The monodromy theorem for compact Kähler manifolds and smooth quasi-projective varieties

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Overview

1. Motivation
   - Milnor fibration
   - The classical Monodromy Theorem

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2. Main Results

Singularities

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Definition

We say that two analytic germ f and g have the same topological type if there is a homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V_f) = V_g$.

Milnor fibration

Theorem (J. Milnor 1968)

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a germ of analytic function. Let* B_ϵ *be a small open ball at the origin in* $\mathbb{C}^n$. *Let* $D_\delta \subset \mathbb{C}$ *be a disc around the origin with* $0 < \delta \ll \epsilon$. *Set* $D_\delta^* = D_\delta \setminus \{0\}$. *Then there exists a fibration*

$$f : B_\epsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*.$$

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$$f : B_\epsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*.$$

The fibre $F = B_\epsilon \cap f^{-1}(\delta)$ is called the Milnor fibre.

Here is the picture of Milnor fibration from D. Massey:

fibration.jpg fibration.jpg

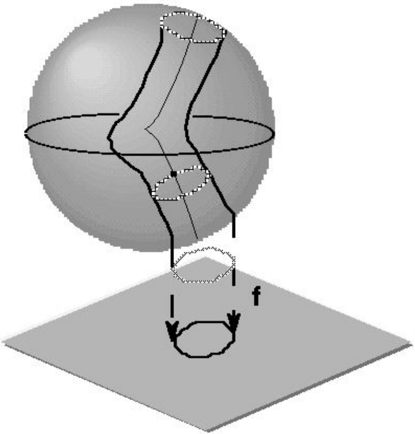

Figure 0.1. The Milnor Fibration inside a ball

Example

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Then the Milnor fibre F is diffeomorphic to the total space of the tangent bundle of the sphere S^{n-1}, hence

$$F \simeq S^{n-1}.$$

Monodromy

Parallel translation along the path

$$\gamma : [0,1] \mapsto D_\delta,$$
$$t \mapsto \delta e^{2\pi i t}$$

gives a homeomorphism

$$h : F \to F$$

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The total space can be identified with

$$F \times [0,1]/(x,1) \sim (h(x),0) \text{ for any } x \in F.$$

The Monodromy Theorem

The geometric monodromy $h : F \to F$ induces an linear automorphism

$$h_i : H_i(F, \mathbb{C}) \to H_i(F, \mathbb{C}).$$

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Theorem

With the above assumptions and notations, we have that

(1) *the eigenvalues of h_i are all roots of unity for all i.*

(2) *the sizes of the blocks in the Jordan normal form of h_i are at most $i+1$.*

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There are many different proofs of this theorem by A. Borel, E. Brieskorn, G. M. Greuel, C. H. Clements, P. Deligne, P. A. Griffiths, A. Grothendieck, N. M. Katz, A. Landman, Lê Dũng Tráng, E. Looijenga, B. Malgrange, W. Schmid .... from 70s to 80s.

- F is homotopy equivalent to a finite $(n-1)$-dimensional CW complex.
- The possible maximal size of Jordan block is n.
- Examples of B. Malgrange (1973) show that the bounds on the sizes of the Jordan blocks are sharp.

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Due to the existence of Milnor fibration, one has a short exact sequence:

$$0 \to \pi_1(F) \to \pi_1(X) \to \pi_1(S^1_\delta) \cong \mathbb{Z} \to 0.$$

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The generator of the deck transformation group acts on $F \times \mathbb{R}$ as $(h, +1)$:

$$\begin{aligned}(h, +1) : F \times \mathbb{R} &\mapsto F \times \mathbb{R} \\ (x, s) &\mapsto (h(x), s + 1).\end{aligned}$$

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Let X^ρ denote the corresponding covering space of X, where

$$0 \to \pi_1(X^\rho) \to \pi_1(X) \xrightarrow{\rho} \mathbb{Z} \to 0.$$

Under the deck group $\mathbb{Z}$ action, $H_i(X^\rho, \mathbb{C})$ becomes a finitely generated $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[t, t^{-1}]$-module.

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Let $T_i(X, \rho)$ denote the torsion part of $H_i(X^\rho, \mathbb{C})$, which is a finite dimensional $\mathbb{C}$-vector space.

Theorem

Let X be either a smooth complex quasi-projective variety or a compact Kähler manifold. Then for any epimorphism $\rho : \pi_1(X) \to \mathbb{Z}$, the eigenvalues associated to the t-action on $T_i(X, \rho)$ are roots of unity for any i.

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This theorem built on a long series of partial results due to Green-Lazarsfeld, Arapura, Simpson, Dimca-Papadima, etc.
The compact Kähler manifold case was finished by B. Wang in 2014 and the smooth quasi-projective variety case wad proved by N. Budur and B. Wang in 2015.

Theorem (N. Budur, Y. Liu, B. Wang 2016)

Let X be a connected compact Kähler manifold. Then for any epimorphism $\rho : \pi_1(X) \to \mathbb{Z}$, the size of Jordan block $T_i(X, \rho)$ is at most 1 for any i.

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The real homotopy type of a compact Kähler manifold is formal.

Let X be a connected compact Kähler manifold.
Then there exists pure Hodge structures on $H^i(X, \mathbb{C})$ with pure weight i.

Theorem (N. Budur, Y. Liu, B. Wang 2016)

Let X be a smooth complex quasi-projective variety with complex dimension n. Then for any epimorphism $\rho : \pi_1(X) \to \mathbb{Z}$, the size of Jordan block for $T_i(X, \rho)$ is at most $\min\{i+1, 2n-i\}$.

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Let X be a smooth complex quasi-projective variety with complex dimension n.
Then there exist mixed Hodge structures on $H^i(X, \mathbb{C})$ with weights ranging from i to $\min\{2i, 2n\}$.

The monodromy theorem tells us that the compact Kähler manifold or smooth complex quasi-projective variety has very special topology.

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Question

Is our upper bound of the size of Jordan block sharp for the smooth quasi-projective variety case?

Thank you !