Weighted discrete Morse theory

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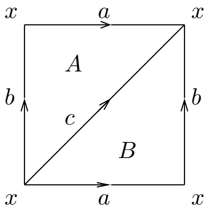
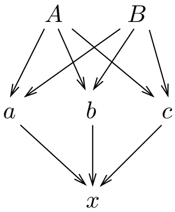
Discrete Morse theory

X – a (finite) CW complex.
G_X – the incidence graph of the cells of X.

Idea: collapse pairs of cells in order to make the complex smaller (preserving homotopy type).





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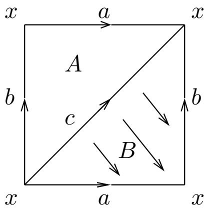

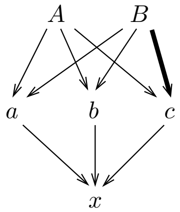

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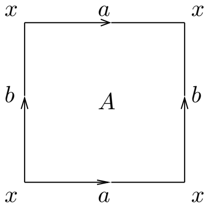

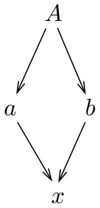

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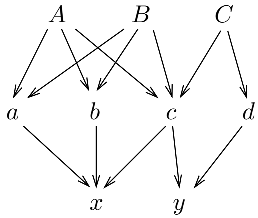

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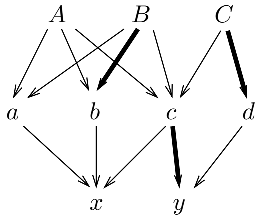

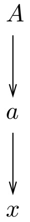

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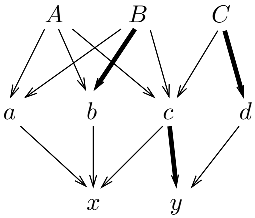
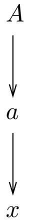


Idea: collapse pairs of cells in order to make the complex smaller (preserving homotopy type).

In general you can choose any matching $\mathcal{M}$ such that:

- if $(c_1 \to c_2) \in \mathcal{M}$, then c_2 is a *regular* face of c_1;
- $\mathcal{M}$ is *acyclic*, i.e. the graph $G_X^{\mathcal{M}}$ obtained from G_X by reversing the arrows in $\mathcal{M}$ is acyclic.



Beware: boundaries usually become more complicated.

Discrete Morse theory

Theorem (Forman '98)

Let X be a (finite) CW complex, and let $\mathcal{M}$ be a matching on the incidence graph G_X such that:

- *if $(c_1 \to c_2) \in \mathcal{M}$, then c_2 is a* **regular** *face of c_1;*
- *$\mathcal{M}$ is* **acyclic***, i.e. the graph obtained from G_X by reversing the arrows in $\mathcal{M}$ is acyclic.*

Then there exists a CW complex $X^{\mathcal{M}} \simeq X$ with cells in one-to-one correspondence with unmatched cells of X.

Algebraic discrete Morse theory

R – a commutative ring with unity.
C_* – a (finitely generated) chain complex of free R-modules:

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

Ω_* – a fixed basis for C_* (elements of Ω_* replace cells).

For $b \in \Omega_n$, write $\partial b = \sum_{a \in \Omega_{n-1}} r_{a,b} \cdot a\,.$

The *incidence graph* G has Ω_* as vertex set, and weighted edges $b \xrightarrow{r_{a,b}} a$ whenever $r_{a,b} \neq 0$.

Idea: "collapse" pairs of elements of the basis so that C_* is chain homotopy equivalent to some $C_*^{\mathcal{M}}$, a chain complex of free R-modules with less generators.

Algebraic discrete Morse theory

Theorem (Jöllenbeck-Welker '05, Kozlov '05, Skjöldberg '06)

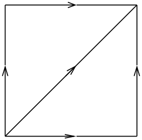
Let C_* be a (finitely generated) chain complex of free R-modules, and let $\mathcal{M}$ be a matching on the incidence graph G such that:

- *if $(b \to a) \in \mathcal{M}$, then $r_{a,b}$ is invertible (**regularity**);*
- *$\mathcal{M}$ is **acyclic**.*

Then there exists a chain complex $C_*^{\mathcal{M}} \simeq C_*$ of free R-modules with a basis in one-to-one correspondence with unmatched elements of the basis of C_*.

Algebraic discrete Morse theory (example)

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}} \mathbb{Z} \longrightarrow 0$$



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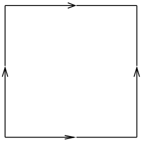

Weighted algebraic discrete Morse theory

R – now a PID.
C_* and Ω_* – as before.

Assign a weight $w_a \in R$ to every $a \in \Omega_*$, in such a way that

$$(b \to a) \quad \Longrightarrow \quad w_a \mid w_b \,.$$

Then there is a natural projection $R/(w_b) \to R/(w_a)$.

Consider the *torsion complex* L_* with $L_n = \bigoplus_{\dim a = n} R/(w_a)$ and boundary induced by the boundary of C_*.

Idea (as usual): "collapse" pairs of elements of the basis so that L_* is chain homotopy equivalent to some $L_*^{\mathcal{M}}$, a torsion complex with less generators.

Weighted algebraic discrete Morse theory

Theorem (Salvetti-Villa '13)

Let L_* be a torsion complex as before, and let $\mathcal{M}$ be a matching on the incidence graph G such that:

- *if $(b \to a) \in \mathcal{M}$, then $r_{a,b}$ is invertible (**regularity**);*
- *$\mathcal{M}$ is **acyclic**;*
- *$\mathcal{M}$ is **weighted**, i.e. if $(b \to a) \in \mathcal{M}$ then $(w_b) = (w_a)$.*

Then there exists a torsion complex $L_*^{\mathcal{M}} \simeq L_*$ with a basis in one-to-one correspondence with unmatched elements of the basis of L_*, and with the same weights.

Application: local homology of Artin groups

(W, S) – a Coxeter system.
A_W – the associated Artin group.
Sal_W – the associated Salvetti complex.
X_W – the quotient complex Sal_W / W, having fundamental group A_W.

Let $R = \mathbb{Q}[q, q^{-1}]$. Consider the representation $\lambda \colon A_W \to \mathrm{Aut}(R)$ sending each standard generator of A_W to the multiplication by $-q$. This determines a local system $\mathcal{L}_\lambda$ on X_W.

The computation of the homology $H_*(X_W; \mathcal{L}_\lambda)$ can be reduced to the computation of the homology of the torsion complex

$$L_* = \bigoplus_{\sigma \in S^f} \frac{R}{(W_\sigma(q))} \cdot e_\sigma \; = \bigoplus_{d \geq 2} \underbrace{\left(\bigoplus_{\sigma \in S^f} \frac{R}{\left(\varphi_d^{w_d(\sigma)}\right)} \cdot e_\sigma \right)}_{(L_{\varphi_d})_*}.$$

Braid groups and independence complexes

In the case of braid groups (Artin groups of type A_n), suitable matchings allow to find a connection between homology of braid groups and homology of independence complexes of graphs.

Given a graph $\mathcal{G} = (V, E)$, the independence complex $\mathrm{Ind}_k(\mathcal{G})$ is the simplicial complex on the set V containing all simplices $\sigma \subseteq \mathcal{G}$ such that every connected component of $\mathcal{G}|_\sigma$ has at most k vertices.

Theorem (Salvetti '15)

$$H_*(\mathrm{Br}_{n+1}; \mathcal{L}_\lambda) = \bigoplus_{d \geq 2} \widetilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right).$$

Homology of groups of finite and affine type

For finite and affine W, we could construct matchings satisfying the following combinatorial property.

Definition

A matching on $(L_{\varphi_d})_*$ is *precise* if, for any edge $\sigma \to \tau$ of $G^{\mathcal{M}}$, we have that $w_\varphi(\sigma) = w_\varphi(\tau) + 1$.

The existence of such matchings has the following interesting theoretical consequence.

Theorem (P.-Salvetti)

For finite and affine W, each $H_k(X_W; \mathcal{L}_\lambda)$ is a direct sum of terms of the form R or $R/(\varphi_d)$ (φ_d^k-torsion for $k \geq 2$ does not occur).

The End