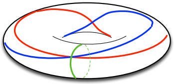
Combinatorics and topology of toric arrangements
III. Epilogue



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Toblach/Dobbiaco
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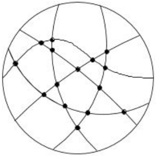
Finite matroids

Rank functions / intersection posets

... of central hyperplane arrangements

Representable m.

Orientable m.



...of pseudosphere arrangements

$|\mathcal{R}(\mathscr{A})| = \chi_{\mathrm{rk}}(-1)$

matroids / geometric lattices

(tropical linear spaces, matroids over the hyperfield $\mathbb{K}$)

Toric arrangements

$$A = [a_1, \ldots, a_n] \in M_{d \times n}(\mathbb{Z})$$
$$\mathscr{A} \text{ in } (\mathbb{Z}_q)^d \subseteq (S^1)^d \subseteq (\mathbb{C}^*)^d$$

Discrete tori (enumeration)

$\{\chi_i\}_{i|\rho}(q) := |(\mathbb{Z}_q)^d \setminus \cup \mathscr{A}_q|$
is a quasipolynomial in q, with

$\chi_1(t) = (-1)^d T(1-t, 0),$

$\chi_j(t) = ?,$

$\chi_\rho(t) = (-1)^d T_A(1-t, 0)$

[Kamiya–Takemura–Terao '08, Lawrence '11, ...]

Ehrhart theory (of zonotopes)

The zonotope $Z_A := \sum a_i$ has Ehrhart polynomial

$E_{Z_A}(t) = (-1)^d T_A(\frac{t+1}{t}, 1)$

$(= |\mathbb{Z}^d \cap tZ_A|$ for $t \in \mathbb{N})$

[d'Adderio–Moci '13]

Topology (in $(\mathbb{C}^*)^d$)

$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \cup \mathscr{A}$

- $\mathrm{Poin}(M(\mathscr{A}), t) = t^d T_A(\frac{2t+1}{t}, 0)$
  [Looijenga '95,
  De Concini–Procesi 2005]
- $M(\mathscr{A})$ minimal; presentation of the ring $H^*(M(\mathscr{A}), \mathbb{Z})$
  [D.–d'Antonio '13, Callegaro–D. '15]
- Wonderful models

[Moci '12, Gaiffi-De Concini '16]

Dissections of $(S^1)^d$

The complement $(S^1)^d \setminus \cup \mathscr{A}$ has

$T_A(1, 0)$ connected regions.

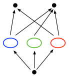
[Lawrence '09 and '11;
Ehrenborg–Readdy–Slone '09]

The "Coxeter case"

[Moci '08, Aguiar-Petersen '14,
D.-Girard '16+]

Poset of layers

$\mathcal{C}(\mathscr{A})$:



Matroid over $\mathbb{Z}$

$$M(I) := \mathbb{Z}^d / \langle a_i \rangle_I$$

Arithmetic matroid

$$m(I) := |\operatorname{Tor}(M(I))|$$

Ar. Tutte Poly.

$$T_A(x, y)$$

Definition 1.19 (Compare Section 2 of [5]). Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a locally ranked triple. A *molecule* of $\mathcal{S}$ is any triple (R, F, T) of disjoint sets with $R \cup F \cup T \in \mathcal{C}$ and such that, for every A with $R \subseteq A \subseteq R \cup F \cup T$,

$$\mathrm{rk}(A) = \mathrm{rk}(R) + |A \cap F|.$$

Remark 1.20. Once a total ordering of the ground set S is fixed, the notion of basis activities for matroids briefly recapped in Proposition 1.17 above allows us to associate to every basis B a molecule $(B \setminus I(B), I(B), E(B))$.

Definition 1.21 (Extending Moci and Brändén [5]). Let $\mathcal{S} = (S, \mathcal{C}, \mathrm{rk})$ be a finite locally ranked triple and $m : \mathcal{C} \to \mathbb{R}$ any function. If (R, F, T) is a molecule, define

$$\rho(R, R \cup F \cup T) := (-1)^{|T|} \sum_{R \subseteq A \subseteq R \cup F \cup T} (-1)^{|R \cup F \cup T| - |A|} m(A).$$

We call the pair $(\mathcal{S}, m)$ *arithmetic* if the following axioms are satisfied:

(P) For every molecule (R, F, T),

$$\rho(R, R \cup F \cup T) \geq 0.$$

(A1) For all $A \subseteq S$ and $e \in S$ with $A \cup e \in \mathcal{C}$:

(A.1.1) If $\mathrm{rk}(A \cup \{e\}) = \mathrm{rk}(A)$ then $m(A \cup \{e\})$ divides $m(A)$.

(A.1.2) If $\mathrm{rk}(A \cup \{e\}) > \mathrm{rk}(A)$ then $m(A)$ divides $m(A \cup \{e\})$.

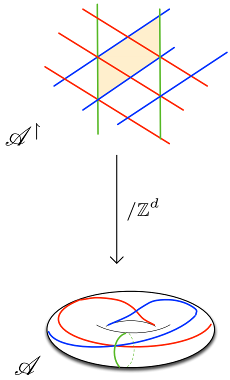
(A2) For every molecule (R, F, T)

$$m(R)m(R \cup F \cup T) = m(R \cup F)m(R \cup T).$$

Following [5] we use the expression *pseudo-arithmetic* to denote the case where m only satisfies (P). An *arithmetic matroid* is an arithmetic pair $(\mathcal{S}, m)$ where $\mathcal{S}$ is a matroid.

Combinatorial framework

Ansatz: "periodic arrangements"



$\mathcal{L}(\mathscr{A}^{\upharpoonright})$

Poset of intersections

$\downarrow$ $/\mathbb{Z}^d$(as posets)

$\mathcal{C}(\mathscr{A})$

$\mathcal{F}(\mathscr{A}^{\upharpoonright})$

Poset (category) of polyhedral faces

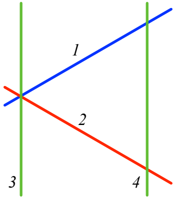
$\downarrow$ $/\mathbb{Z}^d$ (as acyclic categories)

$\mathcal{F}(\mathscr{A})$

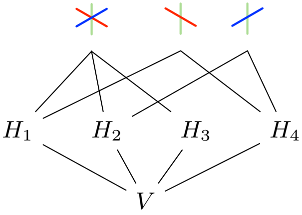
Characterize axiomatically the involved posets and the group actions.

RECALL BY WAY OF EXAMPLE

$\mathscr{A}$:



$\mathcal{L}(\mathscr{A})$:



$\mathcal{K} := \{I \text{ such that } \cap_{i \in I} H_i \neq \emptyset$

$\{\}, \{1\}, \{2\}, \{3\}, \{4\}$

$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}$

$\{1,2,3\}\}$

These are the *central sets.*

$\mathrm{rk} : \mathcal{K} \to \mathbb{N},\ I \mapsto \mathrm{codim}(\cap_{i \in I} H_i)$

$(S, \mathcal{K}, \mathrm{rk})$ is a **semimatroid**.

[today: *loopless*]

$\mathcal{L}$ is a **geometric semilattice**

Cryptomorphism

Group actions on semimatroids

Let G be a group

A G-semimatroid

$$\mathfrak{S} : G \circlearrowright (S, \mathcal{K}, \mathrm{rk})$$

is an action of G
on a semimatroid $(S, \mathcal{K}, \mathrm{rk})$
by rank- and centrality- preserving bijections of S.

A G-geometric semilattice

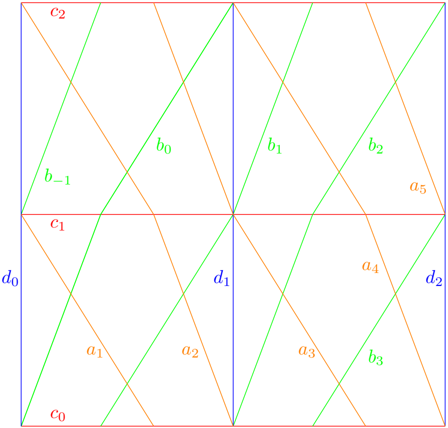
$$\mathfrak{S} : G \circlearrowright \mathcal{L}$$

is an action of G
on a geometric semilattice $\mathcal{L}$
by poset-automorphisms.

Cryptomorphism! ✓

Group actions on semimatroids

$\mathbb{Z}^2 \circlearrowright$



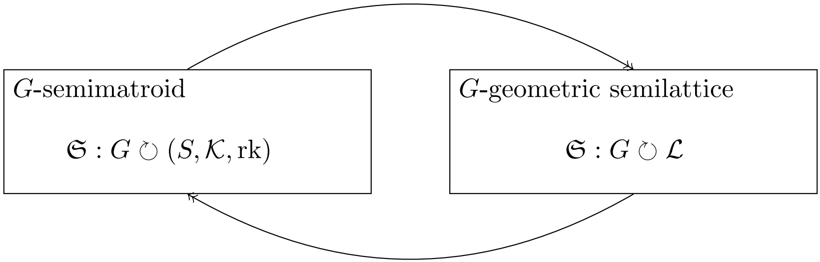
$$S := \{a_i, b_j, c_k, d_l\}_{i,j,k,l \in \mathbb{Z}}, \qquad \mathcal{L} := \text{poset of intersections},$$

$$\mathcal{K} := \{\emptyset,\ a_1,\ b_0,\ a_1 b_0,\ b_1,\ a_1 b_1, \dots\} \not\ni a_1 b_0 c_0$$

$$\text{for } X \in \mathcal{K},\ \text{rk}(X) := \text{codim} \cap X$$

Quotient posets

Let G be a group

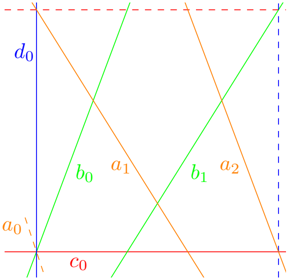


$\mathcal{C}_{\mathfrak{S}} := \mathcal{L}/G$, the set $\{Gx \mid x \in \mathcal{L}\}$ ordered by

$Gx \leq Gy$ iff $x \leq_{\mathcal{L}} gy$ for some g

(This *is* a poset)

Example ($G = \mathbb{Z}^2$)



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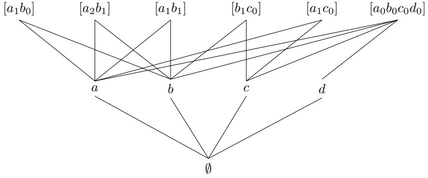
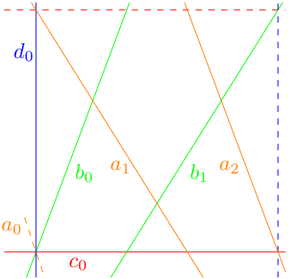
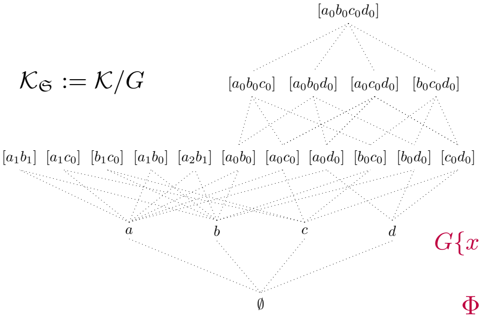


Example ($G = \mathbb{Z}^2$)



$\mathcal{K}_{\mathfrak{S}} := \mathcal{K}/G$



$G\{x_1 \ldots x_k\}$

$\Phi \downarrow$

$\{Gx_1 \ldots Gx_k\}$

$E_{\mathfrak{S}} := S/G = \{a, b, c, d\}$

$$\underline{\mathcal{K}} := \{\{Gx_1 \ldots Gx_k\} | \{x_1 \ldots x_k\} \in \mathcal{K}\} = \left\{ \begin{matrix} \{a,b,c,d\}^{(1)} \\ \{a,b,c\}^{(1)} \quad \{a,b,d\}^{(1)} \quad \{a,c,d\}^{(1)} \quad \{b,c,d\}^{(1)} \\ \{a,b\}^{(4)} \quad \{a,c\}^{(2)} \quad \{b,c\}^{(2)} \quad \{a,d\}^{(1)} \quad \{b,c\}^{(1)} \quad \{b,d\}^{(1)} \\ a^{(1)} \quad b^{(1)} \quad c^{(1)} \quad d^{(1)} \\ \emptyset^{(1)} \end{matrix} \right\}$$

$\underline{\mathrm{rk}}(A) := \max\{\mathrm{rk}(X) \mid \Phi(GX) \subseteq A\}$

$m_{\mathfrak{S}}(A) := |\Phi^{-1}(A)|.$

$$T_{\mathfrak{S}}(x,y) := \sum_{A \subseteq E_{\mathfrak{S}}} m_{\mathfrak{S}}(A)(x-1)^{\underline{\mathrm{rk}}(\mathcal{S}) - \underline{\mathrm{rk}}(A)}(y-1)^{|A| - \underline{\mathrm{rk}}(A)}$$

Translative actions

$\mathfrak{S}$ is called *translative* if, for all $x \in S$ and $g \in G$,

$$\{x, g(x)\} \in \mathcal{K} \text{ implies } x = g(x).$$

Theorem The function $\underline{\mathrm{rk}} : 2^{E_{\mathfrak{S}}} \to \mathbb{N}$ always defines a semimatroid. It defines a matroid if, and only if, $\mathfrak{S}$ is translative.

In the 'realizable' case, this corresponds to the arrangement $\mathscr{A}_0$,

(remember?)

Theorem If $\mathfrak{S}$ is translative, the triple $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ satisfies axiom (P)

"pseudo-arithmetic"

Translative actions

$\mathfrak{S}$ is called *translative* if, for all $x \in S$ and $g \in G$,

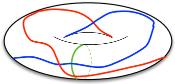
$$\{x, g(x)\} \in \mathcal{K} \text{ implies } x = g(x).$$

Theorem. If $\mathfrak{S}$ is translative, the *characteristic polynomial* of the poset $\mathcal{C}_{\mathfrak{S}} = \mathcal{L}/G$ is

$$\chi_{\mathcal{C}_{\mathfrak{S}}}(t) = (-1)^r T_{\mathfrak{S}}(1 - t, 0).$$

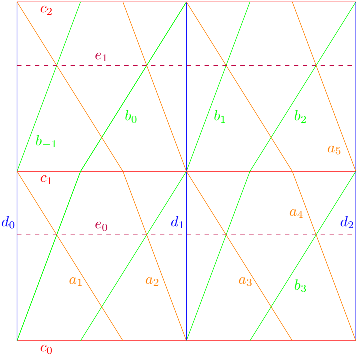
Corollary. If $\mathfrak{S}$ arises from a translative $\mathbb{Z}^r$-action on a rank r oriented semimatroid ("periodic wiggly arrangement"), then the number of regions of the associated toric *pseudoarrangement* is

$$|\mathcal{R}(\mathscr{A})| = (-1)^r T_{\mathfrak{S}}(1, 0)$$



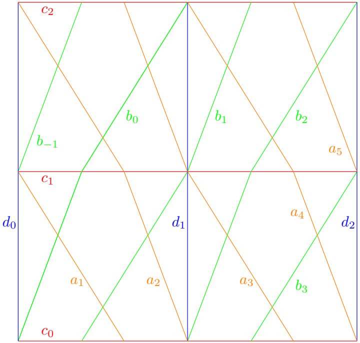
Deletion / Contraction

$\mathfrak{S}$:

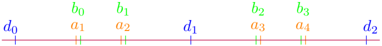


$e := Ge_0; \quad \mathrm{stab}(e) := \mathrm{stab}(e_i)$

$\mathfrak{S} \setminus e$:



$\mathfrak{S}/e$: $\mathrm{stab}(e) \circlearrowright$



Translative actions

Theorem If $\mathfrak{S}$ is translative, for all $e \in E_{\mathfrak{S}}$ we have the recursion

$$T_{\mathfrak{S}}(x,y) = (x-1)T_{\mathfrak{S}\setminus e}(x,y) + (y-1)T_{\mathfrak{S}/e}(x,y),$$

according to whether e is a coloop or a loop of $(E_{\mathfrak{S}}, \underline{\mathcal{K}}, \underline{\mathrm{rk}})$, where

$$\mathfrak{S} \setminus e := G \circlearrowright (S, \mathcal{K}, \mathrm{rk}) \setminus e, \quad \mathfrak{S}/e := \mathrm{stab}(e) \circlearrowright (S, \mathcal{K}, \mathrm{rk})/e.$$

Think: "removing an orbit of hyperplanes", resp. considering the $\mathrm{stab}(H_e)$-periodic arrangement induced in H_e

(NRDC)

Towards arithmetic matroids

A translative $\mathfrak{S}$ is called *normal* if, for all $X \in \mathcal{K}$, $\mathrm{stab}(X)$ is normal in G.

This allows, given $X \in \mathcal{K}$, to consider the *group*

$$\Gamma^X := \prod_{x \in X} G/\,\mathrm{stab}(x)$$

Theorem. If $\mathfrak{S}$ is translative and normal, $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ satisfies (P), (A.1.2) and (A.2).

For the "initiated": moreover, $T_{\mathfrak{S}}(x, y)$ satisfies an "activity decomposition theorem" *à la* Crapo.

Towards arithmetic matroids

A translative $\mathfrak{S}$ is called *normal* if, for all $X \in \mathcal{K}$, $\operatorname{stab}(X)$ is normal in G.

This allows, given $X \in \mathcal{K}$, to consider $\Gamma^X := \prod_{x \in X} G/\operatorname{stab}(x)$, and

$$W(X) := \{(g_x)_{x \in X} \in \Gamma^X \mid \{g_x x\}_{x \in X} \in \mathcal{K}\}$$

$\mathfrak{S}$ is called *arithmetic* if, for all $X \in \mathcal{K}$, $W(X)$ is a subgroup of Γ^X.

Theorem: If $\mathfrak{S}$ is arithmetic, then $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ is an arithmetic matroid.

Remark 1. There are translative and not normal, and normal but not arithmetic $\mathfrak{S}$'s. In general, it seems *very* restrictive to require arithmeticity.

Towards arithmetic matroids

A translative $\mathfrak{S}$ is called *normal* if, for all $X \in \mathcal{K}$, $\operatorname{stab}(X)$ is normal in G.

This allows, given $X \in \mathcal{K}$, to consider $\Gamma^X := \prod_{x \in X} G/\operatorname{stab}(x)$, and

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$\mathfrak{S}$ is called *arithmetic* if, for all $X \in \mathcal{K}$, $W(X)$ is a subgroup of Γ^X.

Theorem: If $\mathfrak{S}$ is arithmetic, then $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ is an arithmetic matroid.

Remark 2. $W(X)$ parametrizes all elements of $\Phi^{-1}(\Phi(GX))$. In the case of periodic arrangements, this induces a group structure on the set of connected components of the intersection of the "subtori" in $\Phi(GX) \subseteq E_{\mathfrak{S}}$.

Representable cases

Call $\mathfrak{S}$ *representable* if it arises as an action by translations on an affine rank d arrangement $\mathscr{A}$ of hyperplanes. In this case, $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ is an arithmetic matroid and

$$\mathcal{C}_{\mathfrak{S}} \simeq \mathcal{C}(\mathscr{A}).$$

$$G = \{\mathrm{id}\} \quad \to \quad \text{(Central) arrangements of hyperplanes,}$$

$$G = \mathbb{Z}^d \quad \to \quad \text{(Centered) toric arrangements*}$$

$$G = \mathbb{Z}^{2d} \quad \to \quad \text{Elliptic arrangements}$$

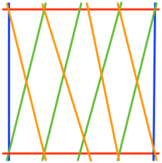
(*) in this case, the arithmetic matroid $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ is *dual* to that associated to the list of defining characters by d'Adderio–Brändén–Moci

Coarse overview

G-semimatroids / G-geometric semilattices

... of periodic hyperplane arrangements

Representable

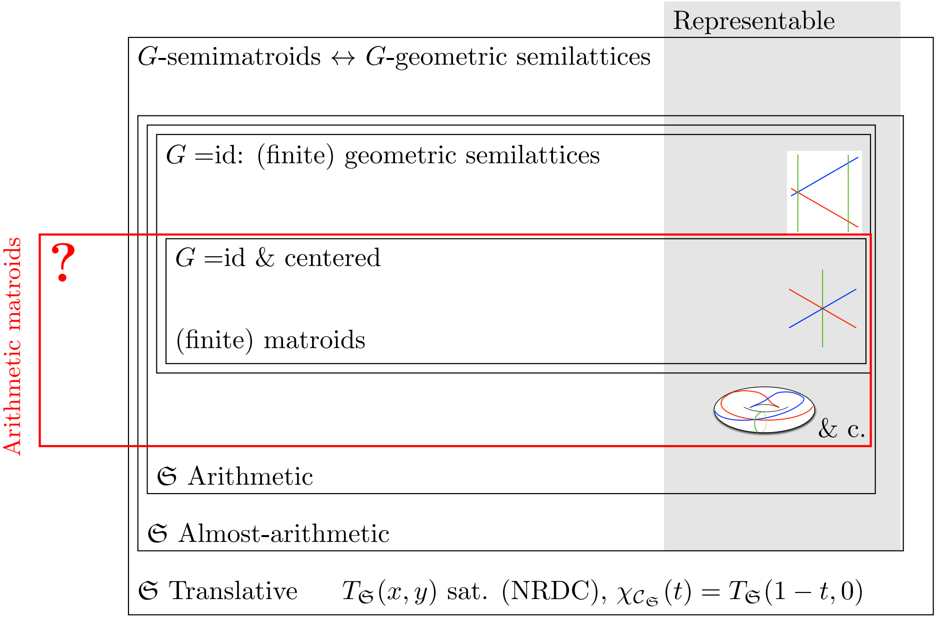


Orientable

...of pseudoarrangements

$\mathfrak{S}$

“Finer” overview



Your turn!

Your turn!

1. Does the theory of AM's fully embed in G-semimatroids?

   Construct, for every arithmetic matroid (E, rk, m) a G-semimatroid $\mathfrak{S}$ such that $(E_{\mathfrak{S}}, \underline{\mathrm{rk}}, m_{\mathfrak{S}})$ is isomorphic to (E, rk, m)

   – or find obstructions (!).

Your turn!

1. Does the theory of AM's fully embed in G-semimatroids?

2. Structure of the posets $\mathcal{C}_{\mathfrak{S}}$

   – are these posets *shellable*? At least Cohen-Macauley?

   ($\mathcal{C}(\mathscr{A})$ shellable for toric Weyl type A_n, B_n, C_n [D.-Girard '17+])

   – characterize intrinsically the class of these posets

   (cf. "developability" in Bridson-Häfliger)

Your turn!

1. Does the theory of AM's fully embed in G-semimatroids?
2. Structure of the posets $\mathcal{C}_{\mathfrak{S}}$
3. Duality theory

   Construct, for a given arithmetic $\mathfrak{S}$, a $\mathfrak{S}^*$ such that $(\mathfrak{S}^*)^* \simeq \mathfrak{S}$ and, for instance, $T_{\mathfrak{S}}(x,y) = T_{\mathfrak{S}^*}(y,x)$.

   Can one do it for general translative $\mathfrak{S}$?

   One motivation for developing duality is the following item.

Your turn!

1. Does the theory of AM's fully embed in G-semimatroids?
2. Structure of the posets $\mathcal{C}_{\mathfrak{S}}$
3. Duality theory
4. Partition functions, Dahmen-Micchelli spaces

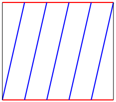
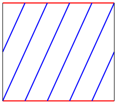
   Recent motivation for the study of toric arrangements is De Concini, Procesi and Vergne's theory of partition functions and splines, see [De Concini – Procesi, Topics in hyperplane arrangements, polytopes and box splines, Springer Universitext 2011]

   Can one describe the combinatorics of this situation (e.g. wall-crossing of partition functions, etc.) in terms of the associated $\mathfrak{S}$?

Your turn!

1. Does the theory of AM's fully embed in G-semimatroids?
2. Structure of the posets $\mathcal{C}_{\mathfrak{S}}$
3. Duality theory
4. Partition functions, Dahmen-Micchelli spaces
5. Topology

   Does $\mathfrak{S}$ determine the cohomology ring in the toric case?

   E.g.: $\mathfrak{S}_1$:  and $\mathfrak{S}_2$:  are not isomorphic.

   Ans: how about nonrealizable toric Salvetti complexes?

Your turn!

1. Does the theory of AM's fully embed in G-semimatroids?
2. Structure of the posets $\mathcal{C}_{\mathfrak{S}}$
3. Duality theory
4. Partition functions, Dahmen-Micchelli spaces
5. Topology

"Und jedem Anfang wohnt ein Zauber inne..."