_2 \\ x & y \end{pmatrix}\begin{pmatrix} C_2 & D_1 \\ x & y \end{pmatrix},$$

$$\forall y \in C \cap D, \quad 0 \in \operatorname{pconv}\left(\left\{\begin{pmatrix} C & D \\ x & y \end{pmatrix} \,\middle|\, x \in C \cap D\right\}\right).$$

Definition. $\mathcal{P}_{\mathbb{C}}(M)$ is the set of projective phasings of M.

Tutte groups $\mathcal{H}_{\mathbb{C}}(M)$

The *extended Tutte group* $\mathbb{T}_M^{\mathfrak{C}}$ of a matroid M is defined in [DW89] to be the finitely generated abelian group with formal generators ϵ and a symbol $C(x)$ for every $C \in \mathfrak{C}$ and all $x \in C$. The relations are $\epsilon^2 = 1$ and

$$\frac{C_1(x_2)C_2(x_3)C_3(x_1)}{C_1(x_3)C_2(x_1)C_3(x_2)} = \epsilon$$

for $C_i \in \mathfrak{C}$, $L = C_1 \cup C_2 \cup C_3 = C_i \cup C_j$ $(i \neq j)$, $x_i \in L \setminus C_i$ and $|L| = \operatorname{rk}(L) + 2$.

The *inner Tutte group* $\mathbb{T}_M^{(0)}$ of M is the subgroup of $\mathbb{T}_M^{\mathfrak{C}}$ generated by ϵ and all products of the form

$$\frac{C_1(x)C_2(y)}{C_1(y)C_2(x)}$$

for circuits C_1, C_2 with $x, y \in C_1 \cap C_2$ and $|C_1 \cup C_2| = \operatorname{rk}(C_1 \cup C_2) + 2$.

Dress and Wenzel [DW89] computed the groups $\mathbb{T}_M^{(0)}$ for M uniform.

Problem 2. Compute the group $\mathbb{T}_M^{(0)}$ for M non uniform.

Definition. $\mathcal{H}_{\mathbb{C}}(M)$ is the set of homomorphisms $\Phi : \mathbb{T}_M^{(0)} \longrightarrow S^1$ which satisfy $\epsilon \mapsto -1$ and

$$0 \in \operatorname{pconv}\left(\left\{1, -\Phi\left(\frac{C_1(x_3)C_2(x_4)}{C_1(x_4)C_2(x_3)}\right), -\Phi\left(\frac{C_4(x_3)C_2(x_1)}{C_4(x_1)C_2(x_3)}\right)\right\}\right)$$

where L and all C_i, x_i are as above with C_i pairwise distinct circuits.

Generalized cross-ratios $\mathcal{G}_{\mathbb{C}}(M)$

Let M be a matroid without minors of Fano or dual-Fano type. Let E be the finite ground set of M and let $\mathfrak{C}$ be the set of circuits of M. A *flat* of M is any set of the the form $F = C_1 \cup \dots \cup C_k$ with $C_i \in \mathfrak{C}$ for all i. For brevity, we define the *dimension* of a flat F as

$$\dim(F) = |F| - \operatorname{rk}(F) - 1.$$

The *generalized cross-ratios* of M are the values

$$\psi(C_1C_2|C_3C_4)$$

defined for $C_1, C_2, C_3, C_4 \in \mathfrak{C}$ with $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$ and $\{C_1, C_2\} \cap \{C_3, C_4\} = \emptyset$, ranging in S^1, satisfying

- $0 \in \operatorname{pconv}(\{1, -\psi(C_1C_2|C_3C_4), -\psi(C_1C_3|C_4C_2)\})$

  for pairwise distinct circuits, $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$

and conditions:

- $\psi(C_1C_2|C_3C_3) = 1$
- $\psi(C_1C_2|C_3C_4) = \psi(C_3C_4|C_1C_2)$
- $\psi(C_1C_2|C_3C_4)\psi(C_1C_4|C_2C_3)\psi(C_1C_3|C_4C_2) = -1$

  for pairwise distinct circuits, $\dim(C_1 \cup C_2 \cup C_3 \cup C_4) = 1$
- $\psi(C_1C_2|C_3C_4)\psi(C_1C_2|C_4C_5)\psi(C_1C_2|C_5C_3) = 1$
- $\psi(C_1C_2|C_6C_9)\psi(C_2C_3|C_4C_7)\psi(C_3C_1|C_5C_8) = 1$

  for any family of circuits $C_1, \dots, C_9 \in \mathfrak{C}$ such that

  ◇ $\dim(L_i) = 1$ for $L_i = C_j \cup C_k$, where $\{i, j, k\} = \{1, 2, 3\}$

  ◇ $\dim(P) = 2$ where $P = C_1 \cup C_2 \cup C_3$

  ◇ $C_{i+3}, C_{i+6} \subseteq L_i$ for $i = 1, 2, 3$

  ◇ $\dim(L_h) = 1$ for $L_h = C_{3+h} \cup C_{4+h} \cup C_{5+h}$, $h \in \{1, 4\}$

  ◇ $\{C_1, C_2, C_3\} \cap \{C_4, \dots, C_9\} = \emptyset$

Definition. $\mathcal{G}_{\mathbb{C}}(M)$ is the set of generalized cross-ratios of M.

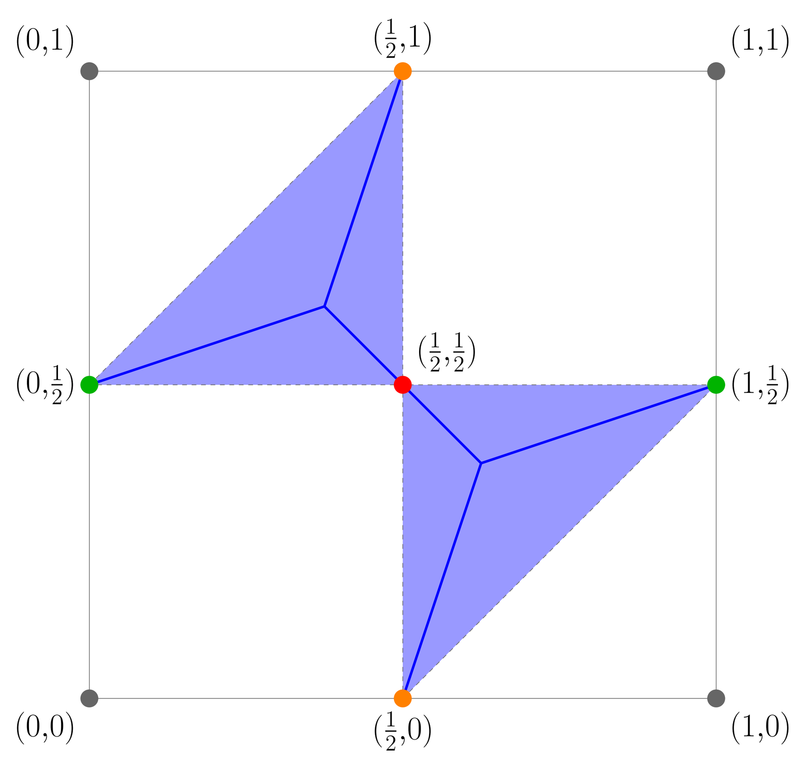
Main result

Let M be a matroid without minors of Fano or dual-Fano type. There exist one-to-one correspondences

$$\mathcal{R}_{\mathbb{C}}(M) \longleftrightarrow \mathcal{P}_{\mathbb{C}}(M) \longleftrightarrow \mathcal{H}_{\mathbb{C}}(M) \longleftrightarrow \mathcal{G}_{\mathbb{C}}(M)$$

Example: $U_2(4)$

We denote by $U_2(4)$ the uniform rank 2 matroid with 4 elements.
The set $\mathcal{R}_{\mathbb{C}}(U_2(4))$ is in bijection with the topological subspace of the torus given by $\left\{(f, g) \in S^1 \times S^1 \mid 0 \in \operatorname{pconv}\{1, f, g\}\right\}$, which corresponds to the two coloured (i.e., not gray) part in the picture. Topologically, this space is homotopy equivalent to a wedge of two circles.



Application

Theorem. *Let M be a matroid without minors of Fano or dual-Fano type and let A be the matrix associated to the lift of the system of "green equations" (box on the right hand side).*
If b_M denotes the rank of the free part of $\mathbb{T}_M^{(0)}$, then

$$b_M = \dim \ker(A).$$

The same result holds with a reduced matrix B. This, together with [Wen89], allows us a computationally tractable solution to Problem 2 for matroids with up to 7 elements. The following are some samples.

Matroid	Inner Tutte Group
F_7^-	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$
$O7$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^2$
$P6$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^9$
$P7$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^2$
$Q6$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^5$
$R6$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^4$

References

[AD12] Laura Anderson and Emanuele Delucchi, *Foundations for a theory of complex matroids*, Discrete Comput. Geom. **48** (2012), no. 4, 807–846.

[DW89] Andreas W. M. Dress and Walter Wenzel, *Geometric algebra for combinatorial geometries*, Adv. Math. **77** (1989), no. 1, 1–36.

[GRS95] Israel M. Gel'fand, Grigori L. Rybnikov, and David A. Stone, *Projective orientations of matroids*, Adv. Math. **113** (1995), no. 1, 118–150.

[Wen89] Walter Wenzel, *A group-theoretic interpretation of Tutte's homotopy theory*, Adv. Math. **77** (1989), no. 1, 37–75.