Higher topological complexity of Eilenberg-MacLane spaces

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Short History

1. Lazar Lusternik-Lev Schnirelmann, 1934; category (cat) of a topological space.

2. Albert Schwarz, 1966: the Schwarz genus of a fibration.

3. Michael Farber, 2002: topological complexity (TC) of a space.

4. Yuli Rudyak, 2010: higher (s-th) topological complexity (TC_s).

Definition of TC_s

Definition

Let X be a path-connected topological space. Then $\mathrm{TC}_s(X)$ is the Schwarz category of the fibration

$$\phi_s : X^{[0,1]} \to X^s,$$

$$\phi_s(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{s-1}\right), \gamma\left(\frac{2}{s-1}\right), \ldots, \gamma\left(\frac{s-2}{s-1}\right), \gamma(1)\right).$$

Remarks

In other words it is the smallest number n such that X^s is partitioned into $n + 1$ not too bad pieces with the property that every s-tuple of distinct points determines a path on X that passes through those points (in order) and depends on them containuously on each piece.
We use the reduced (or normalized) version of TC, i.e. $\mathrm{TC}_2 =$TC-1 in Farber's definition of TC.

Properties of TC we will use

(1) TC(X) is an invariant of the homotopy type of X.

(2) $\mathrm{TC}_s(X) \leq s \cdot \mathrm{hdim}(\mathrm{X})$ where hdim is the homotopy dimension.

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(2) $\mathrm{TC}_s(X) \leq s \cdot \mathrm{hdim(X)}$ where hdim is the homotopy dimension.

(3) $\mathrm{TC}_s(X \times Y) \leq \mathrm{TC}_s(X) + \mathrm{TC_s(Y)}$.

Cohomological lower bound

(4) The cohomological lower bound.

Definition

Let d_s be the diagonal embedding $X \to X^s$. Denote by $cl(X, s)$ the cup length in $\ker d_s^*$, i.e., the largest integer k for which there exist k elements $u_i \in H^*(X^s)$ such that $d_s^* u_i = 0$ and $u_1 u_2 \cdots u_k \neq 0$.

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We have then

$$\mathrm{TC_s(X) \geq cl(X, s).}$$

In the rest of the talk we will use cohomology with coefficients in $\mathbb{C}$ omitting coefficients from the notation.

Groups generated by reflections

Definition

Let V be a complex linear space of dimension r. A (complex) reflection is a finite order invertible linear transformation $\tau : V \to V$ whose fixed point set is a hyperplane H_τ. A finite subgroup of $GL(V)$ is a reflection group if it is generated by reflections.

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For a reflection group W the set $\mathcal{A}_W = \{H_\tau\}$ is called the reflection arrangement of W.

A reflection group W is irreducible if its tautological representation to $GL(V)$ is irreducible. Then the rank of W is r.

Eilenberg-MacLane spaces

Theorem (V.Arnold, P.Deligne, D.Bessis)

Let $M_W = V \setminus \bigcup_{H \in \mathcal{A}_W} H$ *for an arbitrary reflection group* W*.*
Then M_W *is a* $K[\pi, 1]$*.*

Example

For $\ell > 1$ let the hyperplanes H_{ij} ($1 \leq i < j \leq \ell$) are given in $\mathbb{C}^\ell$ by the equations $x_i = x_j$. The set of all H_{ij} is the reflection arrangement of the permutation group $W = \Sigma_\ell$. Here $\pi_1(M_W)$ is the pure Braid group on ℓ strings, that is the pure Artin group of type $\mathrm{A}_{\ell-1}$.

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Similarly, for any (complexified) finite Coxeter group W the group $\pi_1(M_W)$ is the pure Artin group of the respective type. Because of that $\pi_1(M_W)$ for an arbitrary reflection group W is called the pure Artin-type group for W (or the generalized pure Braid group associated to W).

The algebra $H^*(M; \mathbb{C})$

We will always denote by $\mathcal{A}$ a central complex hyperplane arrangement in a linear space V and by M its complement. We assume that the algebra $A = H^*(M; \mathbb{C})$ is known. The set of generators of it, denoted by $\{e_1, \ldots, e_n\} \subset H^1$, is in a fixed one-to-one correspondence with $\mathcal{A}$.

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The ‘independent’ square-free monomials in e_i linearly generate A but are not linearly independent in general (over $\mathbb{C}$). Each linear order on $\mathcal{A}$ determines a monomial (Gröbner) basis.

Upper bound of $\mathrm{TC}_s(M)$

We have

$$\mathrm{TC}_s(M) \leq sr - 1.$$

Indeed $M = \bar{M} \times \mathbb{C}^*$ where $\bar{M}$ is the projectivization of M and has homotopy type of a CW-complex of dimension $r - 1$ whence

$\mathrm{TC}_s(M) \leq \mathrm{TC}_s(M_0) + \mathrm{TC}_s(S^1) \leq s(r-1) + s - 1 = sr - 1.$

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To calculate a lower bound we need some preparation.

Degree one elements in ker d_s^*

Fix an integer $s \geq 2$; for each generator $e_i \in H^1(M)$, and each j $(1 < j \leq s)$ put

$$e_i^{(j)} = e_i \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes e_i \otimes 1 \otimes \cdots \otimes 1$$

where e_i in the second summand is in the jth position. Clearly each $e_i^{(j)} \in \ker d_s^*$.

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where e_i in the second summand is in the jth position. Clearly each $e_i^{(j)} \in \ker d_s^*$.

For every $I \subset \{e_i^{(j)}\}$ let π_I be the product of all elements from I. If π_I does not vanish then $|I|$ is a lower bound for TC_s.

Products over pairs

In the rest of the talk we will identify subsets of $\bar{n}$ with the respective subarrangements of $\mathcal{A}$.
Let $Q = (B, C)$ be an ordered pair of disjoint independent subsets of $\bar{n}$. The product over Q is

$$\pi_Q = e_B \cdot e'_C$$

where

$$e_B = \prod_{i \in B} \prod_{j=2}^{s} e_i^{(j)}, \quad e'_C = \prod_{i \in C} e_i^{(2)}.$$

We put $\bar{Q} = B \cup C$.

Balanced sets and pairs

A subset $S \subset \bar{n}$ is balanced if for every its non-empty subset S' we have $|S'| < 2\,\mathrm{rk}(S')$. A pair $Q = (B, C)$ is balanced if $|B| = r$ and $\bar{Q}$ is balanced.

Lower bound

Theorem

Let $\mathcal{A}$ be a central arrangement. Then for every integer s, $s \geq 2$, and every balanced pair $Q = (B, C)$ we have $\pi_Q \neq 0$.

Idea of proof

Clearly π_Q is a linear combination of pure s-tensors of degree $(s-1)r+|C|$. Among them there is $\mu = e'_C \otimes e_B \otimes \cdots \otimes e_B$ whose all factors are independent.

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It follows from the property of being balanced that all the monomials in μ belong to a monomial basis for some order on $\mathcal{A}$. This implies that μ cannot be canceled whence π_Q does not vanish.

Lower bound

Theorem

For the complement M of a complex central arrangement of hyperplanes, every integer s ($2 \leq s$), and every basic pair (B, C) we have

$$\mathrm{TC}_s(M) \geq (s-1)r + |C|.$$

Large arrangements

There is a substantial class of arrangements for which the lower bound coincides with the upper bound.

Definition

We call an arrangement large if there exists a balanced pair (B, C) with $|C| = r - 1$ whence for every s we have $\mathrm{TC}_s(M) = sr - 1$.

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Example. A simple subclass of large arrangements is formed by the irreducible arrangements of rank 2 (i.e., all non-Boolean rank 2 arrangements).

Sufficient condition

Large arrangements are easy to find due to the following sufficient condition.

Definition

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Theorem

Every well-balanced pair is balanced.

Indeed suppose (B, C) is well-balanced but there is a non-empty $D \subset B \cup C$ with $|D| \geq 2\,\mathrm{rk}\,D$. Then $D \cap B$ and $D \cap C$ are independent whence both are bases of D. Hence every $b \in D \cap B$ depends of $D \cap C$ which contradicts the condition.

Another sufficient condition

Let $L(\mathcal{A})$ be the lattice of all intersections of hyperplanes from $\mathcal{A}$ ordered opposite to inclusion. For $X \in L(\mathcal{A})$ we put $\mathcal{A}_X = \{H \in \mathcal{A} | H \geq X\}$.

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Definition

$L(\mathcal{A})$ is well-balanced if there exists $X \in L(\mathcal{A}), \operatorname{rk} X = r - 1$ such that for no $Y \in L(\mathcal{A}) \setminus \{0\}$ we have $\mathcal{A} = \mathcal{A}_X \cup \mathcal{A}_Y$.

This definition makes sense for an arbitrary finite geometric lattice.

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Proof.

Let C be a base of $\mathcal{A}_X$ from the definition. Then $|C| = r - 1$. Put $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_X$. By definition $\operatorname{rk} \mathcal{A}' = r$. Let B be a base of $\mathcal{A}'$ whence also a base of $\mathcal{A}$. Since B is disjoint with $\mathcal{A}_X$ no $b \in B$ depends on C. □

Corollary

Corollary

Suppose for all $X \in L(\mathcal{A}) \setminus \{0\}$ *we have*

$$|\mathcal{A}(X)| < \frac{n}{2}. \qquad (1)$$

Then $\mathcal{A}$ *is large.*

Clearly it suffices to check the inequality (1) for X of rank $r - 1$ only.

$\mathrm{TC}_s(M_W)$

Here is the main theorem of the talk.

Theorem

For every irreducible reflection group W of rank r and every $s > 1$ the arrangement $\mathcal{A}_W$ is well-balanced whence $TC_s(M_W) = sr - 1$.

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Theorem

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Our proof consists of four parts.

1. W of rank 2; this case is immediate since $r = 2$.
2. Infinite series.
3. The exceptional groups different from Coxeter types E_m.
4. Types E_m.

Case (2)

For the infinite series well-balanced pairs can be exhibited explicitly. Here we identify hyperplanes with their defining linear forms and $\mathcal{A}$ with the product of them.

(a) **Types $\mathbf{A}_r$ ($m = 1$) and full monomial types** $G(m, 1, r)$ ($m > 1$) : $Q = \prod_{i=1}^{r} x_i \prod_{1 \leq i < j \leq r}(x_i^m - x_j^m)$. Put $B = \{x_1, \ldots, x_r\}$ and $C = \{x_1 - x_2, \ldots, x_1 - x_r\}$.

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(a) **Types A$_r$ ($m = 1$)** and **full monomial types** $G(m, 1, r)$ $(m > 1)$: $Q = \prod_{i=1}^{r} x_i \prod_{1 \le i < j \le r} (x_i^m - x_j^m)$. Put $B = \{x_1, \ldots, x_r\}$ and $C = \{x_1 - x_2, \ldots, x_1 - x_r\}$.

(b) **Special monomial types** $G(m, m, r)$ $m \ge 2$:
$Q = \prod_{1 \le i < j \le r} (x_i^m - x_j^m)$. Put
$B = \{x_1 - \zeta x_2, \ldots, x_1 - \zeta x_r, x_2 - \zeta x_3\}$ and
$C = \{x_1 - x_2, \ldots, x_1 - x_r\}$ where ζ is a primitive root of 1 of order m.

Proof of case (2)

In (a), the result is clear. In (b), B is independent since it generates the basis $\{x_1, \ldots, x_r\}$ of V^*. Besides C lies in the kernel of the index (the linear map $ind : V^* \to \mathbb{C}, ind(x_i) = 1$) while no $b \in B$ does.

Case (3)

In this case, we check case-by-case that $L(\mathcal{A}_W)$ is well-balanced using Tables C.1-C.23 from the book: Orlik and Terao, Arrangements of Hyperplanes.

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We use Corollary 6.28 from this book that $\mathcal{A}_X$ is the reflection arrangement for a reflection subgroup W_X of W. The numbers $n_X = |\mathcal{A}_X|$ can be found from Table B.1 as the sums of covariants for W_X.

Table

The table below is organized as follows. The first row consists of the Shephard-Todd classification numbers (23-34) of exceptional groups of ranks greater than 2 (no types E_m). The second row consists of the cardinalities n of the respective arrangements. The third row consists of the maximal cardinalities of $\mathcal{A}_X$. It suffices to check inequality (1): $|\mathcal{A}_X| < \frac{n}{2}$.

23	24	25	26	27	28	29	30	31	32	33	34
15	21	12	21	45	24	40	60	60	40	45	126
5	4	4	5	5	9	12	15	15	12	12	45

Case (4)

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E_6	E_7	E_8
36	63	120
(20,15)	(36,21)	(63,42)

The second row has the same meaning as in the previous table.The last row consists of pairs combining the maximal cardinality of $\mathcal{A}_Y$ with rk $Y = r - 1$ and the cardinality of another $\mathcal{A}_X$ also with rk $X = r - 1$. One needs to check that the sum in each pair is less than the entry of the second row. This shows that $L(\mathcal{A}_W)$ is well-balanced.

Examples

There are other classes of examples of large arrangements. The most significant consists of all generic arrangements with $|\mathcal{A}| \geq 2r - 1$.

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$$\mathrm{TC}_s(M) = \min\{sr - 1, (s-1)n\}.$$

For instance, if $r = 3,\ n = 4,\ s = 2$ then $\mathrm{TC}_2(M) = 4 < 2r - 1$.

Conjecture

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For every complex hyperplane arrangement the topological complexity of its complement equals the cohomological lower bound (for every s).

THANK YOU FOR YOUR ATTENTION!