The Bounded Negativity Conjecture and Harbourne indices

Piotr Pokora
Leibniz Universität Hannover

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Definition

We say that a surface X has the *Bounded Negativity Property* if there exists a number $b(X)$ such that

$$C^2 \geq -b(X)$$

holds for all reduced (and irreducible) curves $C \subset X$.

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Example

- For $\mathbb{P}^2$ it suffices to take $b(\mathbb{P}^2) = 0$.
- For the Hirzebruch surface $\mathbb{F}_n$, $b(\mathbb{F}_n) = n$ suffices.

Conjecture

Every **complex** *surface has the Bounded Negativity Property.*

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Remark

This conjecture **fails** in positive characteristic!

Problem

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Of course, if BNC is true, then the above Problem has positive answer.

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Even if BNC is true, it is interesting to compare the numbers

$$b(X) \quad \textit{and} \quad b(Y)$$

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Let $f : X \to \mathbb{P}^2$ be the blow up of s **general** points. Then the (-1)-curve conjecture due to De Fernex predicts that $b(X) = 1$ is independent of the number of points blown up.

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Remark

The above statement fails completely for **arbitrary** points.

Definition

Let X be a smooth projective surfaces and let $\mathcal{C} = \{C_1, ..., C_k\}$ be a configuration of curves in X. Then we say that $\mathcal{C}$ is a transversal configuration if

1. all curves are (irreducible) smooth,
2. all pairwise intersection points are transversal (locally look like $x_1 x_2 = 0$),
3. there is no point where all curves meet.

Definition

Let X be a smooth complex projective surface and let $\mathcal{C} = \{C_1, ..., C_k\}$ be a configuration of curves in X (here we do not assume anything about types of singularities). Denote by $C = C_1 + ... + C_k$ the associated divisor to $\mathcal{C}$. The H-index of $\mathcal{C}$ is defined as

$$H(X; \mathcal{C}) := \frac{C^2 - \sum_{p \in \mathrm{Sing}(\mathcal{C})} m_p^2(C)}{s},$$

where s is equal to the number of singular points of $\mathcal{C}$ and m_p denotes the multiplicity of $p \in \mathrm{Sing}(\mathcal{C})$.

Definition

Let X be a smooth complex projective surface. Then the global Harbourne index of X is defined as

$$H(X) := \inf_{\mathcal{C}} H(X;\mathcal{C}).$$

Remark

No example of a surface with $H(X) = -\infty$ is known.

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For an arbitrary surface X one has always $H(X) \leq -2$.

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If $H(X)$ is finite, then the BNC holds on blow ups of X at $\mathrm{Sing}(\mathcal{C})$.

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Remark

Even if $H(X) = -\infty$, the Bounded Negativity Property might still hold for X and its blow ups.

Conjecture

$$H(\mathbb{P}^2) = -4.5$$

Theorem (Linear global H-index of $\mathbb{P}^2$, [1])

Let us denote by $H_L(\mathbb{P}^2)$ the global Harbourne index of $\mathbb{P}^2$ in the class of line arrangements. Then

$$H_L(\mathbb{P}^2) \geq -4.$$

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Theorem (Hirzebruch)

Let $\mathcal{L}$ be an arrangement of d lines in the complex projective plane $\mathbb{P}^2$. Then

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k \geq 5}(k-4)t_k, \tag{1}$$

provided $t_d = t_{d-1} = 0$.

Here $t_k = t_k(\mathcal{L})$ denotes the number of points where exactly k lines from $\mathcal{L}$ meet, for $k \geq 2$.

Example (Wiman's configuration)

There exists a configuration of **45** lines with

- $t_3 = 120$
- $t_4 = 45$
- $t_5 = 36$

With $\mathcal{P}$ the set of all singular points of the configuration, this configuration gives

$$H(\mathbb{P}^2; \mathcal{L}) = -\frac{225}{67} \approx -3.36.$$

Theorem (Conical global H-index of $\mathbb{P}^2$, [5])

Let us denote by $H_C(\mathbb{P}^2)$ the global Harbourne index of $\mathbb{P}^2$ in the class of transversal conic arrangements. Then

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Theorem (X. Roulleau, [3])

There exist a configuration of cubic curves with the H-index equal to (asymptotically) -4.

Theorem (Degree d global H-index of $\mathbb{P}^2$, [4])

Let us denote by $H_d(\mathbb{P}^2)$ the global Harboune index of $\mathbb{P}^2$ in the class of transversal curve configurations such that each irreducible component has degree $d \geq 3$. Then

$$H_d(\mathbb{P}^2) \geq -4 - \frac{5}{2}d^2 + \frac{9}{2}d.$$

Let $S_n \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 4$ containing a line configuration $\mathcal{L}$ with $s \geq 1$ singular points.

Theorem ([6])

One has

$$H(S_n, \mathcal{L}) > -4 - \frac{2n(n-1)^2}{s}.$$

Definition

Let X be a projective surface and assume that D is a pseudoeffective $\mathbb{Z}$-divisor. Then D can be written uniquely as a sum $D = P + N$ of $\mathbb{Q}$ divisors such that

1. P is nef,
2. N is effective and has negative definite intersection matrix if $N \neq 0$,
3. $P.N_i = 0$ for every component of N.

Conjecture (Bounded Denominators Conjecture)

Let X be a smooth projective surface and assume that D is an integral pseudoeffective divisor on X. Let $D = P + \sum_i a_i N_i$ be the Zariski decomposition with $a_i \in \mathbb{Q}$. Then there exists an integer $d(X)$ such that denominators of all a_i are bounded from above by $d(X)$ for all D.

Theorem ([2])

For a smooth projective surface X over an algebraically closed field the following two statements are equivalent:

- *X has bounded Zariski denominators,*
- *X has bounded negativity.*

Remark

One of the most important applications of the presented equivalence is the following. It can be shown that if $D = P + N$ is the Zariski decomposition, then for sufficiently divisible integers $m \geq 1$ one has

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)).$$

Sufficiently divisible means then we need to pass to multiple mD in order to clear denominators. For minimal models with the Kodaira dimension 0 it is easy to see that $d(X) = 2^{\rho-1}!$, and thus we have obtained (according to our best knowledge) the first effective description of m.

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Thank you for your attention.