Asymptotic Euler-Maclaurin formula for semi-rational polyhedra

Common work with Nicole Berline.

Semi-rational polyhedra

V real vector space of dimension d ;
Λ lattice in V, dual lattice $\Lambda^* \subset V^*$;
$\eta_1, \eta_2, \ldots, \eta_N$ elements of Λ^* ;
$h_1, h_2, \ldots, h_N$ **real** numbers.
Then the polyhedron

$$\mathfrak{p}(h) = \{x \in V; \langle \eta_i, x \rangle \leq h_i\}$$

is called semi rational.
Examples :
$a \leq x \leq b$: any interval with real end points.
$x \geq 0, y \geq 0, x + 2y \leq \sqrt{2}$

Riemann Sums over Semi-rational Polyhedra

Let $\mathfrak{p}(h)$ be a semi rational polytope of dimension d. Consider the sum of values of a smooth function *test* on the integral points of $\mathfrak{p}$:

$$\langle F(\mathfrak{p}(h), \Lambda), test \rangle = \sum_{x \in \Lambda \cap \mathfrak{p}(h)} test(x).$$

When *test* is a polynomial, we (Baldoni, Berline, Koeppe, De Loera, Vergne) have given "formulae" for $\langle F(\mathfrak{p}(h), \Lambda), test \rangle$ depending of the real parameter h in a "semi-quasi-polynomial" way. Here we show that we can give asymptotic formulae for *test* smooth of the same kind.

Asymptotic Riemann Sums over Semi-rational Polyhedra

Consider a real parameter t. Assume $\mathfrak{p}$ is a semi-rational polyhedron, of dimension d, and *test* be a smooth function with compact support. Define the Riemann sum :

$$F(t) = \frac{1}{t^d} \sum_{x \in t\mathfrak{p} \cap \Lambda} test(x/t)$$

It is clear that when $t \to \infty$, $F(t)$ tends to $\int_{\mathfrak{p}} test$. Thus we want to evaluate at what rate the Riemann sum converges to the integral.

The Euler-MacLaurin formula in dimension 1

Let a be a real number. Let

$$\mathfrak{c} = \{x \in \mathbb{R}; x \geq a\}.$$

Let $\{t\} \in [0, 1[$ be the fractional part of $t \in \mathbb{R}$. Let *test* be a smooth function with compact support, $B_k(x)$ the Bernoulli polynomial, $B_k = B_k(0)$ the Bernoulli number.
Then

$$\sum_{x \geq ta, x \in \mathbb{Z}} \frac{1}{t} test(x/t) = \int_{x \geq a} test(x)dx - \sum_{j=1}^{k-1} \frac{B_j(\{-ta\})}{t^j} test^{(j-1)}(a)$$

$$+ \frac{1}{t^k}(-1)^{k-1} \int_a^{\infty} (B_k(\{tx\}) - B_k) test^{(k)}(x)dx.$$

Riemann sum minus the integral

So we obtain an asymptotic formula in the form

$$\sum_{j=1}^{infty} f_j(t)\frac{1}{t^j}$$

where $f_j(t)$ are polynomial functions of the function $t \to \{-at\}$. Furthermore, an explicit formula for the rest is given as an integral of a derivative of *test* of order k against a bounded and continuous function of tx over $\mathfrak{c} = [a, \infty[$.
If $a = p/q$ is rational, the function $f_j(t)$ is a periodic function of t with period q.
If a and t are integers, then $f_j(t) = -B_j$ is just the Bernoulli number. So we obtain an asymptotic formula in $\frac{1}{t^j}$.

Semi-quasi polynomials

A function $f(t)$ of $t \in \mathbb{R}$ will be called semi-quasi polynomial if $f(t)$ can be expressed as $P(\{c_1 t\}, \{c_2 t\}, \ldots, \{c_K t\})$, a polynomial function of a number of functions $t \to \{c_j t\}$ where c_a are real numbers. If the numbers $c_j = p_j/q_j$ are rationals, then $\{c_j t\}$ is a periodic function of period q_j, so a semi-quasi polynomial is a periodic function of t mod some period Q.

Form of the asymptotic for semi-rational polytope

Theorem • Let $\mathfrak{p}$ be a semi-rational polytope, then when $t \to \infty$, t real,

$$F(t) = \frac{1}{t^d} \sum_{x \in t\mathfrak{p} \cap \Lambda} test(x/t)$$

is equivalent to

$$\int_{\mathfrak{p}} test + \sum_{j=1}^{\infty} \frac{f_j(t)}{t^j} \langle D_j, test \rangle$$

where $f_j(t)$ are semi-quasi polynomial functions of t and $\langle D_j, test \rangle$ are **integrals of derivatives of test on faces of $\mathfrak{p}$.**

• If $\mathfrak{p}$ is rational, $f_j(t)$ are periodic functions of t

• If $\mathfrak{p}$ is with integral vertices, $f_j(t)$ are constants.

Comments

We have explicit forms of the D_j, but unfortunately no nice formula for the rest, meaning as a distribution supported on $\mathfrak{p}$. For the standard simplex, there are some formulae with rest obtained in numerical analysis (clearly an important problem in numerical analysis).

Arrangement of hyperplanes

In the case where $\mathfrak{p}$ is a polytope with integral vertices, two asymptotic formulae were previously obtained : one by Guillemin-Sternberg (extending Khovanskii-Pukhlikov formula for polynomials), one by Tatsuya Tate (extending Berline-Vergne local Euler-MacLaurin formula for polynomials). Our work was motivated by a recent question of Le Floch-Pelayo : How to see directly that these two formulae are the same ?.

Easy to do via renormalization of rational functions with poles on an arrangement of hyperplanes.

Guillemin-Sternberg formula

Let $\mathfrak{p}(h)$ be a semi-rational polytope. Assume that $\mathfrak{p}_0 = \mathfrak{p}(h_0)$ is a Delzant polytope with integral vertices (corresponding to a smooth toric variety M with ample line bundle L). Consider

$$F(t) = \frac{1}{t^d} \sum_{x \in t\mathfrak{p}_0 \cap \Lambda} test(x/t).$$

Define $Todd(z) = \frac{z}{1-e^{-z}} = \sum_{j=0}^{\infty} (-1)^j B_j z^j / j!$.
We can consider the series of differential operators

$$Todd(\partial_h/k) := \prod_{i=1}^{N} Todd(\partial_{h_i}/k)$$

Then : For k integer, $k \to \infty$, we have

$$F(k) \equiv \left(Todd(\partial_h/k) \int_{\mathfrak{p}(h)} test \right)|_{h=h_0}$$

But we want a ”Concrete formula”

Pretty clear that $F(k) \equiv \sum_j \frac{1}{k^j} \langle D_j, test \rangle$ where $\langle D_j, f \rangle$ are integrals of derivatives of *test* on faces of the polytope $\mathfrak{p}$.
We had obtained (Berline-Vergne) exact formulae with explicit operators D_j when test was a polynomial. Tatsuya Tate showed (directly) that these formulae hold in the asymptotic sense. It is in fact easy to deduce any of these asymptotic formulae from Euler-MacLaurin formula in dimension one, and we can formulate a result for any semi-rational polytope. **Our method is as usual the Brianchon-Gram decomposition. and ”renormalization” in the space of rational functions with poles on an arrangement of hyperplanes.**

Integrals and Sums over a cone

V real vector space of dimension d, with a lattice Λ.
$\mathfrak{c}$ a cone, dual cone $\mathfrak{c}^*$ with non empty interior.

$$I(\mathfrak{c})(\xi) = \int_{x\in\mathfrak{c}} e^{\langle \xi, x\rangle} dx,$$

$$S(\mathfrak{c})(\xi) = \sum_{x\in\mathfrak{c}\cap\Lambda} e^{\langle \xi, x\rangle}.$$

These functions are defined when ξ is the opposite of the dual cone $\mathfrak{c}^*$ to $\mathfrak{c}$. For example if $\mathfrak{c} = \mathbb{R}_{\geq 0}$, and $\xi < 0$, then

$$I(\mathfrak{c})(\xi) = \int_{x\geq 0} e^{\xi x} dx = -\frac{1}{\xi}.$$

$$S(\mathfrak{c})(\xi) = \sum_{n\geq 0} e^{n\xi} = \frac{1}{1-e^{\xi}}.$$

Laurent series of $S(\mathfrak{c})$

$I(\mathfrak{c})$ is a rational function of degree $-d$.
Define $[S(\mathfrak{c})] = \sum_{j=-d} S(\mathfrak{c})_{[\ell]}$ the decomposition of $S(\mathfrak{c})$ in sum of rational functions of homogenous degree ℓ.
Example $\mathfrak{c} = \mathbb{R}_{\geq 0} e_1 \oplus \mathbb{R}_{\geq 0}(e_1 + e_2)$

$$
\begin{aligned}
S(\mathfrak{c})(z_1, z_2) &= \frac{1}{(1 - e^{z_1})} \frac{1}{(1 - e^{z_1 + z_2})} \\
&= \frac{1}{z_1(z_1 + z_2)} \\
&\quad - \frac{1}{2z_1} - \frac{1}{2(z_1 + z_2)} \\
&\quad + \frac{1}{3} + \frac{1}{12} \frac{z_1}{z_2} + \frac{1}{12} \frac{z_1}{(z_1 + z_2)} \\
&\quad + \cdots
\end{aligned}
$$

Boundary values

If λ is an interior point in $-\mathfrak{c}^*$ (-interior of the dual cone), define

$$\lim_{\lambda} S(\mathfrak{c})(i\xi) = \lim_{\epsilon \to 0} S(\mathfrak{c})(i\xi + \epsilon\lambda)$$

$$\lim_{\lambda} S(\mathfrak{c})_\ell(i\xi) = \lim_{\epsilon \to 0} S(\mathfrak{c})_\ell(i\xi + \epsilon\lambda)$$

Define tempered distributions on V^*.
The Fourier transform of $\lim_\lambda S(\mathfrak{c})_\ell(i\xi)$ is a derivative of a locally polynomial function supported on $\mathfrak{c}$.
For example $\mathfrak{c} = [0, \infty]$; $\epsilon < 0$,

$$\lim_{\epsilon} \frac{1}{-(i\xi + \epsilon)} = \int_0^\infty e^{i\xi x} dx.$$

The distribution $F(t)$ and its Fourier transform

We consider the cone $\mathfrak{c}$ and the tempered distribution on V defined by

$$\langle F(\mathfrak{c})(t), test \rangle = \frac{1}{t^d} \sum_{x \in \Lambda \cap \mathfrak{c}} test(x/t).$$

It is easy to compute the Fourier transform of this distribution This is

$$\lim_{\lambda} S(\mathfrak{c})(i\xi/t)$$

If $t \to \infty$, ξ/t tends to 0, and it is tempting to use the Laurent series of the function $S(\mathfrak{c})(\xi)$.

Theorem When $t \to \infty$,

$$\lim_{\lambda} S(\mathfrak{c})(i\xi/t) \equiv \sum_{j \geq 0} t^{-j} \lim_{\lambda} S(\mathfrak{c})_j(i\xi).$$

Explicit formula

The proof of this theorem is obvious. It is clearly additive over cones, we decompose the cone $\mathfrak{c}$ in unimodular cones and then we are reduced to dimension 1 and we use *EML* formula.
Obvious then to deduce Guillemin-Sternberg formula from computing this, and using Brianchon-Gram decomposition.
We want to be more explicit.

Renormalization

We consider the arrangement of hyperplanes $H = H(G)$ in V^* with equations $g_a(\xi) = 0$ with $g_a \in G \subset V$ the set of generators of the cone $\mathfrak{c}$. We denote by $R_G(V^*)$ the space of rational functions on V^* with poles contained in $H(G)$. Then the homogeneous components of $S(\mathfrak{c})$ are rational functions with poles on this arrangement of hyperplanes.

Let $\mathcal{S}$ be the set of subspaces L of V generated by elements of G. If $[g_1, g_2, \ldots, g_N]$ is a sequence of elements of G, the function $\frac{1}{\prod_{i=1}^{N} g_j(\xi)}$ is a function in $R_G(V^*)$. For a subspace L, consider the space B_L consisting of the linear span of these functions $\frac{1}{\prod_{a \in A} g_a(\xi)}$ where the sequence g_a in the denominator generates the rational subspace $L : L = \oplus \mathbb{R} g_j$. Then B_L is a subspace of $R_G(V^*)$.

Substracting the polar part

Let us choose a scalar product on V. Consider $\mathbb{C}[L^{\perp}]$ the space of polynomial functions on $L^{\perp}$. Using our scalar product, we can embedd $\mathbb{C}[L^{\perp}]$ in $\mathbb{C}[V^*]$, the space of polynomial functions on V^*.

Proposition(De Concini+Procesi) We have the direct sum decomposition

$$R_G(V^*) = \oplus_{L \in \mathcal{S}}(\mathbb{C}[L^{\perp}] \otimes B_L)$$

Thus we write $\phi \in R_G(V^*)$ uniquely as $\phi = \sum_L T_L(\phi)$ where $T_L(\phi) \in (\mathbb{C}[L^{\perp}] \otimes B_L)$.

The holomorphic part

For $L = \{0\}$, the corresponding term $T_L(\phi)$ is thus a polynomial function on V^*. We denote the corresponding term by *Renorm*(ϕ) and we call the polynomial *Renorm*(ϕ) the renormalisation of the rational function ϕ. It depends of the choice of a scalar product. In one variable $f(z) = \sum a_i z^i$, and *Renorm*$(f) = \sum_{i \geq 0} a_i z^i$. In several variables, we need a scalar product to suppress the polar part (all B_L for $L \neq 0$).

Back to Example : $\mathfrak{c} = \mathbb{R}_{\geq 0} e_1 \oplus \mathbb{R}_{\geq 0}(e_1 + e_2)$

$$
\begin{aligned}
S(\mathfrak{c})(z_1, z_2) = & \frac{1}{(1 - e^{z_1})} \frac{1}{(1 - e^{z_1 + z_2})} \\
= & \frac{1}{z_1(z_1 + z_2)} \\
& - \frac{1}{2z_1} - \frac{1}{2(z_1 + z_2)} \\
& + \frac{1}{3} + \frac{1}{12} \frac{z_1}{z_2} + \frac{1}{12} \frac{z_1}{(z_1 + z_2)} \\
& + \cdots
\end{aligned}
$$

BUT WE REWRITE

$$\frac{1}{3}+\frac{1}{12}\frac{z_1}{z_2}+\frac{1}{12}\frac{z_1}{(z_1+z_2)}$$

as

$$\frac{1}{3}+\frac{1}{12}\frac{z_1}{z_2}+\frac{1}{24}\frac{z_1-z_2}{(z_1+z_2)}+\frac{1}{24}$$

and the holomorphic part of $S(\mathfrak{c})$ is

$$renorm(S(\mathfrak{c})(z_1,z_2)) = 1/3+1/24+\cdots = 3/8+\cdots$$

Normal cones and Renormalization

f face of the cone $\mathfrak{c}$; $< f >$ linear span of $p - q$, p, q in f.
Normal cone $T(f, \mathfrak{c})$ in $< f >^{\perp}$ with lattice $< f >^{\perp} \cap < f > + \Lambda$
Definition $\mu(f, \mathfrak{c})(\xi) = \mathit{Renorm}(S(T(f, \mathfrak{c})))(\xi)$. This is a holomorphic function of the variable $\xi \in < f >^{\perp}$.

MAIN THEOREM (Berline-Vergne, Paycha)

$$S(\mathfrak{c})(\xi) = \sum_{faces} \mu(T(f, \mathfrak{c}))(\xi) I(f)(\xi)$$

As a corollary

$$F(t) = \frac{1}{t^d} \sum_{x \in t\mathfrak{c} \cap \Lambda} test(x/t)$$

$$< F(t), test\rangle \equiv \sum_{faces} \frac{1}{t^{codimf}} \sum_{f} \int_f \mu(f, \mathfrak{c})(D/k) * test$$

Here D are the normal derivatives to the face f.

Example : Cone based on a square

EXAMPLE : let $\mathfrak{c}$ with generators $e_3 + e_1, e_3 - e_1, e_3 + e_2, e_3 - e_2$.
Then

$$\langle F(t), test \rangle \equiv \int_c Test$$

$$+\frac{1}{t}\frac{1}{2}\int_{boundary(c)} Test$$

$$+\frac{1}{t^2}(\frac{2}{9}\int_{edges} Test - \frac{1}{36}\int_{boundary} N \cdot Test)$$

where $N \cdot test$ is the normal derivative. N here is the primitive vector pointing inward.

$$+\frac{1}{t^3}((1/6)Test(0) - \frac{1}{24}\int_{edges} U * test)$$

$$+O(1/t^4).$$

with $U = e_3 - e_1$ for the edge $e_3 + e_1$, etc...

For a polytope $\mathfrak{p}$ with integral vertices

The formulae add up nicely using Brianchom-Gram decomposition.
If *h* is a smooth function on $\mathfrak{p}$ then :

$$t^{-d} \sum_{x \in \mathfrak{p} \cap \Lambda / t} h(x) \equiv \sum_{f} t^{-codim(f)} \int_f (\mu(f, \mathfrak{p})(1/t\partial_x)h)$$

when t is an integer going to ∞.
This is the formula of Tatsuya Tate.

Asymptotic EML for a semi-rational polytope

Quite clear that if we use EML in dimension 1 with cones $[a, \infty]$ and a real, we obtain also an asymptotic formula with coefficients $f_j(t)/t^j \langle D_j, h \rangle$ and f_j semi-quasi polynomial and D_j explicit via the renoramization procedure.

Example : simplex dimension 2

For example, for the standard simple
$\mathfrak{p} = \{x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$
Here is the asymptotic series for

$$\langle S(t), test\rangle = \frac{1}{t^2} \sum_{t\mathfrak{p}\cap\mathbb{Z}^2} test(x/t)$$

and *t* **real.**

$\langle S(t), test \rangle - \int_{\mathfrak{p}} test \equiv$

$$= \frac{1}{t}\left(\frac{1}{2}\int_{f1} test + \frac{1}{2}\int_{f2} test + \left(\frac{1 - 2*\{t\}}{2}\int_{f3} test\right)\right)$$

$$+ \frac{1}{t^2}\left(\frac{-1}{12}\int_{f_1} \partial_y test + \frac{-1}{12}\int_{f_2} \partial_x test\right)$$

$$+ + \frac{1}{t^2}\left(\frac{-1}{12} + \frac{1}{2}\{t\} - \frac{1}{2}\{t\}^2\right)\int_{f_3} (-(\partial_x + \partial_y)/2)) * test$$

$$+ \frac{1}{t^2}\left(\frac{1}{4} test(0,0) + (3/8 - (3/4)\{t\} + (1/4)\{t\}^2) test(1,0)\right)$$

$$\frac{1}{t^2}(3/8 - (3/4)\{t\} + (1/4)\{t\}^2) test(0,1)$$

$$+ O(1/t^3)$$

In contrast to the 1-dimensional case, we dont know how to write a nice remainder in general.