COMBINATORICS OF MODULI SPACES OF CURVES

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Contents

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1. Lecture 1

Abstract tropical curves

Definition 1.
A *(weighted) tropical curve* is a triple $\Gamma = (G, \ell, w)$ such that

$G = (V, E)$ is a graph;

$\ell : E \to \mathbb{R}_{>0}$ is a *length* function on the edges;

$w : V \to \mathbb{Z}_{\geq 0}$ is a *weight* function on the vertices.

Convention. Graphs and tropical curves are connected.

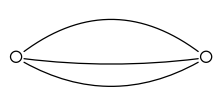
The *genus* of the tropical curve $\Gamma = (G, \ell, w)$ is

$$g(\Gamma) := g(G, w) := b_1(G) + \sum_{v \in V} w(v),$$

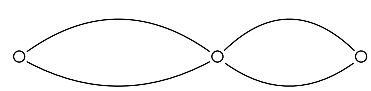
$$b_1(G) = rk_{\mathbb{Z}} H_1(G, \mathbb{Z})$$

Convention. To avoid dealing with special cases, genus ≥ 2.

Definition 2. A tropical curve $\Gamma = (G, \ell, w)$ is *stable* if its underlying graph $G = (V, E)$ is *stable*, i.e.
if every vertex of valency 0 has weight at least 3.



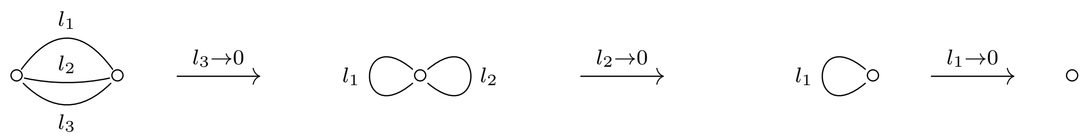
Stable



Not stable

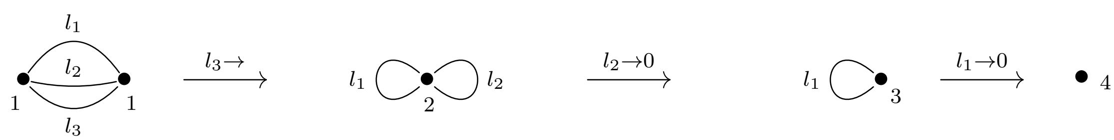
Remark. For any $g \geq 2$ there exist finitely many (non-isomorphic) stable graphs of genus g.

Question. Why a weight on the vertices?



Answer. Because the genus may drop under specialization.

Remedy. ([BMV11]) Add weights to the vertices and refine the concept of specialization.



Specializations of tropical curves correspond to weighted edge-contractions of underlying graphs. we shall denote by

$$(G, w) \longrightarrow (G', w') \quad \text{if} \quad (G', w') \text{ is a contraction of } (G, w)$$

Conclusion. Specializations of tropical curves, or contractions of weighted graphs, preserve the genus.

Remark. Think of a vertex v of positive weight $w(v)$ as having $w(v)$ invisible loops of zero length based at it.

Equivalence of tropical curves

Two tropical $\Gamma = (G, \ell, w)$ and $\Gamma' = (G', \ell', w')$ are *isomorphic* if there is an isomorphism between G and G' which preserves both the weights of the vertices and the lengths of the edges.

Definition 3. Two tropical curves, Γ and $\Gamma'$are *equivalent* if one obtains isomorphic tropical curves, $\overline{\Gamma}$ and $\overline{\Gamma'}$, after performing the following two operations until $\overline{\Gamma}$ and $\overline{\Gamma'}$ are stable.

- Remove all weight-zero vertices of valency 1 and their adjacent edge.

- Remove every weight-zero vertex v of valency 2 and replace it by a point (not a vertex), so that the two edges adjacent to v become one edge.

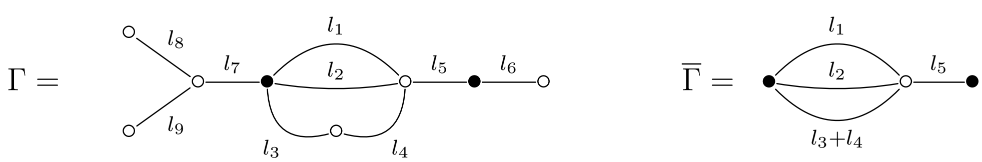


FIGURE 1. A tropical curve Γ and its stabilization, $\overline{\Gamma}$.

Lemma 4. *Let (G, w) be a stable graph. Then G has at most $3g - 3$ edges, and the following are equivalent.*

(1) $|E(G)| = 3g - 3$.
(2) Every vertex of G has weight 0 and valency 3.
(3) Every vertex of G has weight 0 and $|V(G)| = 2g - 2$.

Proof. EXERCISE. ♣

The moduli space of tropical curves of genus g

M_g^{trop} = moduli space of equiv. classes of tropical curves of genus g.

Set theoretically

$$M_g^{\text{trop}} = \bigsqcup_{(G,w)\in \mathrm{S}_g} M^{\text{trop}}(G,w)$$

where

$$\mathrm{S}_g = \text{ set of stable graphs of genus } g$$

and

$$M^{\text{trop}}(G,w) = \frac{\text{tropical curves having } (G,w) \text{ as underlying graph}}{\text{isomorphism}}$$

Remark. From now on we shall assume tropical curves are stable.

Goal. Construction of M_g^{trop} as a topological space (following [Cap12]).

Start from constructing the stratum $M^{\text{trop}}(G,w)$.

Construction of M_g^{trop} as a topological space

Step 1. Construction of the stratum $M^{\mathrm{trop}}(G, w)$.

Set $G = (V, E)$.
Consider the open cone in $\mathbb{R}^{|E|}$ with the euclidean topology:

$$\mathbb{R}_{>0}^{|E|}.$$

There is a natural surjection

$$\begin{array}{ccc} \mathbb{R}_{>0}^{|E|} & \longrightarrow & M^{\mathrm{trop}}(G, w) \\ \ell = (l_1, \ldots, l_{|E|}) & \mapsto & (G, \ell, w) \end{array}$$

$\mathrm{Aut}(G, w)$= automorphism group of (G, w).

$\mathrm{Aut}(G, w)$ acts on $\mathbb{R}_{>0}^{|E|}$ by permuting the coordinates.

The above surjection is the quotient by that action:

$$M^{\mathrm{trop}}(G, w) = \frac{\mathbb{R}_{>0}^{|E|}}{\mathrm{Aut}(G, w)}$$

with the quotient topology.

We are done with $M^{\mathrm{trop}}(G, w)$.

Now look at specializations of curves in $M^{\mathrm{trop}}(G, w)$.

Step 2. Study specializations of curves in $M^{\text{trop}}(G, w)$.

The boundary of the closed cone

$$\mathbb{R}_{\geq 0}^{|E|}$$

parametrizes tropical curves with fewer edges, that are specializations of tropical curves in the open cone.

The closure in M_g^{trop} of a stratum is a union of strata:

$$M^{\text{trop}}(G, w) \subset \overline{M^{\text{trop}}(G', w')} \quad \Leftrightarrow \quad (G', w') \to (G, w).$$

The action of $\text{Aut}(G, w)$ extends to the closed cone so that we have

$$\widetilde{M^{\text{trop}}}(G, w) := \mathbb{R}_{\geq 0}^{|E|}/\text{Aut}(G, w).$$

Step 3. Construct M_g^{trop}.

For every stable graph (G, w) have a natural map

$$\widetilde{M^{\text{trop}}}(G, w) := \mathbb{R}_{\geq 0}^{|E|}/\text{Aut}(G, w) \longrightarrow M_g^{\text{trop}}$$

mapping a curve to its isomorphism class.
Hence we have the following natural map

$$\bigsqcup_{\substack{(G,w)\in \mathrm{S}_g: \\ |E|=3g-3}} \widetilde{M^{\text{trop}}}(G, w) \longrightarrow M_g^{\text{trop}}.$$

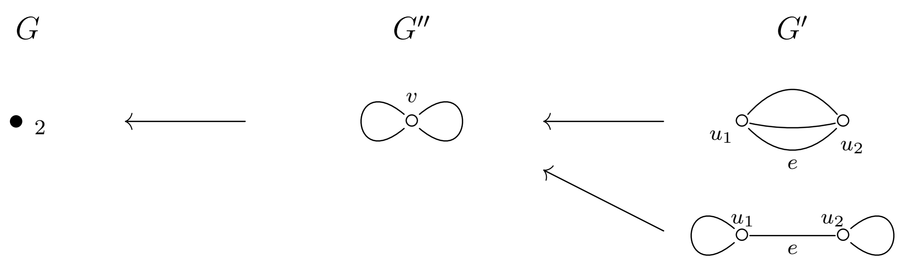
Question. Is the above map surjective?

Answer. Yes, by the following proposition.

Proposition 5. *Let (G, w) be a stable graph of genus g. Then there exists a stable graph (G', w') of genus g with $3g - 3$ edges such that*

$$M^{\text{trop}}(G, w) \subset \overline{M^{\text{trop}}(G', w')}.$$

Example.



We can thus endow M_g^{trop} of the quotient topology.

Theorem 6 ([Mik07], [BMV11], [Cap12])**.** *The topological space M_g^{trop} is connected, Hausdorff, and of pure dimension $3g-3$ (i.e. it has a dense open subset which is a $(3g-3)$-dimensional orbifold over $\mathbb{R}$).*

Extended tropical curves

Remark. M_g^{trop} is not compact.

Definition 7. An *extended tropical curve* is a triple $\Gamma = (G, \ell, w)$ where (G, w) is a stable graph and $\ell : E \to \mathbb{R}_{>0} \cup \{\infty\}$ an "extended" length function.

Compactify $\mathbb{R} \cup \{\infty\}$ by the Alexandroff one-point compactification, and consider its subspaces with the induced topology.

The moduli space of extended tropical curves with (G, w) as underlying graph:

$$\overline{M^{\text{trop}}(G,w)} = \frac{(\mathbb{R}_{>0} \cup \{\infty\})^{|E|}}{\text{Aut}(G,w)}$$

with the quotient topology.

As for M_g^{trop}, we have

$$\bigsqcup_{\substack{(G,w)\in \mathrm{S}_g: \\ |E|=3g-3}} \widetilde{M_\infty^{\text{trop}}}(G,w) \longrightarrow \overline{M_g^{\text{trop}}} = \bigsqcup_{(G,w)\in \mathrm{S}_g} \overline{M^{\text{trop}}(G,w)}.$$

Theorem 8. [Cap12] *The moduli space of extended tropical curves, $\overline{M_g^{\text{trop}}}$, with the quotient topology, is compact, normal, and contains M_g^{trop} as dense open subset.*

Remark. A tropical curve will correspond to families of smooth algebraic curves degenerating to nodal ones.

An extended tropical curve will correspond to families of nodal algebraic curves degenerating, again, to nodal ones.

Under this correspondence an extended tropical curve $\Gamma = (G, w, \underline{\infty})$, all of whose edges have length equal to ∞, corresponds to locally trivial families all of whose fibers have dual graph (G, w).

2. Lecture 2.

From algebraic curves to tropical curves

Algebraic curve = projective variety of dimension one over an algebraically closed field k.

We shall be interested exclusively in

Nodal curves = reduced (possibly reducible) curves admitting at most nodes as singularities.

Convention. Curves will be connected.

To a curve X we associate its (weighted) dual graph, (G_X, w_X)

$$V(G_X) = \text{ irreducible components of } X;$$

for $v \in V(G_X)$

$$w_X(v) = \text{ geometric genus of the corresponding component};$$

$$E(G_X) = \text{ nodes of } X.$$

An edge e joins the vertices v and w if the corresponding components mett at the node e.

X is *stable* if so is its dual graph, (G_X, w_X).

Proposition 9. *A connected curve is stable if and only if it has finitely many automorphisms, if and only if its dualizing line bundle is ample.*

Proof. EXERCISE (if you know some algebraic geometry). ♣

Proposition 10. *The (arithmetic) genus of an algebraic curve X is equal to the genus of its dual graph, (G_X, w_X).*

Proof. $g(X) := h^1(X, \mathcal{O}_X)$.

Now, write $G_X = (V, E)$, and consider the normalization map

$$\nu : X^\nu = \bigsqcup_{v \in V} C_v^\nu \longrightarrow X.$$

The associated map of structure sheaves yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \nu_* \mathcal{O}_{X^\nu} \longrightarrow \mathcal{S} \longrightarrow 0$$

where $\mathcal{S}$ is a skyscraper sheaf supported on the nodes of X.

The associated exact sequence in cohomology is as follows (identifying the cohomology groups of $\nu_* \mathcal{O}_{X^\nu}$ with those of $\mathcal{O}_{X^\nu}$ as usual)

$$0 \longrightarrow \quad H^0(X, \mathcal{O}_X) \longrightarrow \quad H^0(X^\nu, \mathcal{O}_{X^\nu}) \xrightarrow{\tilde{\delta}} k^{|E|} \longrightarrow$$

$$\longrightarrow \quad H^1(X, \mathcal{O}_X) \longrightarrow \quad H^1(X^\nu, \mathcal{O}_{X^\nu}) \longrightarrow 0.$$

Hence

$$g = h^1(X^\nu, \mathcal{O}_{X^\nu}) + |E| - |V| + 1 = \sum_{v \in V} g_v + b_1(G_X) = g(G_X, w_X)$$

where $g_v = h^1(C_v^\nu, \mathcal{O}_{C_v^\nu})$ is the genus of C_v^ν.

Now $g_v = w_X(v)$, hence X and (G_X, w_X) have the same genus. ♣

Families of algebraic curves over local schemes
$K \supset k$

K is a field complete with respect to a non-Archimedean valuation v_K

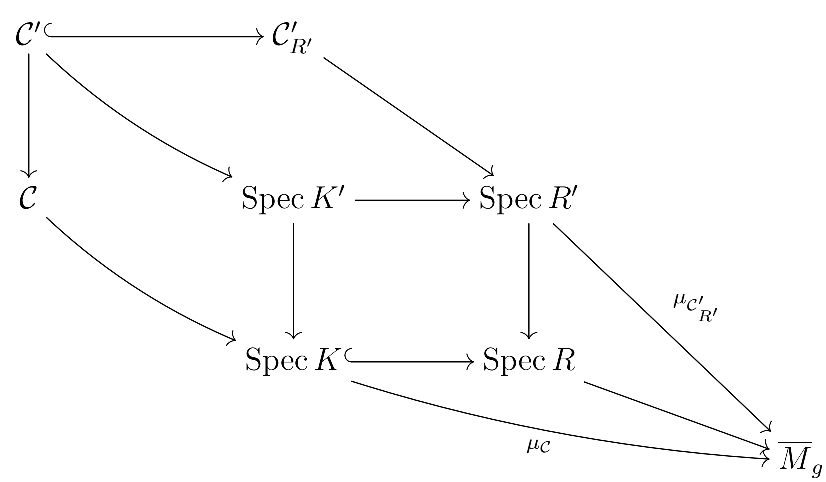
$$v_K : K \to \mathbb{R} \cup \{\infty\}.$$

Such a K is also called a *non-Archimedean field*.
The valuation of K induces on k the trivial valuation $k^* \to 0$.
R is the valuation ring of K.

The (updated) Stable Reduction Theorem of Deligne-Mumford [DM69].

Theorem 11. *Let $\mathcal{C}$ be a stable curve over K.*
Then there exists a finite field extension $K'|K$ such that the base change $\mathcal{C}' = \mathcal{C} \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ admits a unique model over the valuation ring of K' whose special fiber is a stable curve.

The theorem is represented in the following commutative diagram.



The moduli space of algebraic curves of genus g

$$\overline{M}_g = \text{moduli space of stable curves of genus } g.$$

We have

$$\overline{M}_g = \bigsqcup_{(G,w)\in \mathrm{S}_g} M(G,w)$$

where

$$M(G,w) = \text{ locus of stable curves having } (G,w) \text{ as dual graph.}$$

We have

$$M(G,w) \subset \overline{M_{(G',w')}} \quad \Leftrightarrow \quad (G,w) \to (G',w').$$

This is analogous, though <u>reversing the arrow</u>, to what happens in $\overline{M_g^{\mathrm{trop}}}$.

For details about the following statement we refer to [HM98], [ACG11].

Theorem 12. *The moduli space $\overline{M}_g$ of stable curves of genus g is an irreducible, normal, projective variety of dimension $3g-3$.*

For every stable graph (G,w) the locus $M(G,w)$ is quasiprojective, irreducible, of codimension $|E(G)|$.

The locus of smooh curves, written M_g is open in $\overline{M}_g$

Example

The graph with no edges, and one vertex of weight g is denoted by

$$(G,w) = \bullet_g$$

hence

$$M_g = M(\bullet_g).$$

3. Lecture 3.

The poset of stable graphs.

S_g is the set of stable graphs of genus g.
S_g is a *poset* (i.e. partially ordered) with respect to contractions:

$$(G, w) \geq (G', w') \qquad \text{if} \qquad (G, w) \to (G', w')$$

i.e. for some $S \subset E(G)$

$$(G', w') = (G/S, w_{/S}).$$

Remark. $(G/S, w_{/S}) \leq (G/T, w_{/T})$ if and only if $T \subset S$.

S_g is *graded* by the following *rank* function

$$\text{rk} : S_g \longrightarrow \mathbb{N} : \qquad G \mapsto |E(G)|$$

Recall: $0 \leq \text{rk}(G) \leq 3g - 3$, and this is sharp.

Question. What are the maximal elements in S_3?

– –

Let M be a "geometric" space (an algebraic variety, a topological space) and let (S, rk) be a graded poset.

We say M is *stratified* by S if M admits a partition indexed by S:

$$M = \bigsqcup_{s \in S} M(s)$$

such that

$$M(s) \subset \overline{M(s')} \quad \Leftrightarrow \quad s \geq s' \qquad [\text{ or } s' \geq s]$$

and

$$\operatorname{codim} M(s) = \text{rk}(s) \qquad [\text{ or } \dim M(s) = \text{rk}(s)].$$

The moduli space of algebraic stable curves, $\overline{M}_g$ is *stratified* by the poset S_g:

$$\overline{M}_g = \bigsqcup_{(G,w)\in \mathrm{S}_g} M(G,w)$$

with

$$M(G,w) \subset \overline{M(G',w')} \quad \Leftrightarrow \quad (G,w) \geq (G',w').$$

$$\operatorname{codim} M(G,w) = |E(G)|.$$

$-------------------$*Analogously*$-------------------$

The moduli space of extended tropical curves, $\overline{M_g^{\mathrm{trop}}}$, is *stratified* by S_g:

$$\overline{M_g^{\mathrm{trop}}} = \bigsqcup_{(G,w)\in \mathrm{S}_g} \overline{M^{\mathrm{trop}}}(G,w)$$

with

$$\overline{M^{\mathrm{trop}}}(G,w) \subset \overline{\overline{M^{\mathrm{trop}}}(G',w')} \quad \Leftrightarrow \quad (G',w') \geq (G,w).$$

and

$$\dim \overline{M^{\mathrm{trop}}}(G,w) = |E(G)|.$$

Connection between $\overline{M_g^{\mathrm{trop}}}$ and $\overline{M}_g$: the global picture.

$\overline{M_g^{\mathrm{trop}}}$ is constructed by gluing Euclidean cones via of combinatorial rules.

The same combinatorial rules are respected, up to arrow-reversal, by $\overline{M}_g$.

The theory of *Toroidal Embeddings* (Kempf-Knudsen-Mumford-Saint Donat, [KKMSD73]) indicates that M_g^{trop} should be the *skeleton* of $\overline{M}_g$.

The problem is: $\overline{M}_g$ does not have a *toroidal* structure.
But its moduli *stack*, $\overline{\mathcal{M}_g}$, the moduli stack of stable curves, does.

The toroidal structure of $\overline{\mathcal{M}_g}$ enables one to construct such a skeleton as a *generalized cone complex* associated to $\overline{\mathcal{M}_g}$, denoted by

$$\Sigma(\overline{\mathcal{M}_g})$$

and compactified by an *extended generalized cone complex*, written

$$\overline{\Sigma}(\overline{\mathcal{M}_g}).$$

Theorem 13. [ACP15] *There are canonical isomorphisms*

$$\Sigma(\overline{\mathcal{M}_g}) \cong M_g^{\mathrm{trop}} \qquad \text{and} \qquad \overline{\Sigma}(\overline{\mathcal{M}_g}) \cong \overline{M_g^{\mathrm{trop}}}$$

fitting in a commutative diagram

$$\begin{array}{ccc} \Sigma(\overline{\mathcal{M}_g}) & \hookrightarrow & \overline{\Sigma}(\overline{\mathcal{M}_g}) \\ \cong \downarrow & & \downarrow \cong \\ M_g^{\mathrm{trop}} & \hookrightarrow & \overline{M_g^{\mathrm{trop}}} \end{array}$$

This theorem is an explanation of the global geometric analogies between $\overline{M}_g$ and $\overline{M_g^{\mathrm{trop}}}$.

Question. What about the local point of view?

Connection between $\overline{M_g^{\text{trop}}}$ and $\overline{M}_g$: the local picture

$K \supset k$ is a non-Archimedean, i.e. a field complete with respect to a non-Archimedean valuation v_K

$$v_K : K \to \mathbb{R} \cup \{\infty\} \qquad \text{such that} \qquad v_K(0) = \infty.$$

The valuation v_K induces on k the trivial valuation $k^* \to 0$.
R is the valuation ring of K.

Let $\mathcal{C} \to \operatorname{Spec} K$ be a stable curve over K. The ascociated *moduli map* is

$$\mu_{\mathcal{C}} : \operatorname{Spec} K \longrightarrow \overline{M}_g$$

If $\mathcal{C}$ adimits a stable model, $\mathcal{C}_R \to \operatorname{Spec} R$, over R, then the *moduli map* associated to $\mathcal{C}_R$

$$\mu_{\mathcal{C}_R} : \operatorname{Spec} R \longrightarrow \overline{M}_g$$

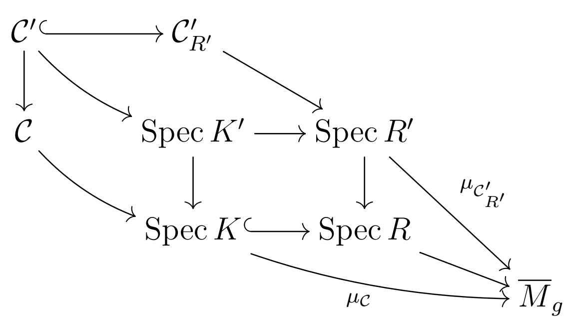
is the extension of $\mu_{\mathcal{C}}$ to $\operatorname{Spec} R$.

Remark. By the vautaive criterion for properness, the map $\mu_{\mathcal{C}}$ admits a unique extension to a map $\operatorname{Spec} R \to \overline{M}_g$. But it may happen that this extension is not the moduli map of a stable curve over $\operatorname{Spec} R$.

Again the Stable Reduction Theorem of Deligne-Mumford [DM69].

Theorem 14. *Let $\mathcal{C}$ be a stable curve over K.*
Then there exists a finite field extension $K'|K$ such that the base change $\mathcal{C}' = \mathcal{C} \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ admits a unique model over the valuation ring of K' whose special fiber is a stable curve.

The theorem is represented in the following commutative diagram.



$\overline{\mathcal{M}}_g(K)$: the set of stable curves of genus g over K.
We can define a *(stable) reduction* map for our field K:

$$\mathrm{red}_K : \overline{\mathcal{M}}_g(K) \longrightarrow \overline{M}_g; \qquad \mathcal{C} \mapsto \mathcal{C}_k.$$

Indeed: the map $\mu_{\mathcal{C}} : \operatorname{Spec} K \to \overline{M}_g$ extends uniquely to a map $\operatorname{Spec} R \to \overline{M}_g$. Hence the image of the special point of $\operatorname{Spec} R$ is uniquely determined by $\mathcal{C}$. This is a stable curve over the residue field, k, of R, denoted by $\mathcal{C}_k$ and called the *stable reduction of* $\mathcal{C}$.

Introduce the set of K-*points* (or K-*rational points*) of $\overline{M}_g$

$$\overline{M}_g(K) := \operatorname{Hom}(\operatorname{Spec} K, \overline{M}_g) = \{\operatorname{Spec} K \to \overline{M}_g\}$$

We have a natural map

$$\mu_K : \overline{\mathcal{M}}_g(K) \longrightarrow \overline{M}_g(K); \qquad \mathcal{C} \mapsto \mu_{\mathcal{C}}$$

It is clear that the map red_K factors through μ, i.e. we have

$$\mathrm{red}_K : \overline{\mathcal{M}}_g(K) \xrightarrow{\mu} \overline{M}_g(K) \longrightarrow \overline{M}_g.$$

So far the valuation of K did not play a specific role; its existence was used to apply the valuative criterion of properness. It will play a more important role in what follows.

The Stable Reduction Theorem implies the following.

Proposition 15. *Let $\mathcal{C}$ be a stable curve over K and let $\mathcal{C}_k$ be the stable reduction of $\mathcal{C}$. Then there exists an extended tropical curve $\Gamma_{\mathcal{C}} = (G_{\mathcal{C}}, \ell_{\mathcal{C}}, w_{\mathcal{C}})$ with the following properties.*

(1) *$(G_{\mathcal{C}}, w_{\mathcal{C}})$ is the dual graph of $\mathcal{C}_k$.*
(2) *$\Gamma_{\mathcal{C}}$ is a non-extended tropical curve (i.e. all edges have finite length) if and only if $\mathcal{C}$ is smooth.*
(3) (Compatibility with base change) *If $K' \supset K$ is a finite extension and $\mathcal{C}'$ the base change of $\mathcal{C}$ over K', then $\Gamma_{\mathcal{C}} = \Gamma_{\mathcal{C}'}$.*

Main point of the proof: to complete definition of the tropical curve $\Gamma_{\mathcal{C}}$ by defining the length function $\ell_{\mathcal{C}}$.

Step 1. Stable Reduction Theorem $\Rightarrow$ can assume $\mathcal{C}_k$ is the special fiber of a family of stable curves over R', for some finite extension $R' \supset R$.

Step 2. Let e be a node of $\mathcal{C}_k$. The equation of the family locally at e has the form

$$xy = f_e$$

with $f_e \in M' \subset R'$ (M' the maximal ideal of R').

Step 3. K is complete and the extension $K' \supset K$ is finite $\Rightarrow$ v_K extends to a unique valuation $v_{K'}$, and K' is complete.

Step 4. Set

$$\ell_{\mathcal{C}}(e) = v_{K'}(f_e).$$

Step 5. $\mathcal{C}$ smooth, $\Rightarrow$ $f_e \neq 0$, hence $\ell_{\mathcal{C}}(e) \in \mathbb{R}_{>0}$ and $\Gamma_{\mathcal{C}}$ is a tropical curve.

$\mathcal{C}$ has some node $\Rightarrow$ this node specializes to some node, e, of $\mathcal{C}_k$, for which $f_e = 0$, because the family is locally reducible. Therefore $\ell_{\mathcal{C}}(e) = v_{K'}(0) = \infty$.

The rest of proof consists in showing independence from the various choices and compatibility with base change; all of that is standard.

(Same proposition as previous slide)

Proposition 16. *Let $\mathcal{C}$ be a stable curve over K and let $\mathcal{C}_k$ be the stable reduction of $\mathcal{C}$. Then there exists an extended tropical curve $\Gamma_{\mathcal{C}} = (G_{\mathcal{C}}, \ell_{\mathcal{C}}, w_{\mathcal{C}})$ with the following properties.*

(1) $(G_{\mathcal{C}}, w_{\mathcal{C}})$ *is the dual graph of $\mathcal{C}_k$.*
(2) $\Gamma_{\mathcal{C}}$ *is a non-extended tropical curve (i.e. all edges have finite length) if and only if $\mathcal{C}$ is smooth.*
(3) (Compatibility with base change) *If $K' \supset K$ is a finite extension and $\mathcal{C}'$ the base change of $\mathcal{C}$ over K', then $\Gamma_{\mathcal{C}} = \Gamma_{\mathcal{C}'}$.*

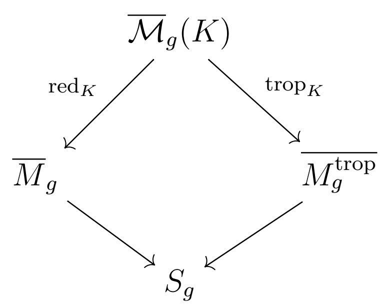
Consequence. We can define a *local tropicalization map*, trop_K, as follows

$$\mathrm{trop}_K : \overline{\mathcal{M}}_g(K) \longrightarrow \overline{M_g^{\mathrm{trop}}}; \qquad \mathcal{C} \mapsto \Gamma_{\mathcal{C}}.$$

As before, trop_K factors as follows

$$\mathrm{trop}_K : \overline{\mathcal{M}}_g(K) \xrightarrow{\mu} \overline{M}_g(K) \longrightarrow \overline{M_g^{\mathrm{trop}}}.$$

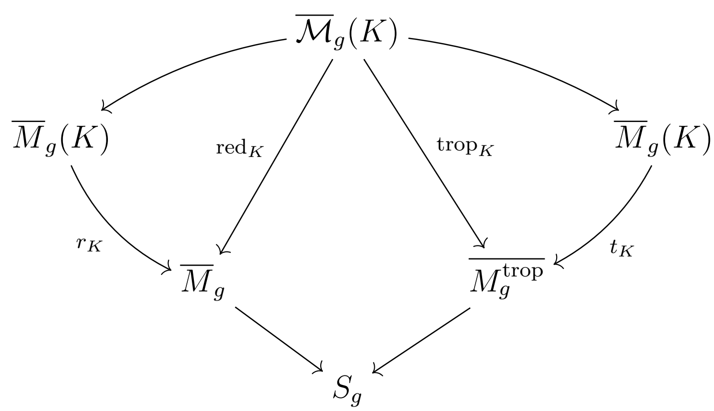
Conclusion. We have a commutative diagram representing the local analogies.



4. Lecture 4,

Connection between $\overline{M_g^{\mathrm{trop}}}$ and $\overline{M}_g$: the local picture

The commutative diagram represents the local analogies:



$K|k$ a non-Archimedean field (complete w.r.t. a non-Arch. valuation).

$\overline{\mathcal{M}}_g(K)$ = the set of stable curves of genus g over K.

$\overline{M}_g(K) = \mathrm{Hom}(\mathrm{Spec}\, K, \overline{M}_g) =$ the set of K-points of $\overline{M}_g$.

S_g= the graded poset of stable graphs of genus g.

The Berkovich analytification, $\overline{M}_g^{\mathrm{an}}$, of $\overline{M}_g$.

A theory due to Berkovich ([Ber90]) provides, for any algebraic variety X over k, an analytic space, X^{an}, the *analytification* of X, to which analytic methods can be applied.

We apply this theory to $\overline{M}_g$.

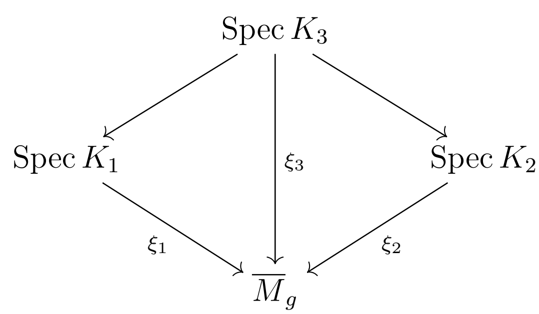
The *analytification* $\overline{M}_g^{\mathrm{an}}$ of $\overline{M}_g$ is, set theoretically, described as follows

$$\overline{M}_g^{\mathrm{an}} := \frac{\bigsqcup_{K|k} \overline{M}_g(K)}{\sim_{\mathrm{an}}}$$

where the union is over all non-Archimedean extensions $K|k$, and

$$\xi_1 \sim_{\mathrm{an}} \xi_2$$

for $\xi_i \in \overline{M}_g(K_i)$ for $i = 1, 2$ if there exists a $\xi_3 \in \overline{M}_g(K_3)$ and a commutative diagram:



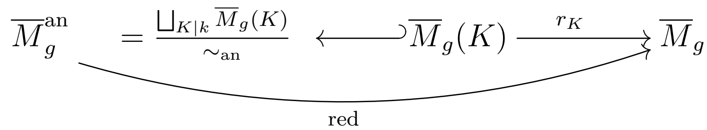
Remark. A point of $\overline{M}_g^{\mathrm{an}}$ is represented by a stable curve $\mathcal{C}$ over a non-Archimedean field K.

By the Stable Reduction Theorem, we can assume, up to field extension, that $\mathcal{C}$ admits a stable model over the valuation ring of K.

The local stable reduction maps, $r_K : \overline{M}_g(K) \to \overline{M}_g$, define a *reduction* map

$$\mathrm{red} : \overline{M}_g^{\mathrm{an}} \longrightarrow \overline{M}_g$$

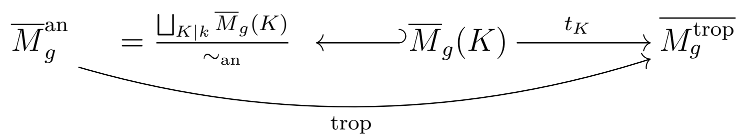
such that the restriction to $\overline{M}_g(K)$ coincides with the map r_K:



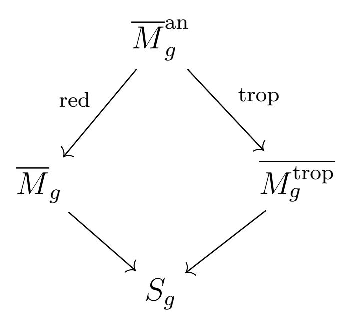
The local tropicalization maps, $t_K : \overline{M}_g(K) \to \overline{M_g^{\mathrm{trop}}}$, define a *tropicalization* map

$$\mathrm{trop} : \overline{M}_g^{\mathrm{an}} \longrightarrow \overline{M_g^{\mathrm{trop}}}$$

such that the restriction to $\overline{M}_g(K)$ coincides with the map t_K:



Conclusion. The local analogies between $\overline{M}_g$ and $\overline{M_g^{\mathrm{trop}}}$ described so far derive from the following commutative diagram



[Tyo12], [BPR16], [Viv13]

The <u>local</u> analogies derive from the following commutative diagram

$$\begin{array}{ccccc} & & \overline{M}_g^{\mathrm{an}} & & \\ & \swarrow^{\mathrm{red}} & & \searrow^{\mathrm{trop}} & \\ \overline{M}_g & & & & \overline{M_g^{\mathrm{trop}}} \\ & \searrow & & \swarrow & \\ & & S_g & & \end{array}$$

In Lecture 3 - Theorem 1, we had a canonical isomorphism of *extended generalized cone complexes*:

$$\overline{\Phi} : \overline{\Sigma}(\overline{\mathcal{M}_g}) \xrightarrow{\cong} \overline{M_g^{\mathrm{trop}}}$$

explaining the <u>global</u> analogies between $\overline{M}_g$ and $\overline{M_g^{\mathrm{trop}}}$.

Question. Is the isomorphism $\overline{\Phi}$ connected to the diagram above?

— —

Answer. Yes. The connection is achieved using results of Thuillier [Thu07], which enable us to construct a *retraction* of $\overline{M}_g^{\mathrm{an}}$ onto the extended skeleton of $\overline{\mathcal{M}_g}$. More precisely, there is a *Homotopy*

$$H : [0,1] \times \overline{M}_g^{\mathrm{an}} \longrightarrow \overline{M}_g^{\mathrm{an}}$$

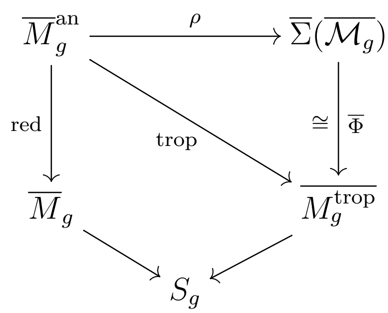
connecting $\mathrm{id}_{\overline{M}_g^{\mathrm{an}}}$ to an idempotent map $\rho : \overline{M}_g^{\mathrm{an}} \longrightarrow \overline{M}_g^{\mathrm{an}}$. Hence the image, $\rho(\overline{M}_g^{\mathrm{an}}) \subset \overline{M}_g^{\mathrm{an}}$ is a retraction of $\overline{M}_g^{\mathrm{an}}$.

Now, this retraction, $\rho(\overline{M}_g^{\mathrm{an}})$ can be identified with the *extended skeleton* of $\overline{\mathcal{M}_g}$. We thus have a retraction:

$$\rho : \overline{M}_g^{\mathrm{an}} \longrightarrow \overline{\Sigma}(\overline{\mathcal{M}_g}).$$

Conclusion. The following statement contains all the facts described so far.

Theorem 17. *(*[ACP15]*) We have a commutative canonical diagram:*



$\overline{M}_g^{\mathrm{an}}$ is the *Berkovich analytification* of $\overline{M}_g$.

$\overline{\Sigma}(\overline{\mathcal{M}_g})$ is the *Skeleton* of the stack $\overline{\mathcal{M}_g}$.

$\rho : \overline{M}_g^{\mathrm{an}} \longrightarrow \overline{\Sigma}(\overline{\mathcal{M}_g})$ is the retraction.

The previous theorem is a special case of the following

Theorem 18 ([ACP15]). *Let g and n be non-negative integers.*

(1) There is an isomorphism of generalized cone complexes with integral structure

$$\Phi_{g,n} : \Sigma(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\sim} M_{g,n}^{\mathrm{trop}}$$

extending uniquely to the compactifications

$$\overline{\Phi}_{g,n} : \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\sim} \overline{M}_{g,n}^{\mathrm{trop}}.$$

(2) The following diagram is commutative:

$$\begin{array}{ccc} \overline{M}_{g,n}^{\mathrm{an}} & \xrightarrow{\rho_{g,n}} & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \\ \downarrow{\scriptstyle \mathrm{red}_{g,n}} & \searrow{\scriptstyle \mathrm{trop}_{g,n}} & \downarrow{\scriptstyle \overline{\Phi}_{g,n}} \\ \overline{M}_{g,n} & & \overline{M}_{g,n}^{\mathrm{trop}} \\ & \searrow \quad \swarrow & \\ & S_{g,n} & \end{array}$$

In particular the map $\mathrm{trop}_{g,n}$ is continuous, proper, and surjective.

Jacobians of algebraic curves.

C = smooth, connected, projective curve C of genus $g \geq 2$ over k.

The *Jacobian* of C is a *principally polarized abelian variety*, i.e. a pair

$$\mathbf{Jac}(C) := (\mathrm{Jac}(C), \Theta(C))$$

$\mathrm{Jac}(C)$ = an abelian variety of dimension g;
$\Theta(C)$ = the *theta divisor*, an irreducible, ample divisor in $\mathrm{Jac}(C)$.
If $k = \mathbb{C}$ we have

$$\mathrm{Jac}(C) := H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z}) \cong \mathbb{C}^g/\mathbb{Z}^{2g}.$$

For any $d \in \mathbb{Z}$ we have an isomorphism

$$\mathrm{Jac}(C) \cong \mathrm{Pic}^d(C) = \frac{\text{line bundles of degree } d}{\cong} \cong \frac{\text{Divisors of degree } d}{\sim}.$$

Remark. If $d = 0$ then $\mathrm{Pic}^0(C)$ is a group, hence $\mathrm{Jac}(C)$ is a group.

The Theta divisor, viewed in $\mathrm{Pic}^{g-1}(C)$, is

$$\Theta(C) := \{L \in \mathrm{Pic}^{g-1}(C) : \ h^0(C, L) \geq 1\}.$$

− −

Recall the following famous Torelli Theorem.

Theorem 19 (Torelli version 1)**.** *Let C_1 and C_2 be two smooth curves; then $C_1 \cong C_2$ if and only if* $\mathbf{Jac}(C_1) \cong \mathbf{Jac}(C_2)$.

Question. What about moduli spaces of Jacobians, or of abelian varieties?

Moduli of Abelian varieties and the Torelli map for stable cuves.

A_g = moduli space of principally polarized abelian varieties of dimension g

A_g is an irreducible, non projective, algebraic variety of dimension $g(g+1)/2$.

The Torelli theorem can be re-stated using moduli spaces:

Theorem 20 (Torelli version 2)**.** *The following* Torelli map

$$\tau : M_g \longrightarrow A_g; \qquad C \mapsto \mathbf{Jac}(C)$$

is injective.

Question. Does the Torelli map extend to $\overline{M}_g$?

Answer. Yes, provided we compactify A_g.

There exist several compactifications for A_g, all of which rely on some type of combinatorial methods. To extend the Torelli map we use [Ale02] and [Ale04]:

$\overline{A_g}$= *Main irreducible component* of the moduli space for *semi-abelic stable pairs.*

The Torelli map extends to $\overline{M}_g$

$$\begin{array}{ccc} \overline{M}_g & \xrightarrow{\overline{\tau}} & \overline{A}_g; \\ \uparrow & & \uparrow \\ M_g & \xrightarrow{\tau} & A_g \end{array} \qquad [X] \longmapsto [\overline{\mathbf{Jac}}(X)] = \left[\mathrm{Jac}(X) \curvearrowright (\overline{P_{g-1}}(X), \overline{\Theta(X)})\right]$$

X is a stable curve;

$\mathrm{Jac}(X)$ is the (generalized) Jacobian of X, i.e.

$$\mathrm{Jac}(X) := \frac{\text{line bundles having degree 0 on every irr. comp. of } X}{\cong}.$$

$\overline{P_{g-1}}(X)$ is the *compactified Jacobian* constructed in [Cap94]. It is the moduli space for *balanced* line bundles of degree $g-1$ on semistable curves stably equivalent to X.
$\overline{P_{g-1}}(X)$ is a connected and reduced projective variety; it may have several irreducible components, all of dimension g.

$\overline{\Theta(X)}$ is the *Theta divisor*, it is an ample Cartier divisor of $\overline{P_{g-1}}(X)$ [Est01].

$\mathrm{Jac}(X)$ is a group, and acts on $\overline{P_{g-1}}(X)$ by tensor product.

Remark. The orbits in $\overline{P_{g-1}}(X)$ under the $\mathrm{Jac}(X)$-action have an interesting combinatorial structure, governed by the dual graph (G_X, w_X) of X.

5. Lecture 5.

Jacobians of stable curves

X a stable curve, (G_X, w_X) its dual graph.
The desingularization of X:

$$\nu : X^\nu = \bigsqcup_{v\in V(G_X)} C_v^\nu \longrightarrow X$$

The Jacobian of X:

$$\mathrm{Jac}(X) = \{L \in \mathrm{Pic}(X) : \deg_{C_v} L = 0, \ \forall v \in V(G_X)\}.$$

The Jacobian of X^ν

$$\mathrm{Jac}(X^\nu) = \Pi_{v\in V(G_X)} \mathrm{Jac}(C_v^\nu)$$

is an abelian variety.

— —

We have an exact sequence of algebraic groups

$$0 \longrightarrow (k^*)^b \longrightarrow \mathrm{Jac}(X) \xrightarrow{\nu^*} \mathrm{Jac}(X^\nu) \longrightarrow 0$$

where

$$b = b_1(G_X) = |E(G_X)| - |V(G_X)| + 1.$$

Remark. $\mathrm{Jac}(X)$ is an abelian variety if and only if $b = 0$ if and only if G_X is a tree. Such curves are called of *compact type*.

$\overline{M}_g^{\mathrm{cpt}}$ is the locus in $\overline{M}_g$ of curves of compact type; it is an open subset and

$$M_g \subset \overline{M}_g^{\mathrm{cpt}} \subset \overline{M}_g$$

The extended Torelli morphism

$$\begin{array}{ccc} \overline{M}_g & \xrightarrow{\overline{\tau}} & \overline{A}_g; \\ \uparrow & & \uparrow \\ M_g & \xrightarrow{\tau} & A_g \end{array} \qquad [X] \longmapsto [\overline{\mathbf{Jac}}(X)] = \big[\mathrm{Jac}(X) \curvearrowright (\overline{P_{g-1}}(X), \overline{\Theta(X)})\big]$$

By the previous Remark

$$\overline{\tau}^{-1}(A_g) = \overline{M}_g^{\mathrm{cpt}}$$

and the restriction of $\overline{\tau}$ to $\overline{M}_g^{\mathrm{cpt}}$ is <u>not</u> injective. Indeed:

Proposition 21. *Let X_1 and X_2 be stable curves of compact type. If $X_1^\nu \cong X_2^\nu$ then $\overline{\tau}(X_1) = \overline{\tau}(X_2)$.*

The converse holds if X_1 and X_2 have the same number of irreducible components.

Proof. EXERCISE. ♣

Towards a combinatorial description of $\overline{P_{g-1}}(X)$

Let $G = (V, E)$ be a graph of genus g.

An orientation on G is *totally cyclic* if it has no *directed cut*, i.e. if there exists no non-empty subset $U \subsetneq V$ such that the edges joining U to $V \smallsetminus U$ are all directed towards U.

$\mathcal{O}(G) := \{O : O \text{ is a totally cyclic orientation on } G\}$.

To an orientation O we associate a $\underline{d}^O \in \mathbb{Z}^V$ defined as follows

$$\underline{d}_v^{\mathcal{O}} := g(v) - 1 + \mathrm{indeg}^O(v)$$

where $\mathrm{indeg}^O(v)$ is the number of edges of G having v as target.

Remark. For any orientation O on G

$$|\underline{d}^O| = g - 1.$$

Recall that $\overline{P_{g-1}}(X)$ parametrizes *balanced line bundles of degree* $g-1$.

Two orientations O and O' on G are equivalent, written $O \sim O'$, if $\underline{d}^O = \underline{d}^{O'}$.

$\overline{\mathcal{O}}(G)$:=Equivalence classes of totally cyclic orientations on G.

Example. If G is a cycle, then

$$|\mathcal{O}(\text{cycle})| = 2 \quad \text{and} \quad |\overline{\mathcal{O}}(\text{cycle})| = 1$$

– –

Fact. *$\mathcal{O}(G)$ is not empty if and only if G is free from bridges.*

G_{br} denotes the set of bridges of the graph G.

The poset of *bridgeless subgraphs* of G is

$$\mathcal{BP}(G) := \{S \subset E : (G - S)_{br} = \emptyset\},$$

ordered by reverse inclusion:

$$S \leq S' \quad \text{if} \quad S' \subset S.$$

$\mathcal{BP}(G)$ is a graded poset with respect to the rank function $S \mapsto g(G - S)$.

A combinatorial stratification of $\overline{P_{g-1}}(X)$.

Let X be a stable curve and (G_X, w_X) its dual graph.

The poset of totally cyclic orientations on G_X is defined as follows

$$\overline{\mathcal{OP}}(G_X) := \bigsqcup_{S \in \mathcal{BP}(G_X)} \overline{\mathcal{OP}}(G_X - S)$$

with, for $S, T \in \mathcal{BP}(G_X)$ and $O_S \in \mathcal{O}(G_X - S)$, $O_T \in \mathcal{O}(G_X - T)$

$$[O_S] \leq [O_T] \quad \text{if} \quad S \leq T \quad \text{and} \quad (O_T)_{|G-S} \sim O_S.$$

$\overline{\mathcal{OP}}(G_X)$ is graded with respect to the rank function $[O_S] \mapsto g(G - S)$.

– –

Back to $\overline{P_{g-1}}(X)$.

Proposition 22. *Let X be a stable curve of genus g. Then*

$$\overline{P_{g-1}}(X) = \bigsqcup_{[O_S] \in \overline{\mathcal{OP}}(G_X)} P_{O_S}$$

with P_{O_S} an irreducible variety of dimension $g(G_X - S)$.

Moreover

$$P_{O_S} \subset \overline{P_{O_T}} \quad \Leftrightarrow \quad [O_S] \leq [O_T].$$

Consequence. The number of irreducible components of $\overline{P_{g-1}}(X)$ is equal to the number of equivalence classes of totally cyclic orientations on $G - G_{br}$.

The minimal stratum of $\overline{P_{g-1}}(X)$ corresponds to $S = E(G_X)$ and is canonically isomorphic to $\mathrm{Jac}(X^\nu)$.

The fibers of the extended Torelli morphism

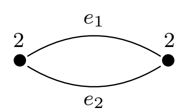
$$\overline{M}_g \xrightarrow{\overline{\tau}} \overline{A}_g; \qquad [X] \longmapsto [\overline{\mathbf{Jac}}(X)] = \big[\mathrm{Jac}(X) \curvearrowright (\overline{P_{g-1}}(X), \overline{\Theta(X)})\big]$$

Surprising Remark. The restriction of $\overline{\tau}$ away from curves of compact type is <u>not</u> injective.

Example. Let C_1 and C_2 be two non-isomorphic smooth curves of genus 2. Pick $p_i, q_i \in C_i$ not mapped to one another by an automorphism of C_i.

$$X_1 := \frac{C_1 \sqcup C_2}{p_1 = p_2, q_1 = q_2} \quad \text{and} \quad X_2 := \frac{C_1 \sqcup C_2}{p_1 = q_2, q_1 = p_2}$$

$$G_{X_1} = G_{X_2} =$$



We have $X_1 \ncong X_2$ but $\overline{\mathbf{Jac}}(X_1) \cong \overline{\mathbf{Jac}}(X_2)$.

Theorem 23 ([CV11])**.** *Let X_1 and X_2 be two stable curves with bridgeless dual graphs.*

(1) Assume that $\overline{\mathbf{Jac}}(X_1) \cong \overline{\mathbf{Jac}}(X_2)$. Then
  - *(a) $(X_1^\nu, \nu^{-1}(\mathrm{sing}(X_1))) \cong (X_2^\nu, \nu^{-1}(\mathrm{sing}(X_2)))$;*
  - *(b) $G_{X_1} \equiv_{cyc} G_{X_2}$.*

(2) Assume G_{X_1} and G_{X_2} are 3-edge connected. Then $\overline{\mathbf{Jac}}(X_1) \cong \overline{\mathbf{Jac}}(X_2)$ if and only if $X_1 \cong X_2$.

Consequence. The restriction of $\overline{\tau}$ to curves with bridgeless dual graph has finite fibers.

The tropical Torelli map

Let $\Gamma = (G, \ell, w)$ be a tropical curve. Its *tropical Jacobian* is the following polarized $\mathbb{R}$-torus:

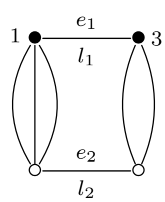
$$\mathbf{Jac}(\Gamma) := \left(\frac{H_1(G,\mathbb{R}) \oplus \mathbb{R}^{g-b_1(G)}}{H_1(G,\mathbb{Z}) \oplus \mathbb{Z}^{g-b_1(G)}} ; (\ ,\ )_\ell \right)$$

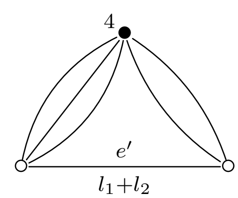
Kotani-Sunada [KS00], Mikhalkin-Zharkov [MZ08], C.V. [CV11], Brannetti-Melo-Viviani [BMV11].

Here is the tropical version of the Torelli theorem.

Theorem 24 ([CV10], [BMV11])**.** *Let* Γ_1 *and* Γ_2 *be tropical curves. Then* $\mathbf{Jac}(\Gamma_1) \cong \mathbf{Jac}(\Gamma_2)$ *if and only if* $\Gamma_1^{(3)} \equiv_{cyc} \Gamma_2^{(3)}$.

Example.

$\Gamma =$ 

$\Gamma^{(3)} =$ 

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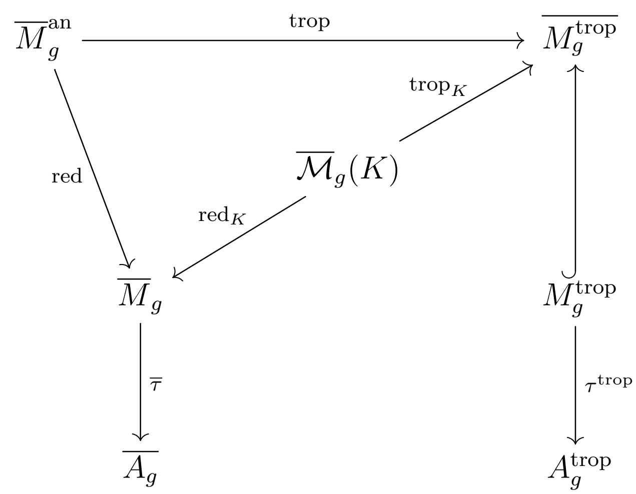
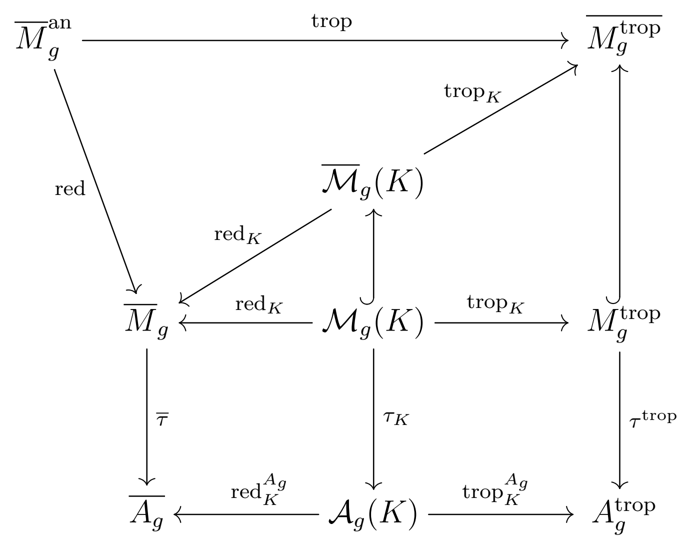
In [BMV11] the authors construct $M_g^{\rm trop}$ and a *moduli space for tropical abelian varieties*, $A_g^{\rm trop}$. They introduce and study the tropical Torelli map

$$\tau^{\rm trop} : M_g^{\rm trop} \longrightarrow A_g^{\rm trop}; \qquad \Gamma \mapsto \mathbf{Jac}(\Gamma).$$

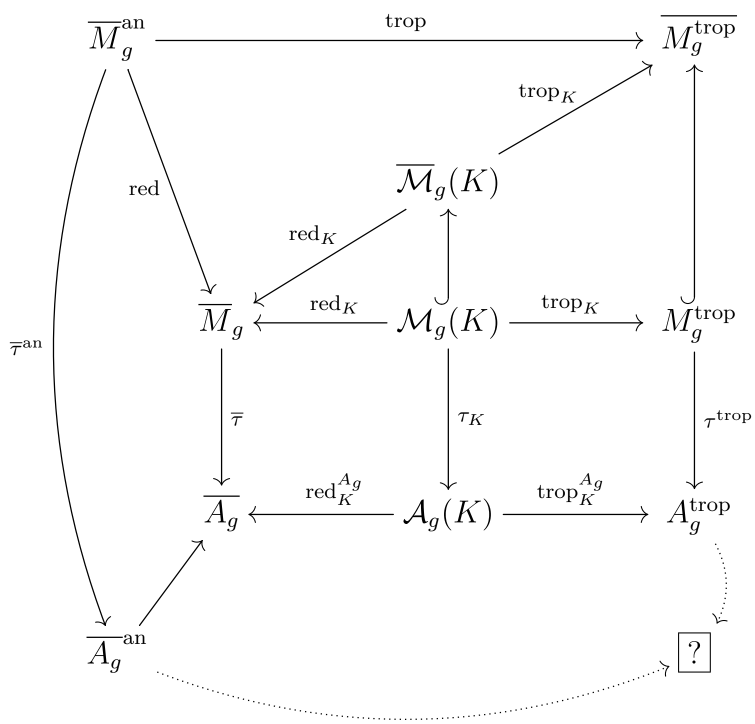
Summarizing commutative diagram 1.



Summarizing commutative diagram 2.



Partly conjectural, summarizing commutative diagram.



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