Combinatorics and topology of small arrangements

Elía Saini

Université de Fribourg - Universität Freiburg
Swiss National Science Foundation SNSF

20-th February 2017

Main definitions

- A **complex hyperplane arrangement** is a finite collection $\mathcal{A} = \{H_1, \ldots, H_m\}$ of affine hyperplanes in $\mathbb{C}^d$.
- The **complement manifold** $M(\mathcal{A})$ is $\mathbb{C}^d \setminus \bigcup_{j=1}^m H_j$.
- **Problem:** study the topology of $M(\mathcal{A})$.

Central arrangements

- A complex hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$ in $\mathbb{C}^d$ is **central** if all the H_j's contain the **origin**.
- **Result:** to understand $M(\mathcal{A})$ we can study the central case.

Underlying matroid of an arrangement

For a complex central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $\mathbb{C}^d$ pick linear forms $\alpha_1, \dots, \alpha_m \in (\mathbb{C}^d)^*$ with $H_j = \ker \alpha_j$.
The **underlying matroid** $M_{\mathcal{A}}$ of $\mathcal{A}$ is the pair $(E_{\mathcal{A}}, \mathfrak{I}_{\mathcal{A}})$ where:

- $E_{\mathcal{A}} = \{1, \dots, m\}$;
- $\mathfrak{I}_{\mathcal{A}} = \{S \subseteq E_{\mathcal{A}} \mid \{\alpha_j\}_{j \in S} \text{ are linearly independent}\}$.

$M_{\mathcal{A}}$ does **not** depend on the choice of the α_j's.

Rank of an arrangement

The **rank** of a complex central hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$ in $\mathbb{C}^d$ is the rank of its underlying matroid $M_{\mathcal{A}}$. We say that $\mathcal{A}$ is **essential** if its rank is maximal.

Main question

Which **topological** information is encoded by the **combinatorics**?

Orlik-Solomon theorem

Theorem (Orlik and Solomon, 1980)

*For a complex central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $\mathbb{C}^d$ the **cohomology ring** $H^*(M(\mathcal{A}), \mathbb{Z})$ depends only on the underlying matroid $M_{\mathcal{A}}$.*

Randell isotopy theorem

Theorem (Randell, 1989)

Let $\mathcal{A}_t$ be a ***smooth one-parameter*** *family of complex central hyperplane arrangements in $\mathbb{C}^d$. If the underlying matroid $M_{\mathcal{A}_t}$ does* ***not*** *depend on t, so does the* ***diffeomorphism*** *type of $M(\mathcal{A}_t)$.*

Rybnikov matroid

Theorem (Rybnikov, 1997)

There exist complex central hyperplane arrangements with ***same*** *underlying matroid but* ***different*** *fundamental group of the corresponding complement manifolds.*

The underlying matroid $M_{\mathcal{A}}$ does **not** completely determine the topology of the complement manifold of an arrangement.

Projective line arrangements

Theorem (Nazir and Yoshinaga, 2012)

Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ and $\mathcal{B} = \{K_1, \ldots, K_m\}$ be complex central essential hyperplane arrangements in $\mathbb{C}^3$ with ***same*** *underlying matroid. If $m \leq 7$, then the complement manifolds $M(\mathcal{A})$ and $M(\mathcal{B})$ are* ***diffeomorphic****.*

Moreover, up to 8 hyperplanes in $\mathbb{C}^3$ the combinatorics **determines** the topology of the complement manifold.

A diffeomorphism result

Theorem (Gallet and S., 2017)

*Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ and $\mathcal{B} = \{K_1, \ldots, K_m\}$ be complex central essential hyperplane arrangements in $\mathbb{C}^d$ with **same** underlying matroid. If $m \leq 7$, then the complement manifolds $M(\mathcal{A})$ and $M(\mathcal{B})$ are **diffeomorphic**.*

Further questions

- Find **wider** classes of matroids for which our statement holds.
- Describe more **refined** combinatorial invariants to study the topology of the complement manifold of an arrangement.

A small bibliography

- Matteo Gallet and Elia Saini, *The diffeomorphism type of small hyperplane arrangements is combinatorially determined*, to appear in Advances in Geometry.
- Peter Orlik and Louis Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), no. 2, 167–189.
- Richard Randell, *Lattice-isotopic arrangements are topologically isomorphic*, Proc. Amer. Math. Soc. **107** (1989), no. 2, 555–559.