Fabrizio Caselli

A classification of special matchings in lower Bruhat intervals

Dobbiaco, 21/02/2017

Symmetric groups

S_n: group of permutations of $\{1, 2, \dots, n\}$

S_n is generated by

$$s_1 = (1,2) \quad s_2 = (2,3) \quad \dots \quad s_{n-1} = (n-1, n)$$

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The s_i's are called simple reflections or Coxeter generators of S_n

$S = \{s_1, \ldots, s_n\}$

(S_n, S) is a Coxeter system

Coxeter groups and systems

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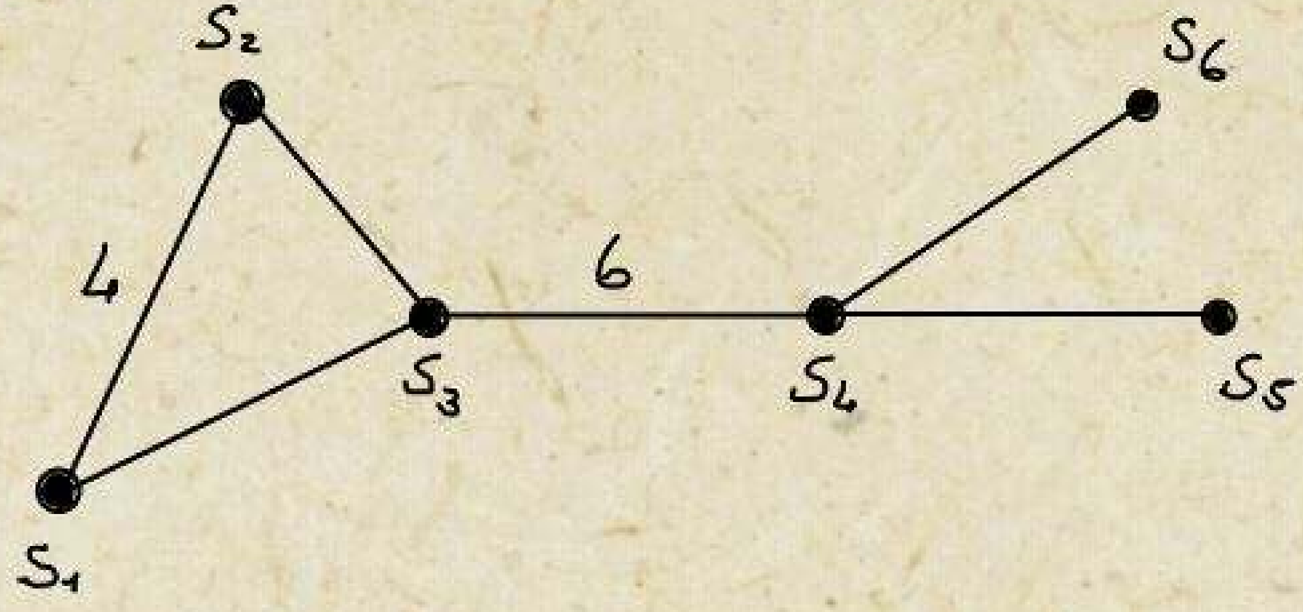
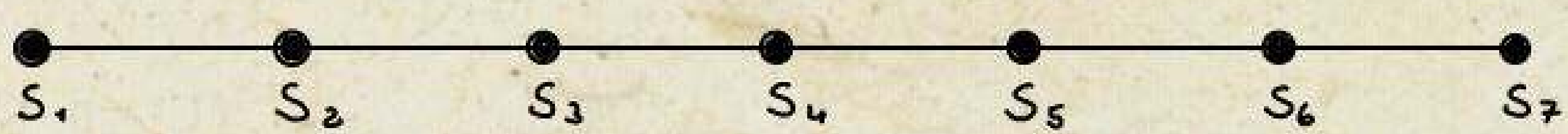
W a group, S a finite set of generators with relations:

- $s^2 = 1$
- $(st)^{m_{s,t}} = 1$ some $s,t \in S$ $m_{s,t} > 2$

Coxeter graph



Coxeter graph of S_8



is its Dynkin diagram

Bruhat order

$w \in W$

$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$

if ℓ is minimal $\ell = \ell(w)$

$u \leq w$

means u is a subword of a reduced expression of v

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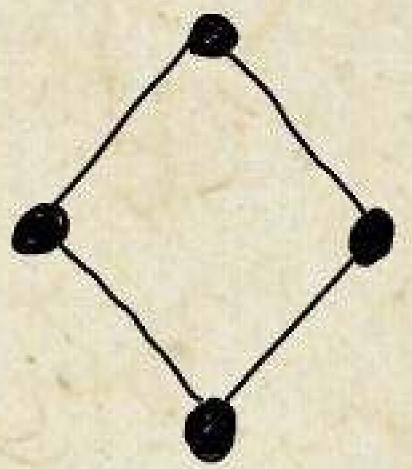
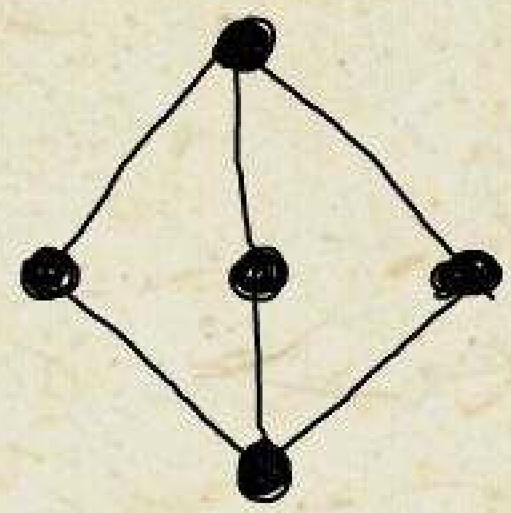
- W is graded by ℓ
- w & ws differ in length by 1 and are always comparable.

Bruhat intervals are Eulerian and in particular Bruhat intervals of rank 2 are necessarily



(we never have



Right and left descents

if $ws < w$ we say s is a right descent of w and let

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similarly define

$D_L(w)$

A geometric interpretation

If $W \cong S_n$

Bruhat decomposition

$$SL_n = \dot{\bigsqcup_{w}} BwB = = \dot{\bigsqcup_{w}} B^{-} w B$$

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$v \leq w \iff \overline{\Omega_v} \subseteq \overline{\Omega_w}$

$\iff \overline{\Omega^\circ_v} \supseteq \overline{\Omega^\circ_w}$

An algebraic interpretation

Hecke algebras

$$\mathcal{H} = \bigoplus_{w \in W} \mathbb{Z}[q, q^{-1}]\, T_w$$

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$T_u \cdot T_v = T_{uv}$ if $\ell(u) + \ell(v) = \ell(uv)$

$T_s^2 = (q-1)T_s + qT_e$ if $s \in S$

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$$T_s^2 = (q-1)T_s + qT_e \quad \text{if } s \in S$$

$$T_{w^{-1}}^{-1} = \bigoplus_{u \leq w} R_{u,w}(q)\, T_u$$

Kazhdan-Lusztig polynomials

There is a unique family of polynomials

$\{ R_{u,v} \in \mathbb{Z}[q] \ : \ u,v \in W \}$ s.t.

$R_{u,u} = 1$

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and for all s : $vs < v$

$$\boxed{R_{u,v} = \left(1 - q^{c_s(u)}\right) R_{u,vs} + q^{1-c_s(u)} R_{us,vs}}$$

where $c_s(u) = \begin{cases} 1 & \text{if } us < u \\ 0 & \text{if } us > u \end{cases}$

A geometric interpretion of R-polys

Schubert varieties over $\mathbb{F}_q$

$R_{u,v}(q) = |\overline{\Omega^{\circ}_{u}} \cap \overline{\Omega_{v}}|$

Number of points in the
Richardson variety

Kazhdan - Lusztig polynomials

uniquely determined by
R-polynomials.

$$q^{\ell(v)-\ell(u)} P_{u,v}(q^{-1}) = \sum_{a\in[u,v]} R_{u,a}(q)\, P_{a,v}(q)$$

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Many applications
IC of Schubert varieties
Decomposition of Verma modules
Soergel bimodules
Representations of Hecke algebras

Some literature

Kazhdan-Lusztig, Invent. Math. 1979

Kazhdan-Lusztig, Proc. Symp. P.M 1980

Beilinson-Bernstein, C.R. Math. Acad. Sci. Paris 1981

Brylinski-Kashiwara, Inv. Math. 1981

Haglund-Haiman-Loehr, JAMS 2005

Elias-Williamson, Ann. of Math. 2014

A combinatorial interpretation
of KL-polynomials

BRENTI-C. Peak algebras, paths in the Bruhat graph and KL-polynomials

Adv. in Mathematics 2017

Combinatorial invariance

$w \in W \qquad w' \in W'$

$$[e, w] \xrightarrow[\Phi]{\cong} [e', w']$$

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$$R_{u,v} = R_{\phi(u),\phi(v)} \quad \text{for all } u, v \le w$$

and hence

$$P_{u,v} = P_{\phi(u),\phi(v)} \quad \text{for all } u, v \le w$$

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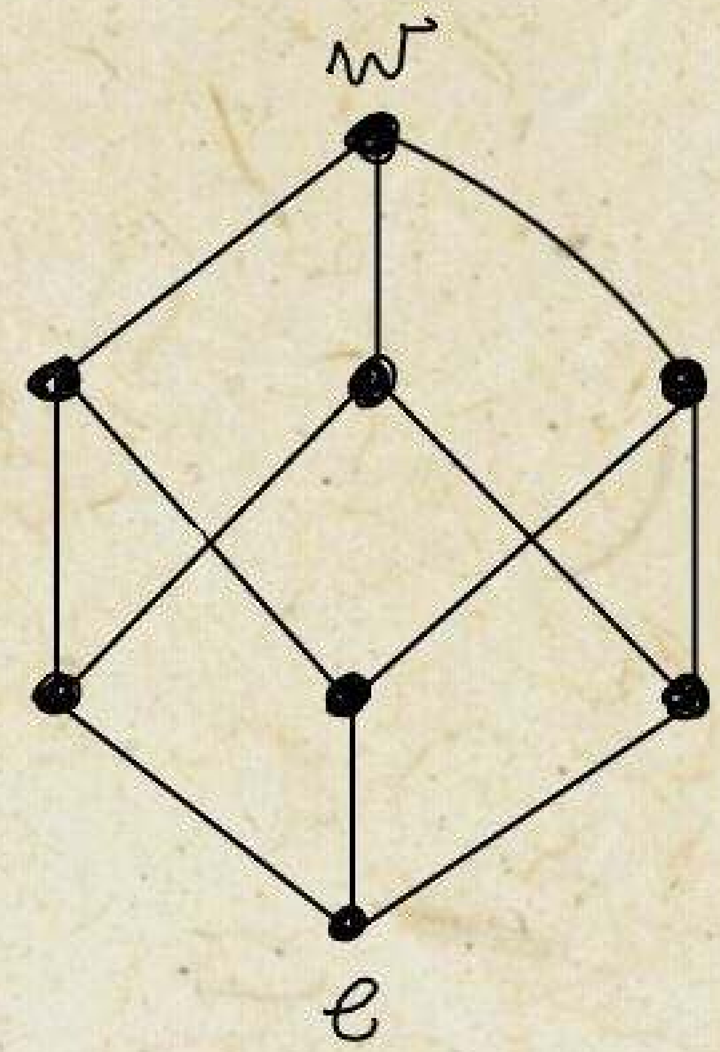
BRENTI - C - MARIETTI
(Adv. in Math. 2006)

Matching of w



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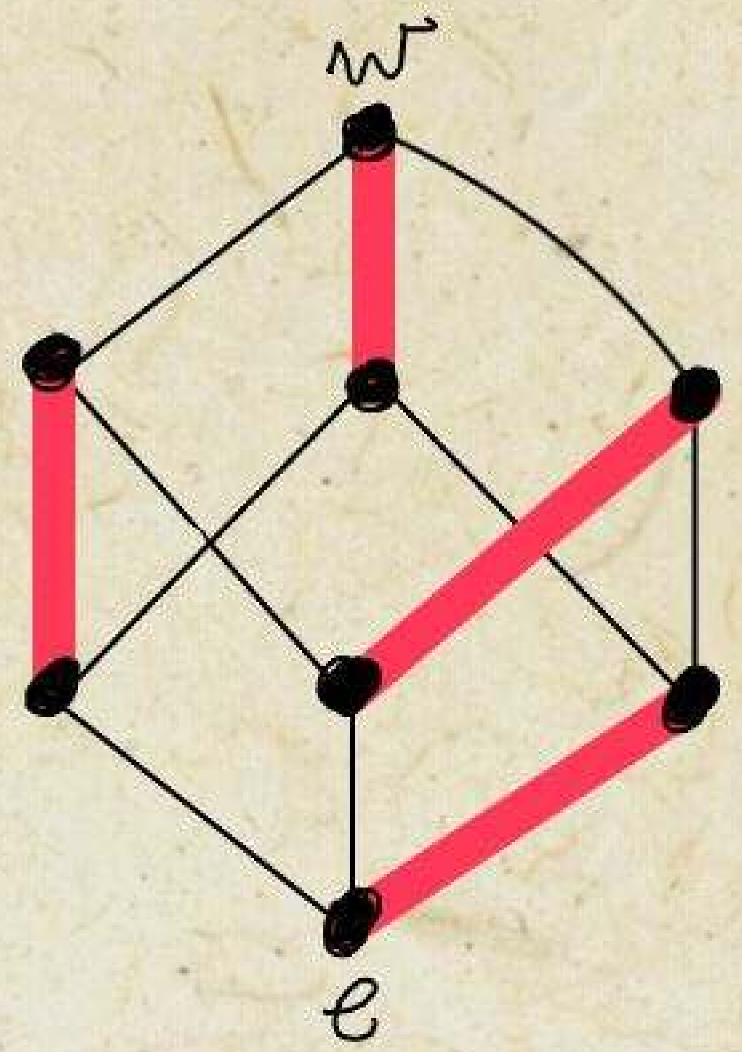


M is a map

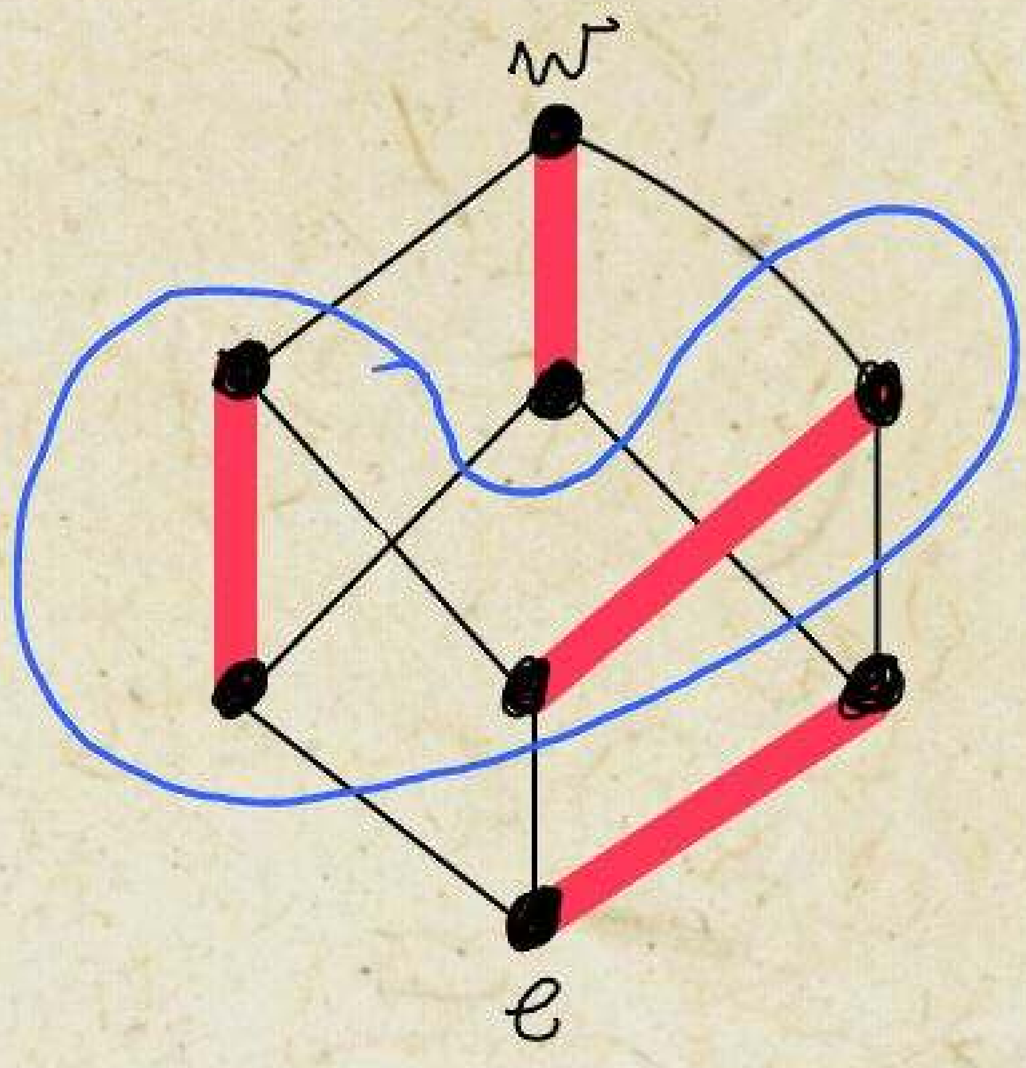
$M : [e, w] \longrightarrow [e, w]$:

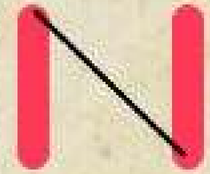
$M(u) \lhd u$ or $u \lhd M(u)$

$\forall u$

& $M^2 = id$

Matching of w



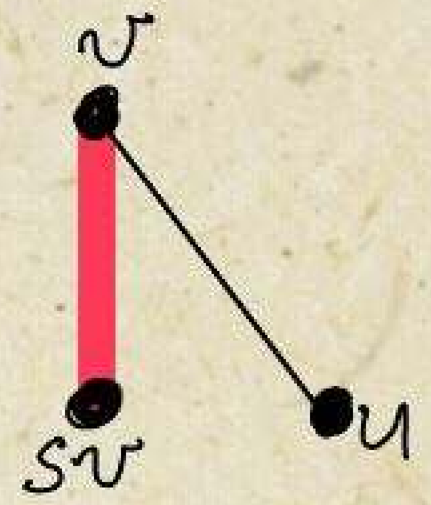


(something we would like to avoid)

Multiplication matchings

$s \in D_L(w)$ $\quad \lambda_s$ is a matching of w

Moreover, if $sv < v$



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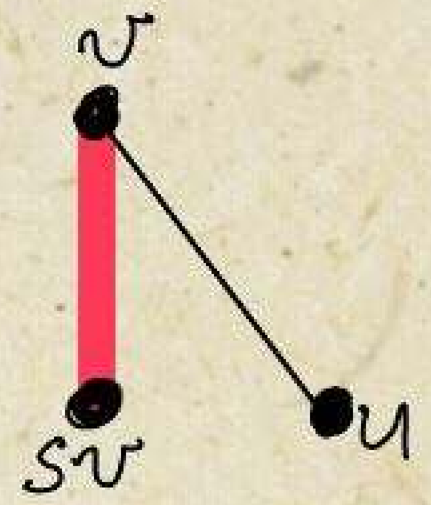

$v = s s_2 \cdots s_\ell$

$u = s s_2 \cdots \hat{s}_i \cdots s_\ell$

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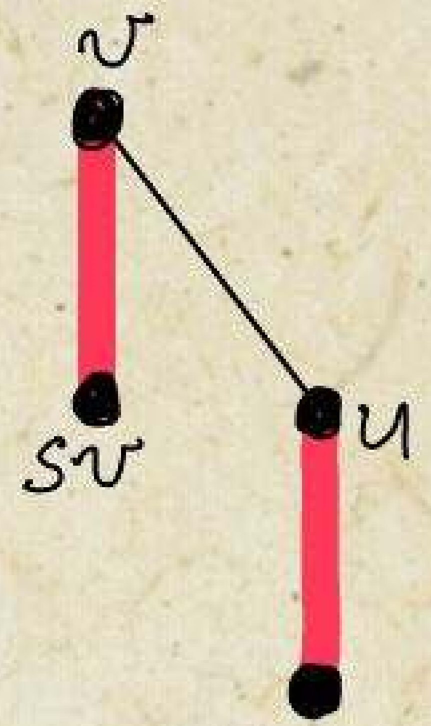

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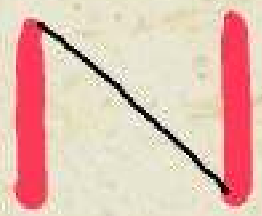
and so λ_s avoids the 

So multiplication matchings avoid N-configuration

Def.: A matching is special if it avoids the N-configuration.

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Remark

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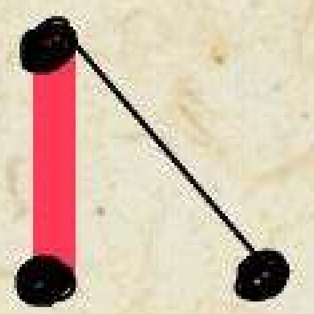
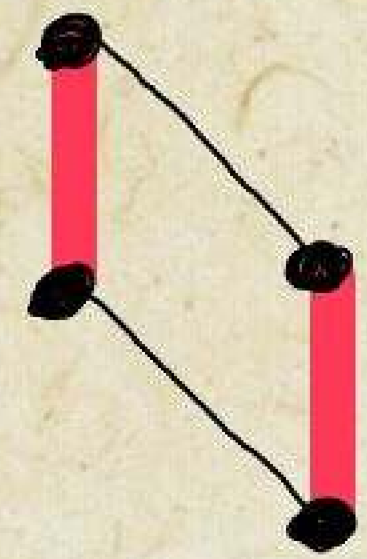


then



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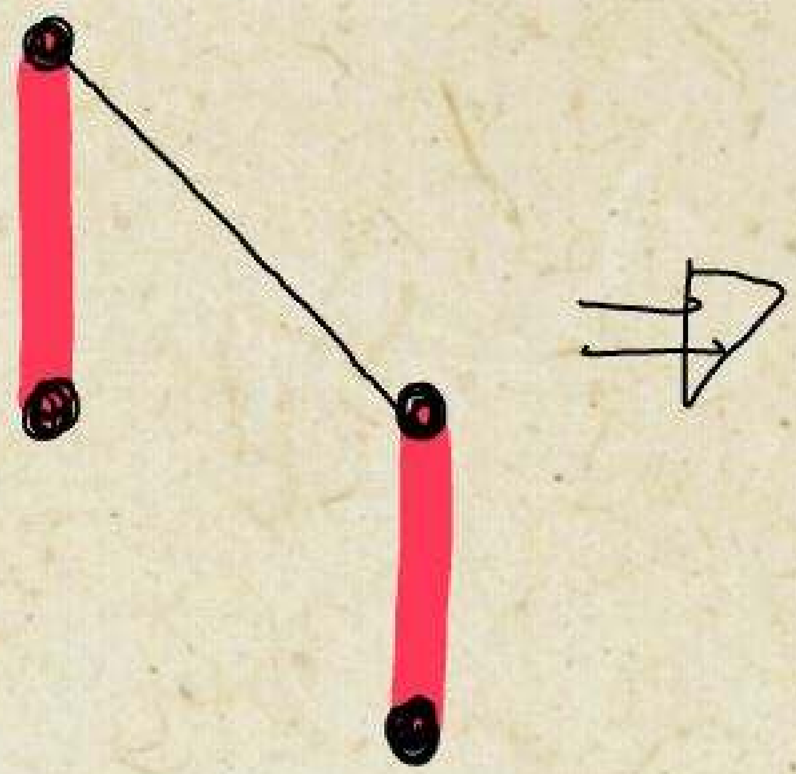

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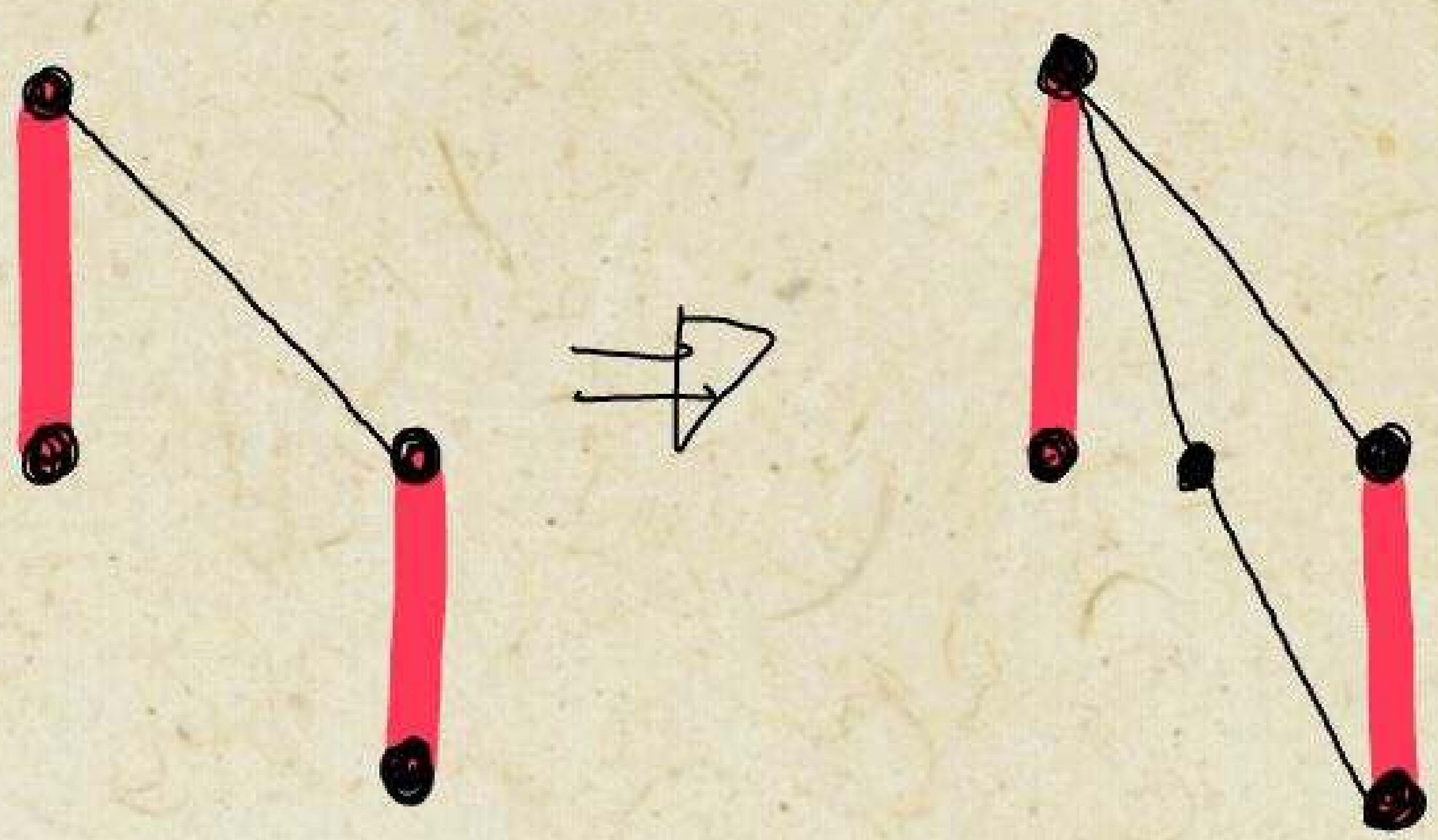

and in any case we have a

$s \in D_L(w) \Rightarrow \lambda_s$ is a special matching of w

$t \in D_R(w) \Rightarrow \rho_t$ is a special matching of w

and, for $M = \lambda_s, \rho_t$

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and, for $M = \lambda_s, \rho_t$

$$\boxed{\begin{gathered} R_{u,w} = \left(q^{c(M,u)} - 1\right) R_{u,M(w)} + q^{c(M,u)} R_{M(u),M(w)} \\ \text{where} \quad c(M,u) = \begin{cases} 0 & M(u) \lhd u \\ 1 & M(u) \rhd u \end{cases} \end{gathered}}$$

We will show

If M is a special matching of w then

$$\boxed{R_{u,w} = \left(q^{c(M,u)} - 1\right) R_{u,M(w)} + q^{c(M,u)} R_{M(u),M(w)}}$$

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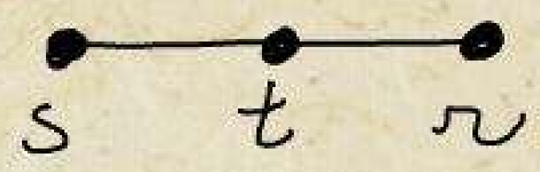
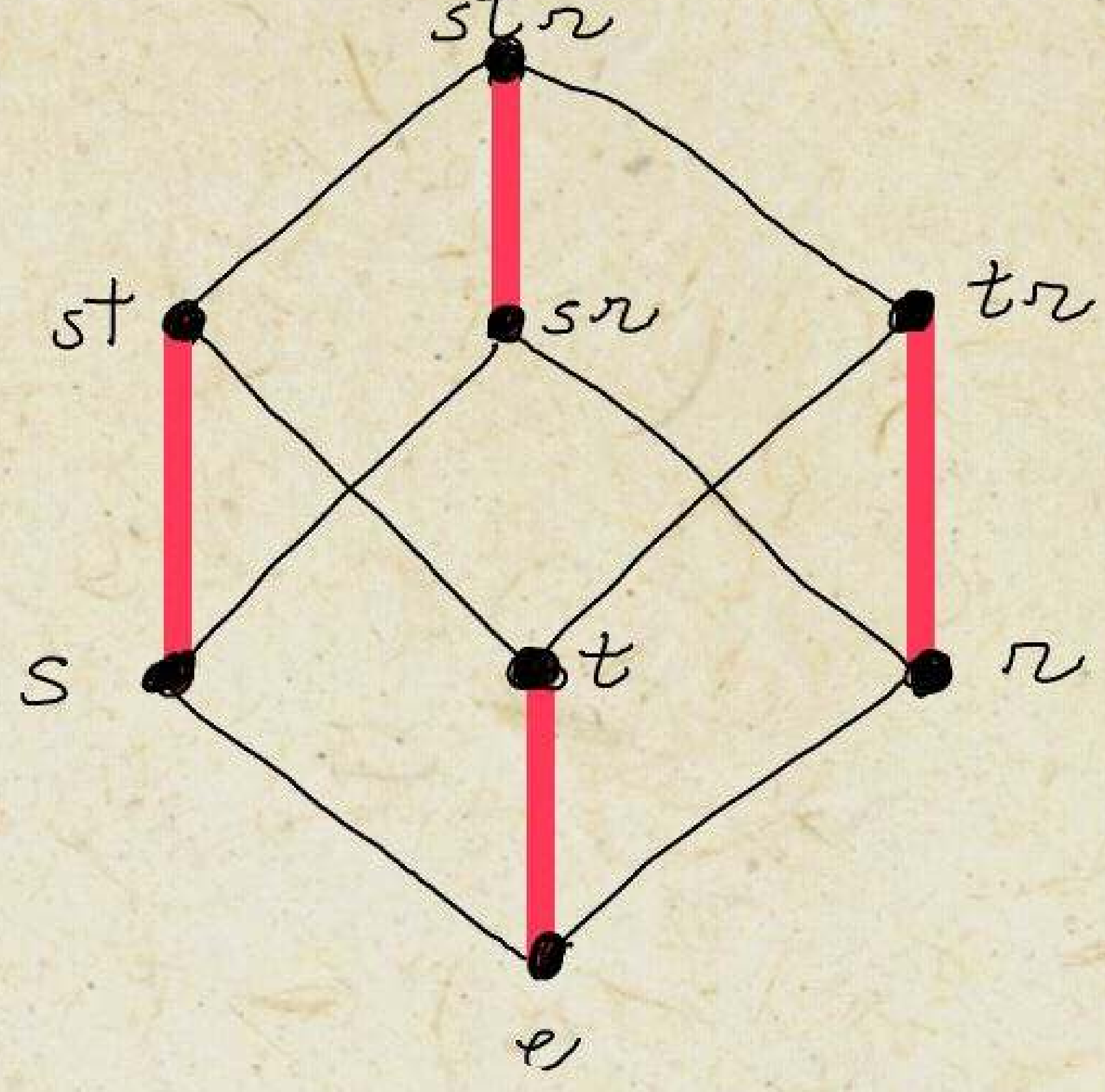
An example of special matching

$w = str$





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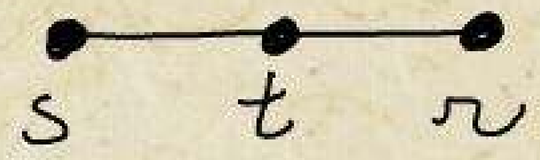
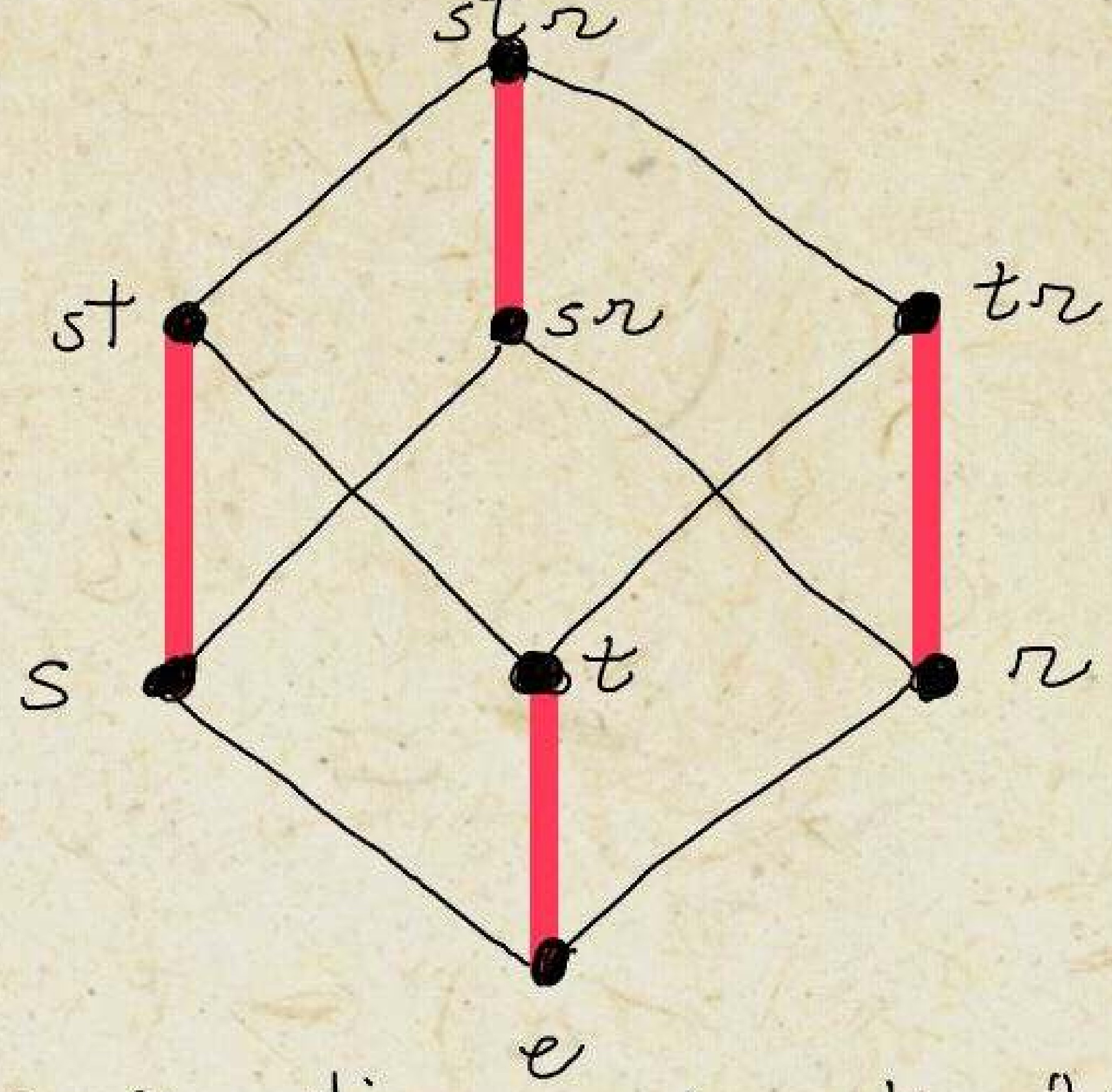
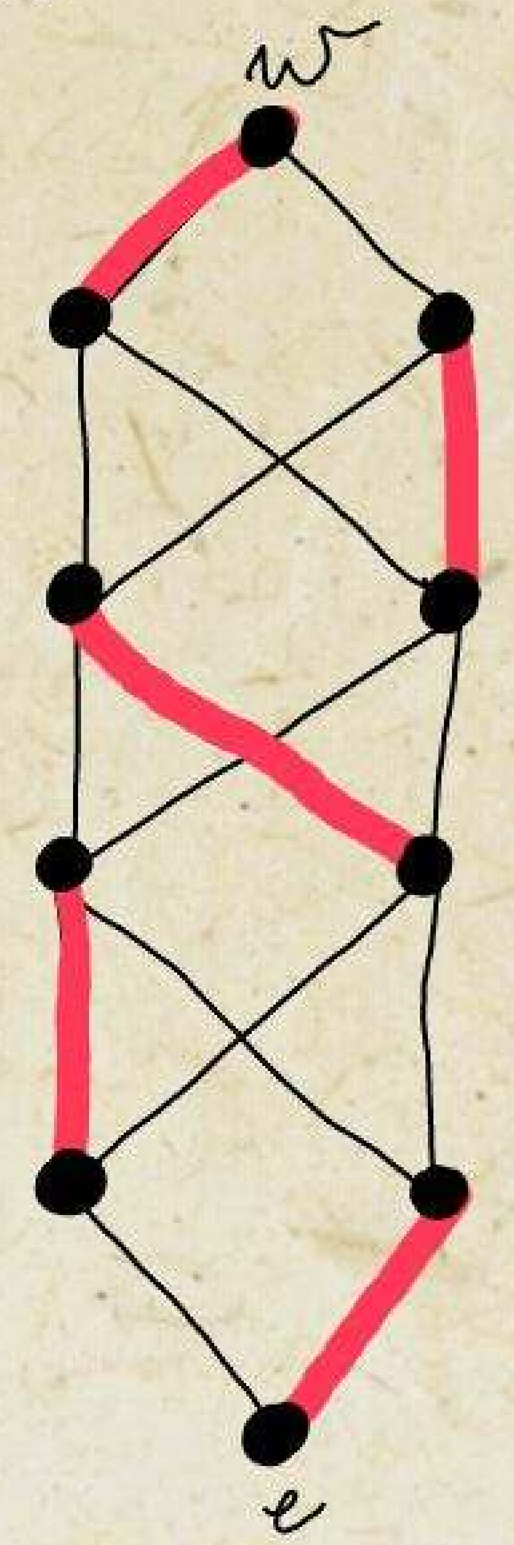




Multiplication by t "in the middle"

Another example

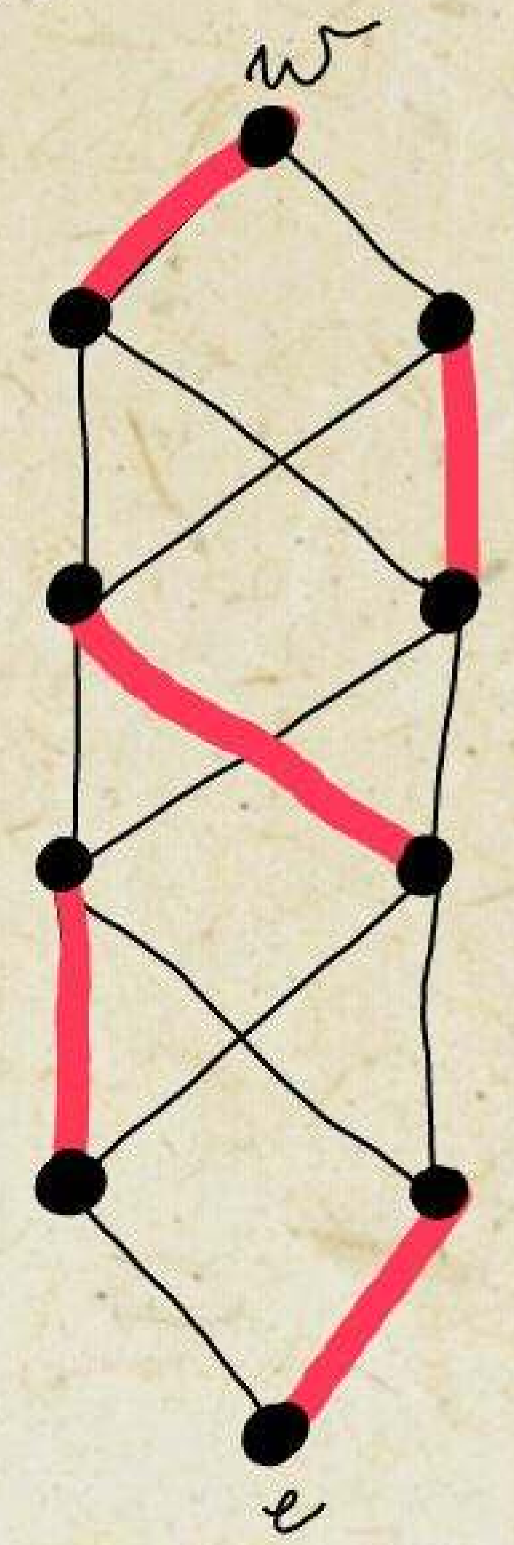


Every matching is special

Much freedom

...

Another example



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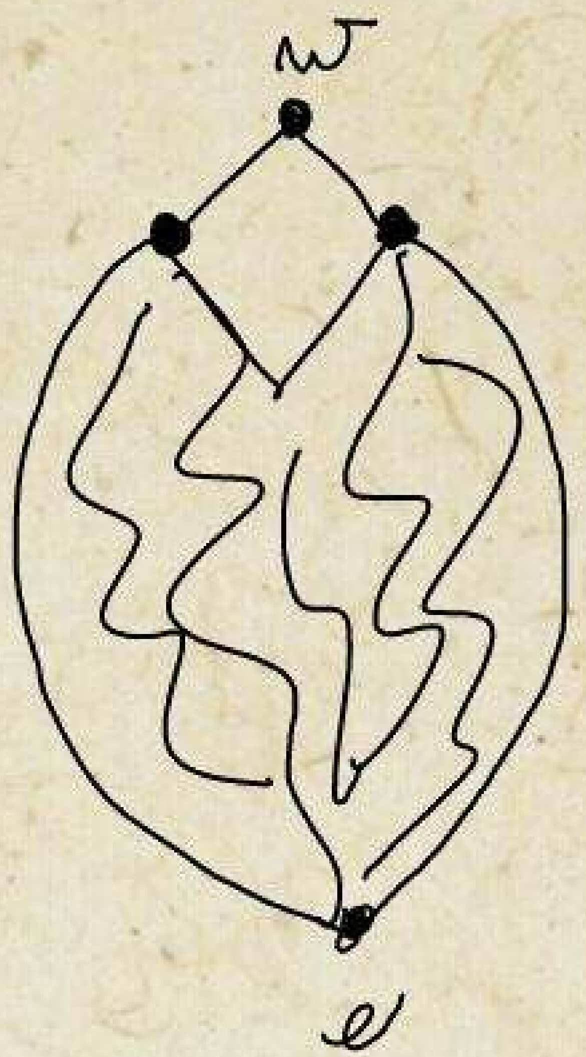
we will see not so much.

Need properties of Bruhat intervals

1 If



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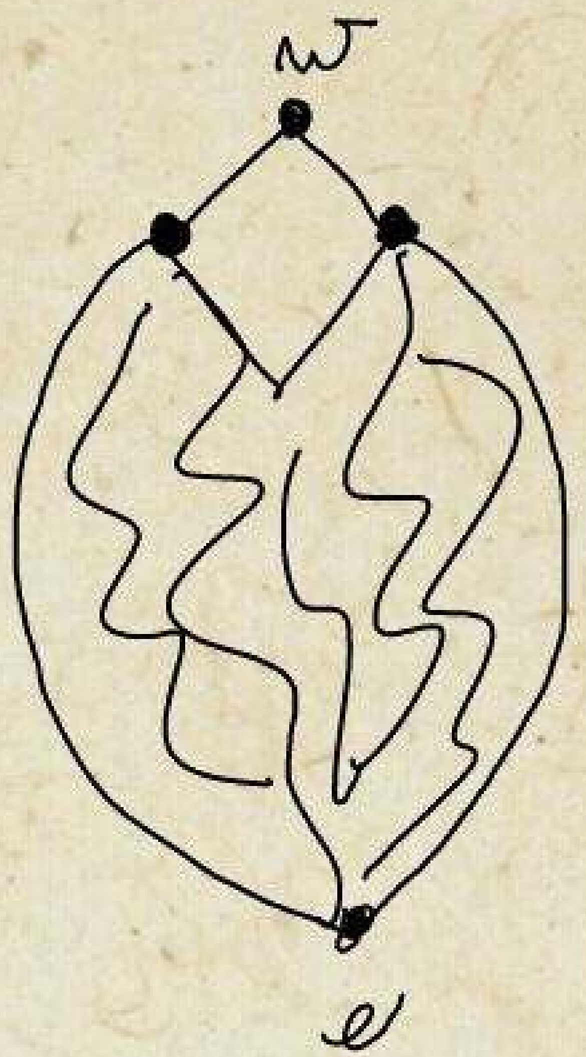

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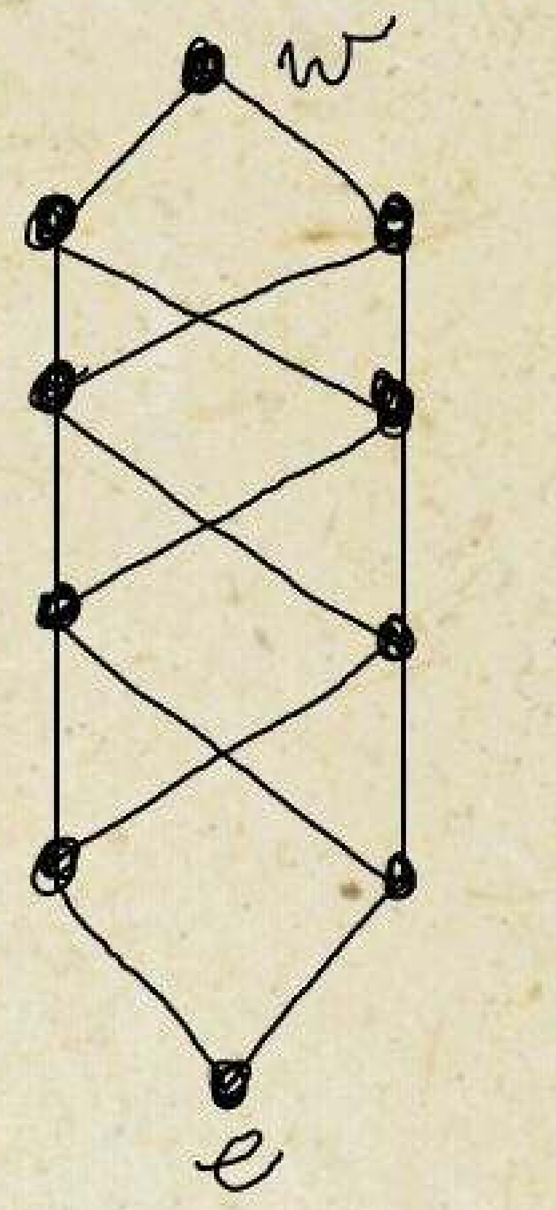

(classical)

$K_{3,2}$ - avoidance

A Bruhat interval never shows

$K_{3,2}$

(new)

Parabolic decompositions

$J \subset S$

$W_J = \langle J \rangle$

$W^J = \{ w \in W : D_R(w) \cap J = \emptyset \}$

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Theorem: $w \in W$. There are unique

$w^J \in W^J \quad {}^J w \in {}^J W \quad w_J \in W_J \quad {}_J w \in W_J$:

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For these decompositions

$$\ell(w) = \ell(w^J) + \ell(w_J) = \ell({}_J w) + \ell({}^J w)$$

Notation

$s \in S$ $\quad C_s = \{ r \in S : rs = sr \}$

If $J \supset C_s$ the s-complement of J is

$$K = (S \setminus J) \cup C_s$$

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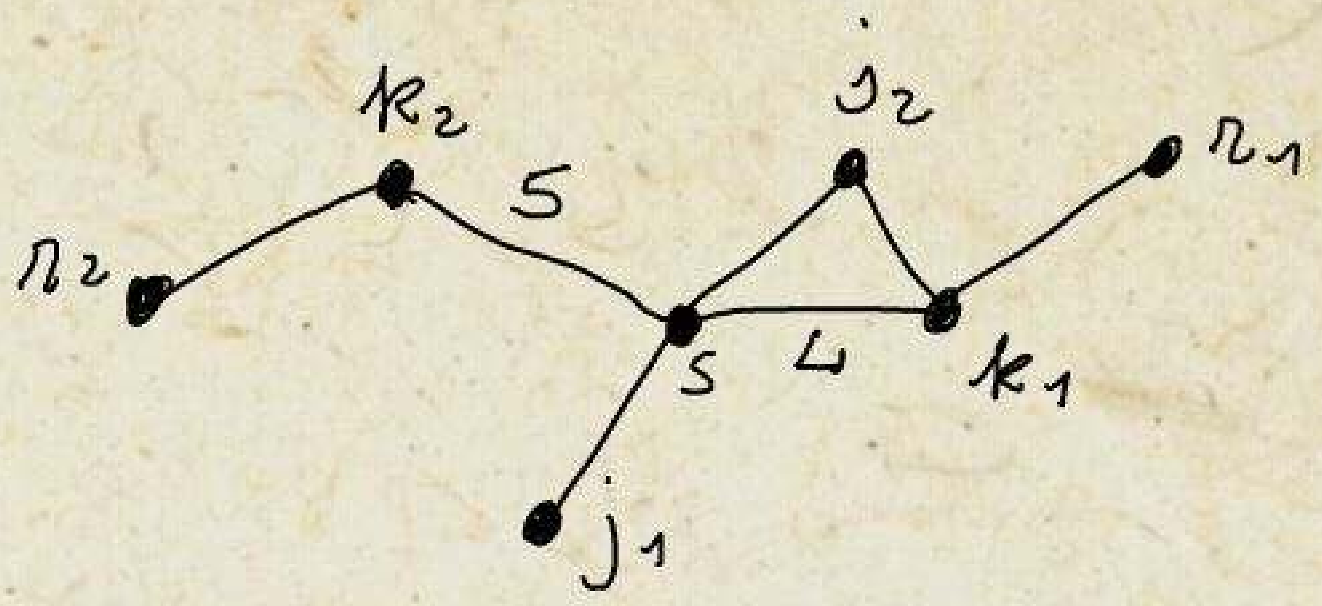

$C_s = \{ s, r_1, r_2 \}$

$J = \{ j_1, j_2 \} \cup C_s$

$K = \{ k_1, k_2 \} \cup C_s$

K is the s-complement of J

Special systems

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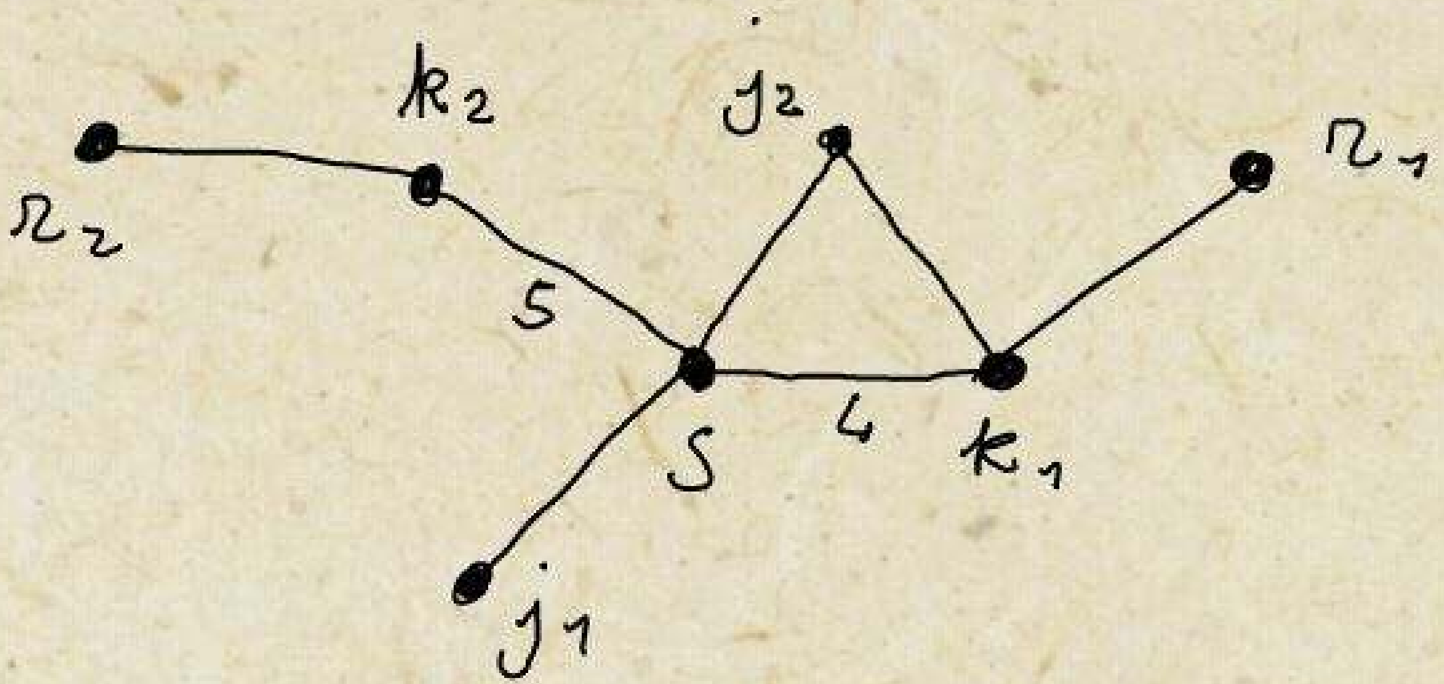
① $w^K \in W_J$

② M is multiplication matching only if $|H| = 1$

③ for $\alpha \in H \qquad \alpha \leq (w^K)^H$ then $M \lambda_\alpha = \lambda_\alpha M$

$\alpha \leq {}^H({}^J w)$ then $M \rho_\alpha = \rho_\alpha M$

Example

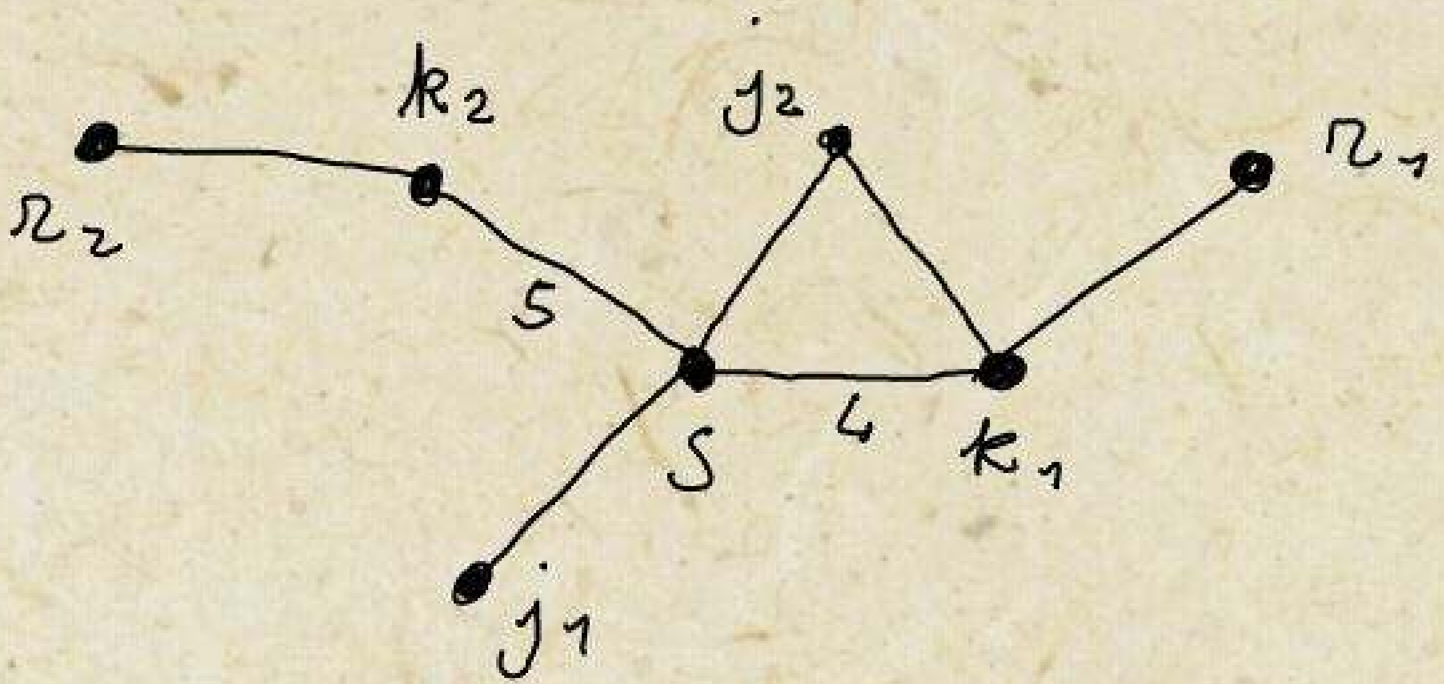


$$w = \underbrace{r_1\, r_2\, k_2\, k_1\, s\, k_1}_{w^J}\ \underbrace{s\ j_2\, j_1\, r_2}_{w_J}$$

$J = \{s, j_1, j_2, r_1, r_2\}$

$H = \{s, k_1\}$

Example



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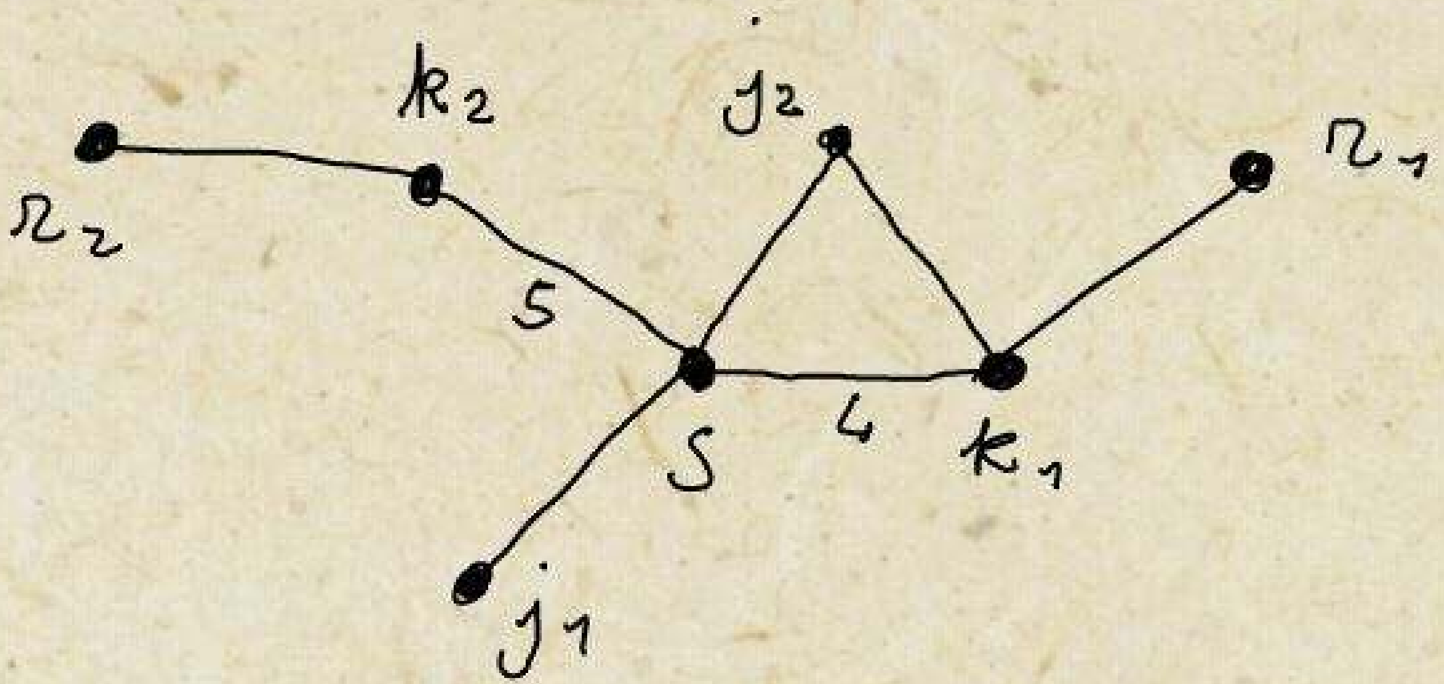
- $w^J \in W_K$
- $(w^J)^H = r_1 r_2 k_2$
- ${}^H({}^K w) = j_2 j_1 r_2$

Example



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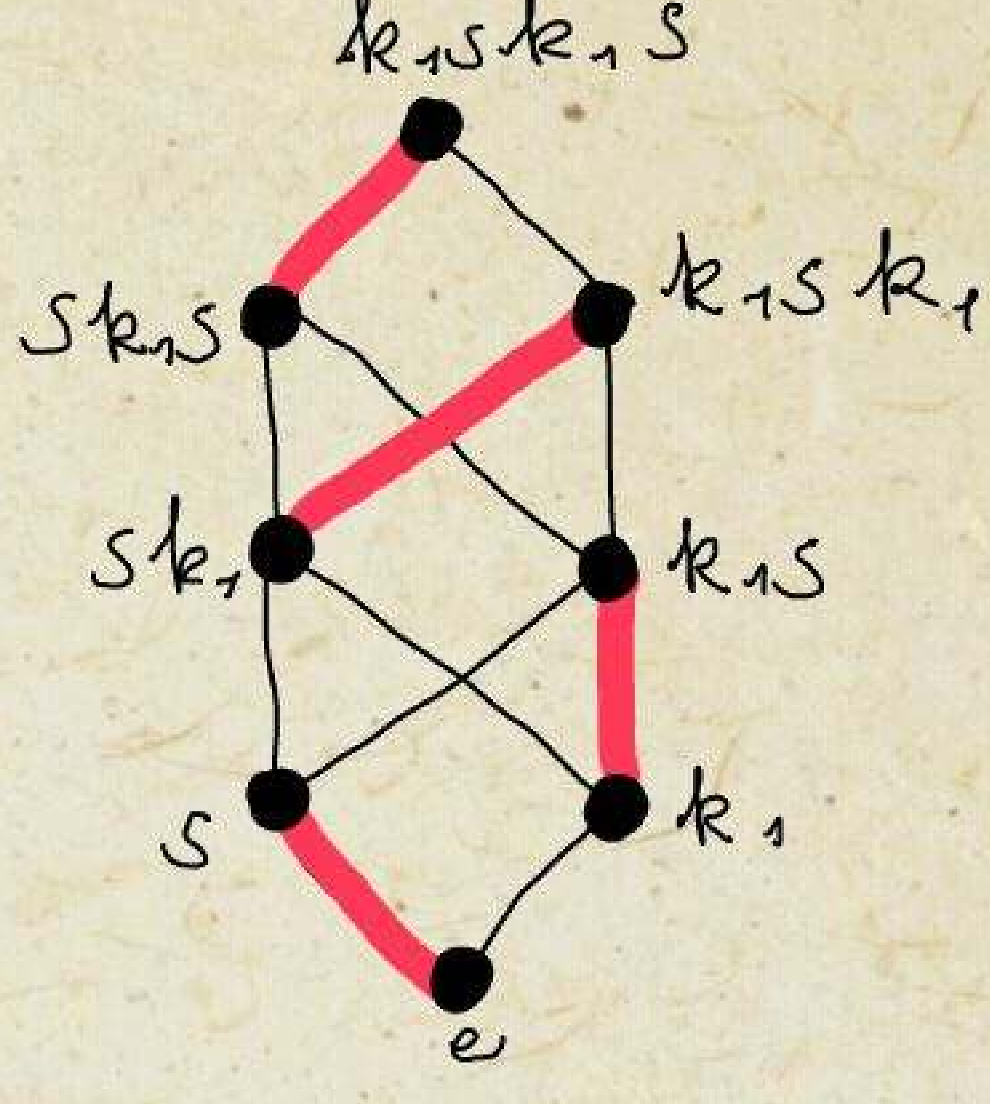
$M =$



$\mathfrak{S}$ special system for w, $\mathfrak{S} = (J, H, M)$

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- $\ell(u) = \ell(a) + \ell(b) + \ell(c)$
- $a \in W_K \cap W^H$
- $b \in W_H$
- $c \in W_J \cap {}^H W$

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- $a \in W_K \cap W^H$
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Every $u \leq w$ has a $\mathcal{S}$-factorization &

$$\boxed{M_{\mathcal{S}}(u) := a\, M(b)\, c}$$

does not depend on the $\mathcal{S}$-factorization.

Classification theorem

$M_{\underset{\sim}{S}}$ is a special matching of w

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&

$\{M_{\underset{\sim}{S}} : \underset{\sim}{S}$ a special system for $w\}$
is a complete list of all special matchings of $[e, w]$

Classification theorem

$M_{\underset{\sim}{S}}$ is a special matching of w

&

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is a complete list of all special matchings of $[e, w]$

C - MARIETTI

European J. Combin, 2017

Now we know special matchings
explicitly.

Why do they satisfy the recursion
for R-polynomials ?

Can assume $[e,w]$ not dihedral

If M is a special matching then $\exists$ a multiplication matching N

$M(w) \neq N(w)$

$MN(u) = NM(u) \quad \forall u \leq w$

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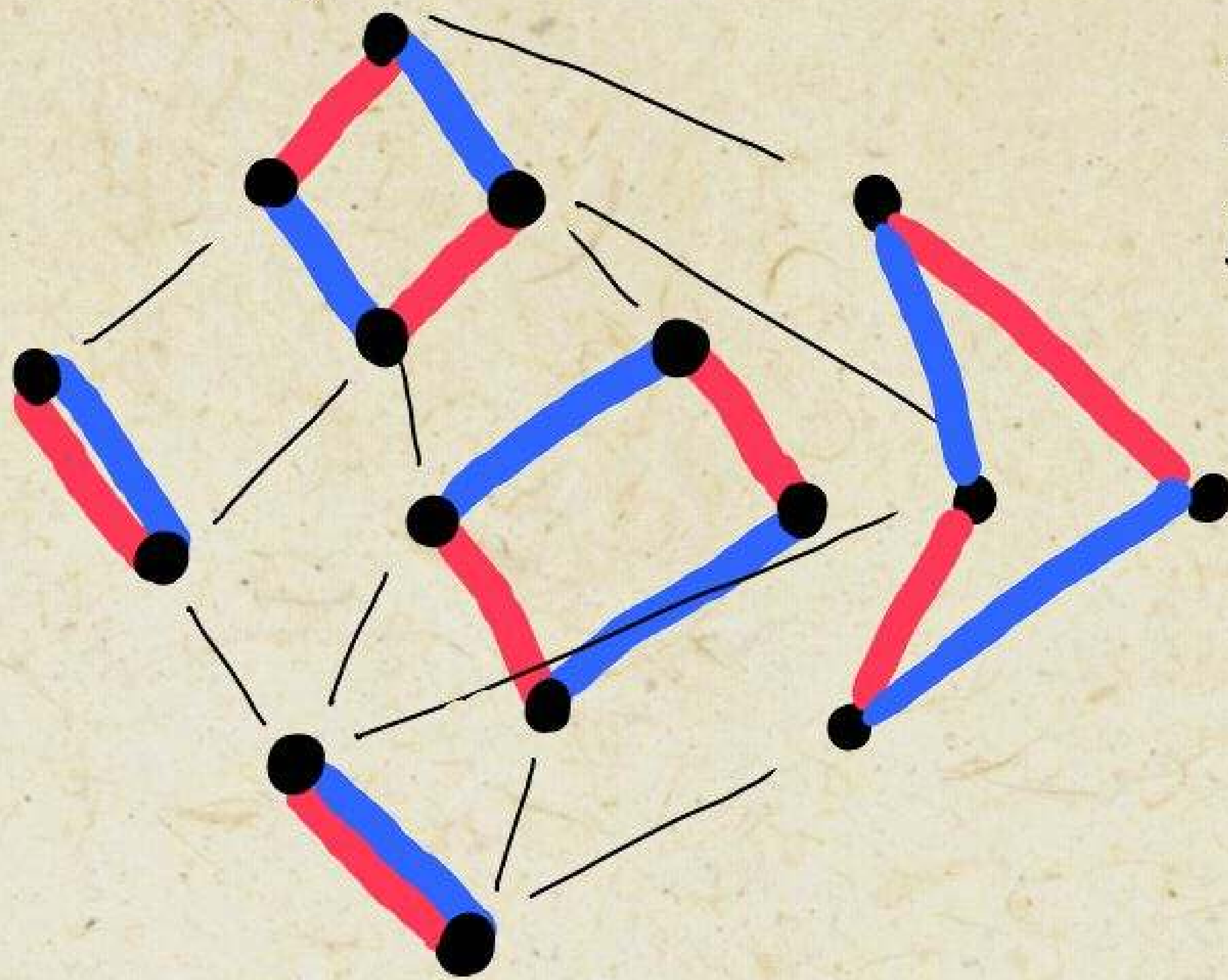

Sketch of proof

$w = a\ b\ c \qquad a \neq e$

$r \in D_L(a) \qquad N = \partial r$

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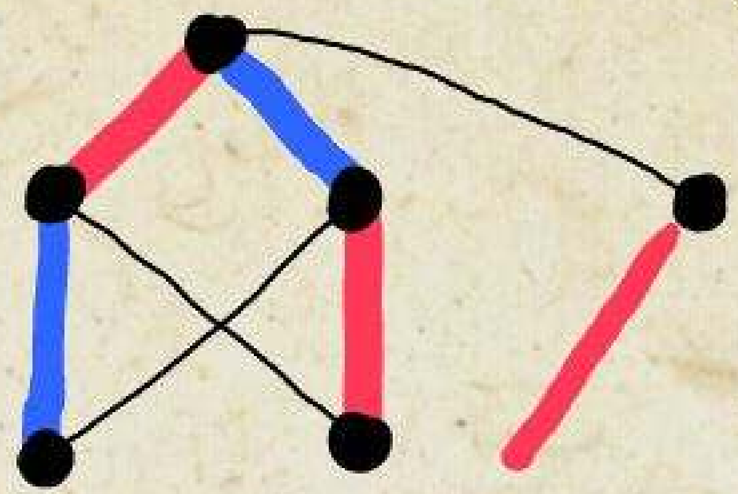

Sketch of proof

$w = a\,b\,c \qquad a \neq 1$

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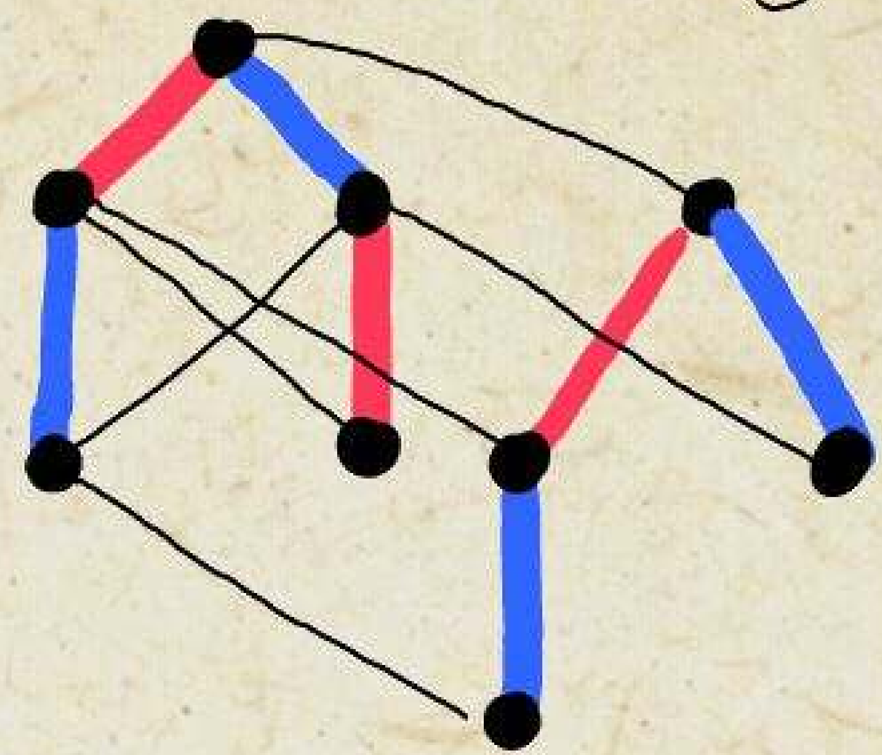

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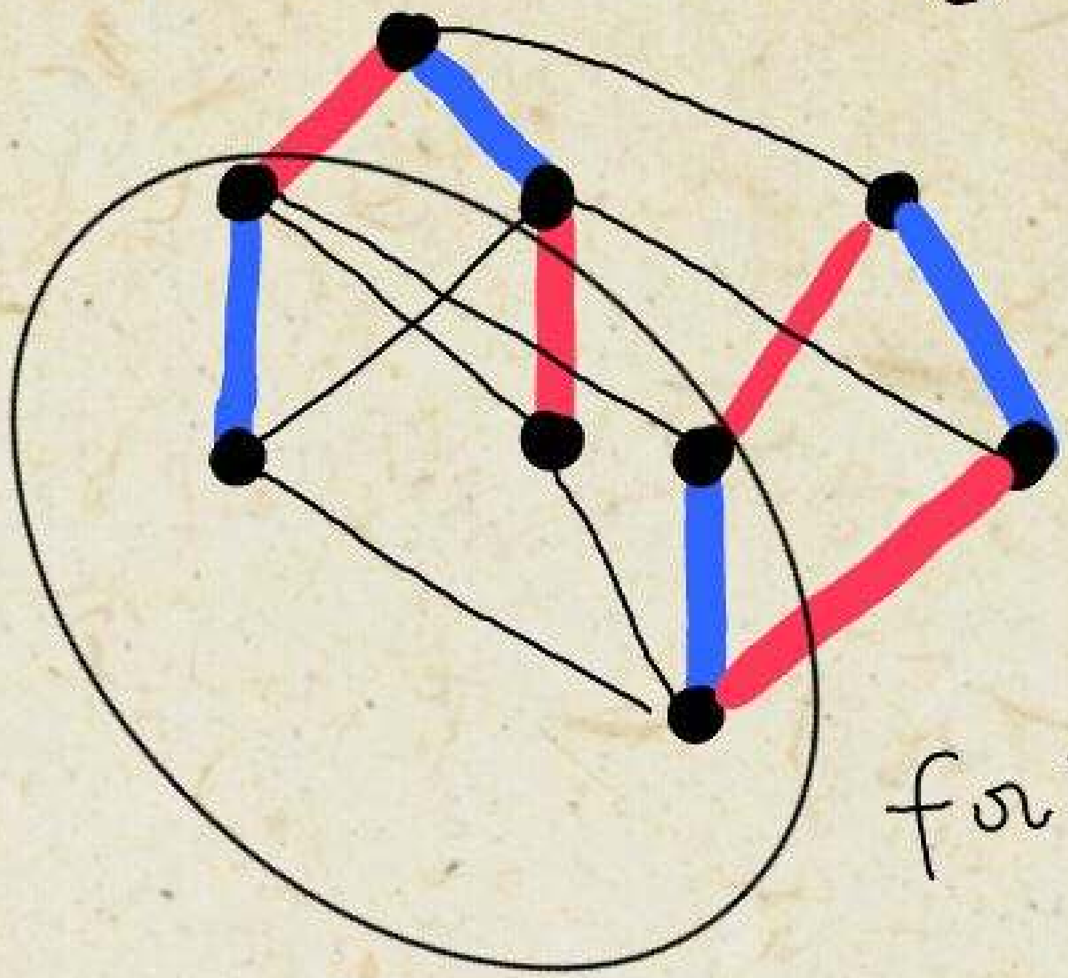

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Completion of the proof

We want to show

$$R_{u,w} = (q^{c(M,u)} - 1) R_{u,M(w)} + q^{c(M,u)} R_{M(u),M(w)}$$

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$$\mathcal{M} = \bigoplus_{x \in \langle M,N \rangle u} \mathbb{Z}[q,q^{-1}] \cdot x$$

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Consider the Hecke algebra $\mathcal{H}$ of $A_1 \times A_1$ generated by T_1, T_2

$\mathcal{H}$ acts on $\mathcal{M}$ by

$$T_1(x) = (q^{c(M,u)} - 1) x + q^{c(M,u)} M(x)$$

$$T_2(x) = (q^{c(N,u)-1}) x + q^{c(N,u)} N(x)$$

$\forall z \in W$

$v_z : \mathcal{M} \longrightarrow \mathbb{Z}[q]$

$v_z(x) = R_{x,z}$ extended by linearity.

Need to prove

$v_w(x) = v_{M(w)}(T_1(x))$

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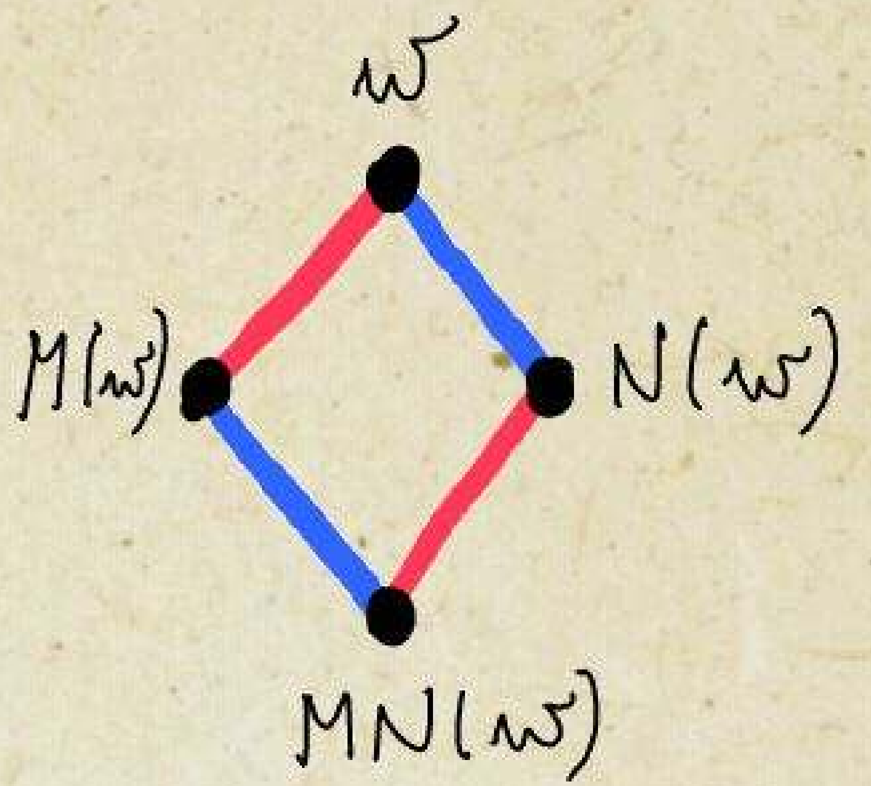
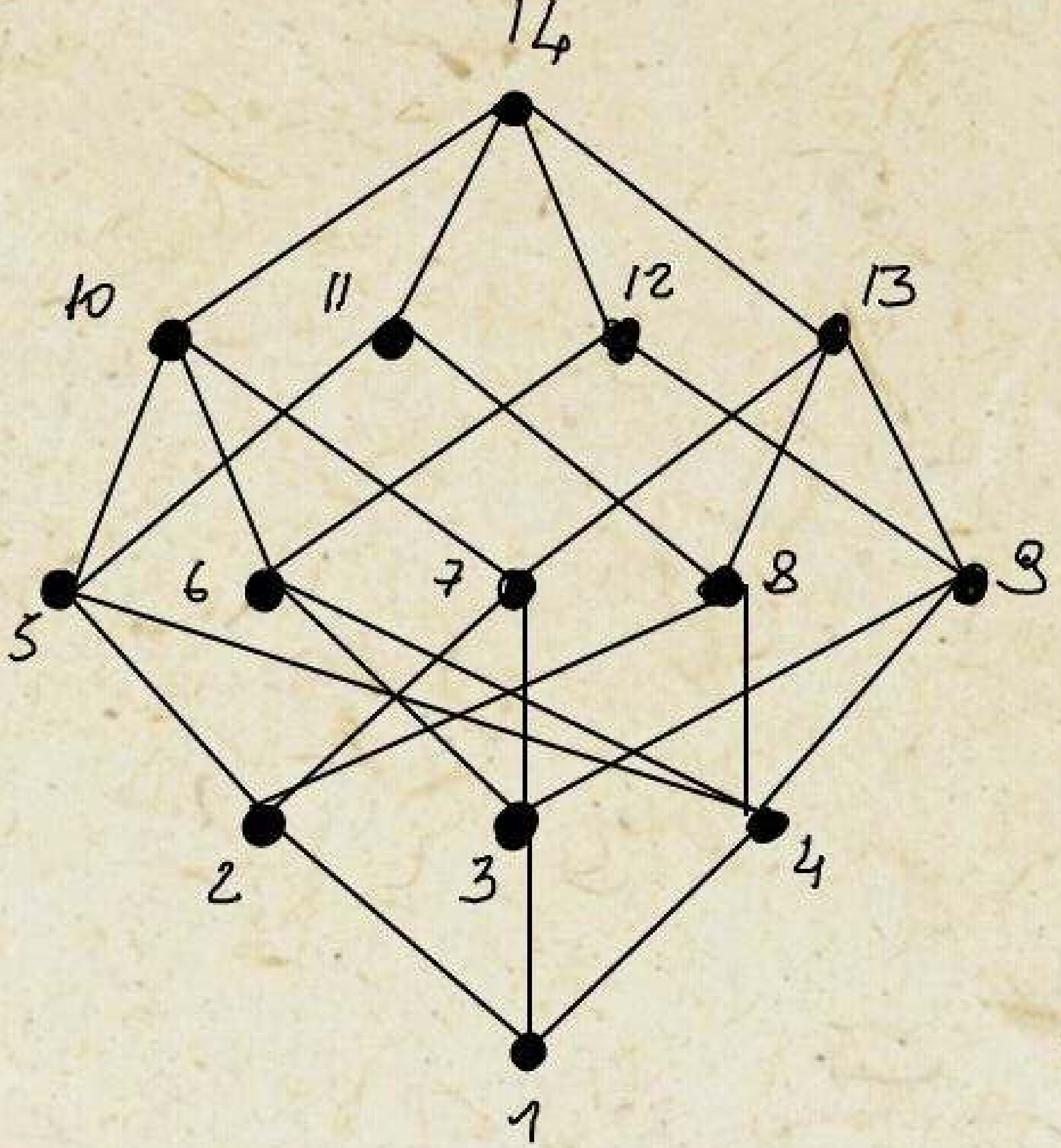


By induction on $\ell(w)$

$$\begin{aligned} v_w(x) &= v_{N(w)}(T_2(x)) \\ &= v_{MN(w)}(T_1 T_2(x)) \\ &= v_{NM(w)}(T_2 T_1(x)) \\ &= v_{M(w)}(T_1(x)) \end{aligned}$$

$\square$

An example



$R_{1,14} =$

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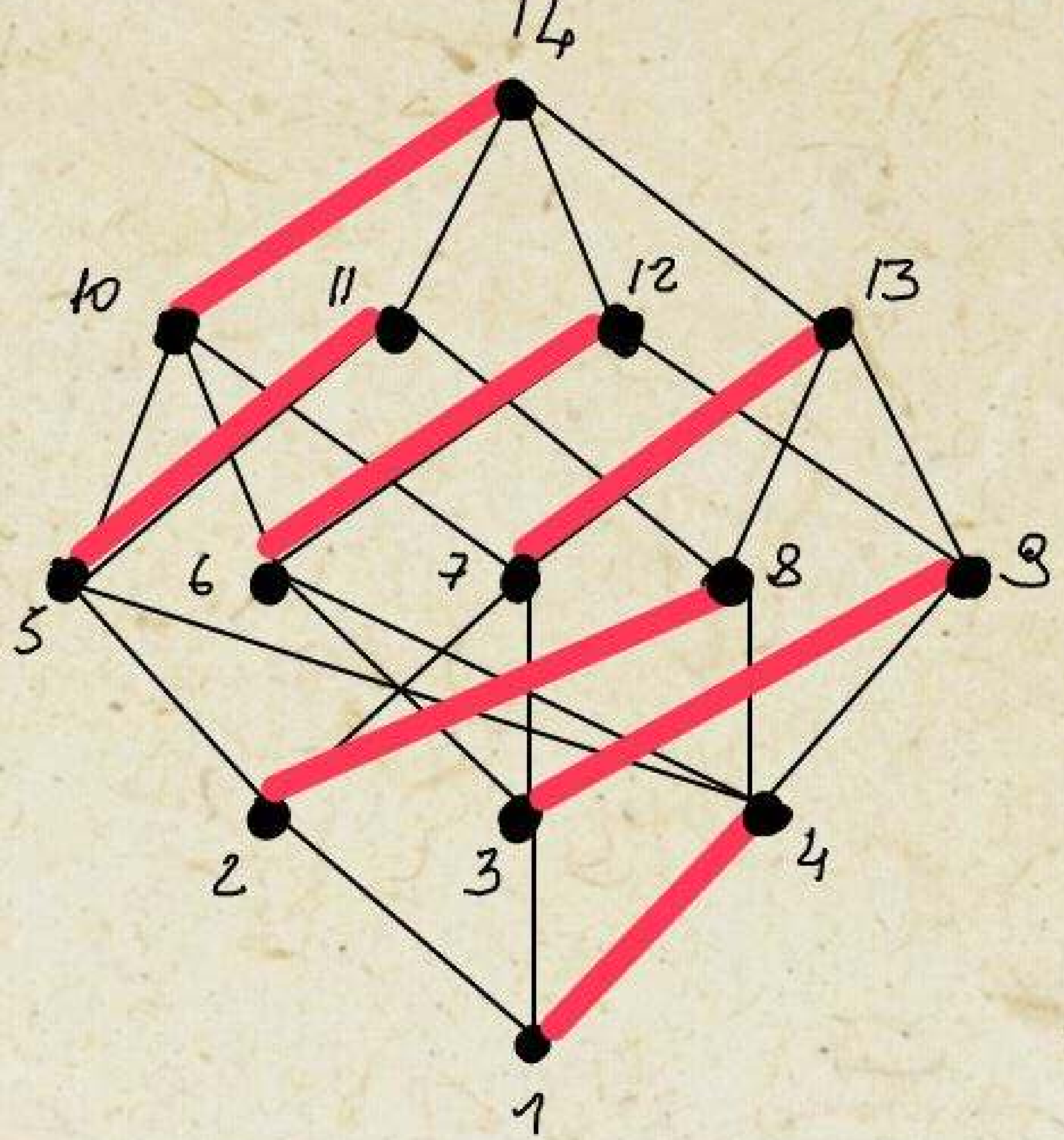

$$R_{1,14} = (q-1) R_{1,10} + R_{4,10}$$

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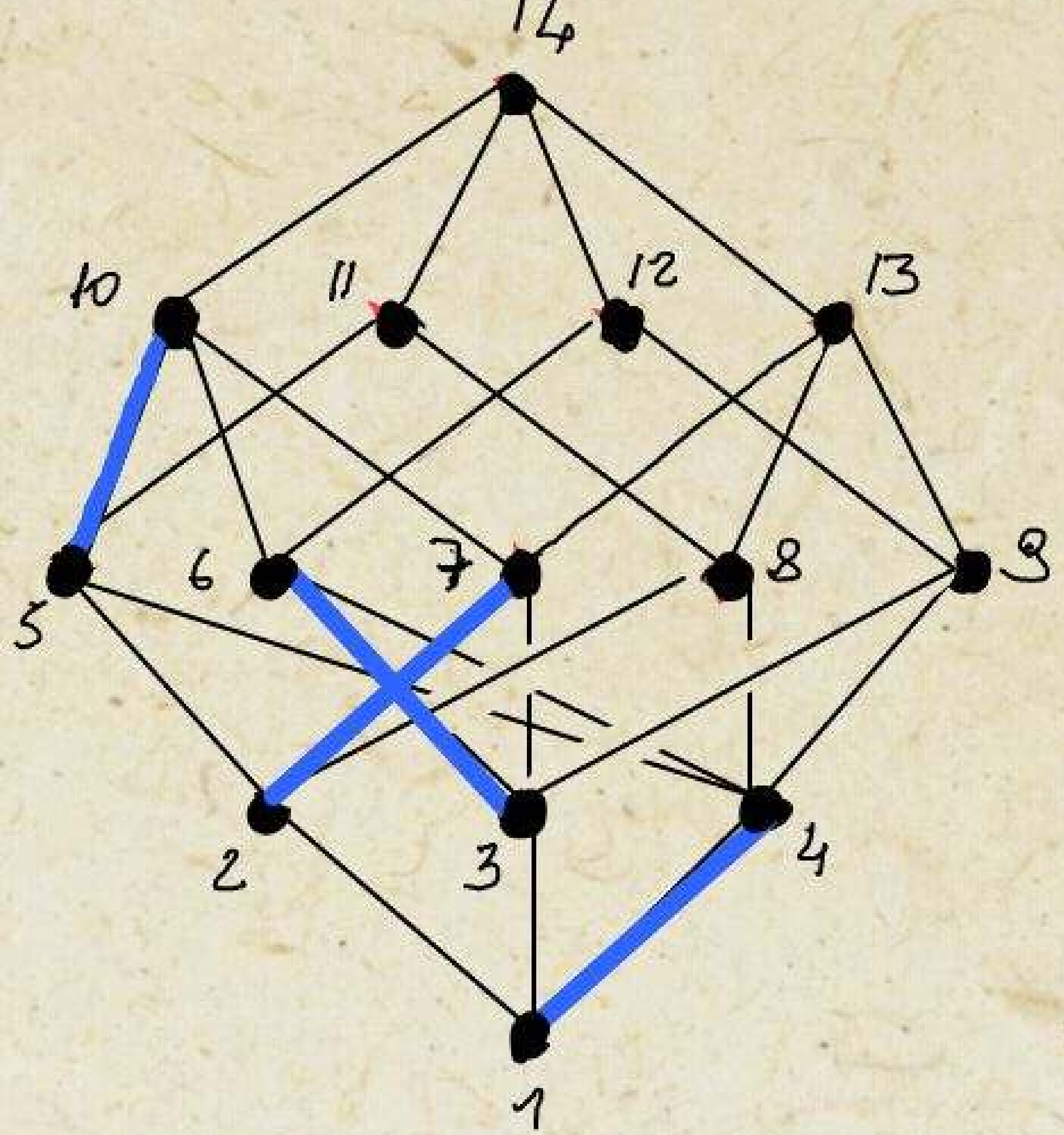

$$R_{1,14} = (q-1) R_{1,10} + R_{4,10}$$

$$= (q-1)\big((q-1) R_{1,5} + q R_{4,5}\big)$$

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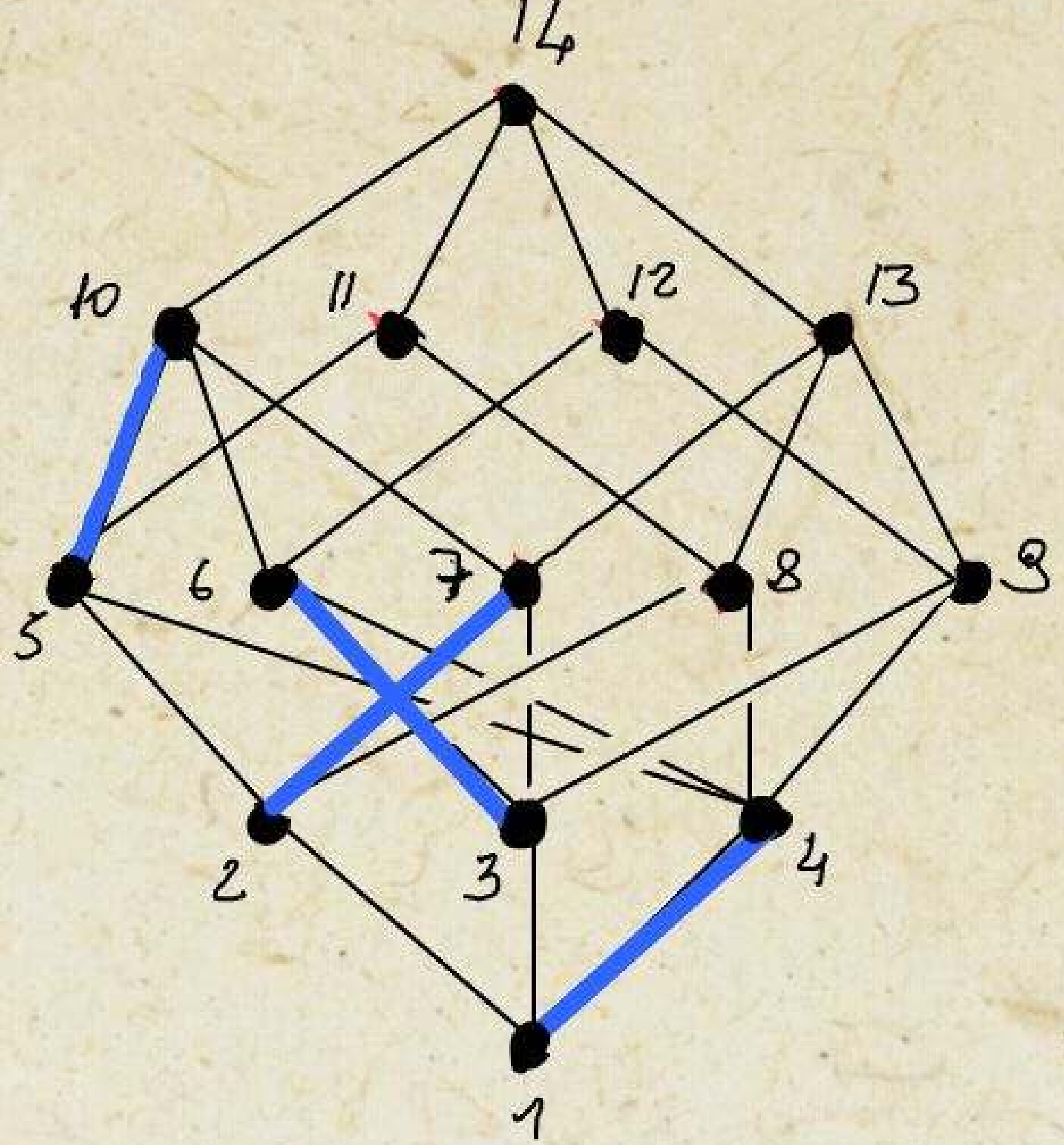

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$$= (q^2 - 2q + 2)(q-1)^2 + (q-1)^2 q$$

$$= (q-1)^2 (q^2 - q + 2)$$