Eigenvalue /Saturated Tensor Product Problem for Reductive Groups

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Abstract: For any $n \times n$ Hermitian matrix A, let $\lambda_A = (\lambda_1 \geq \cdots \geq \lambda_n)$ be its set of eigenvalues written in descending order. We recall the following classical problem.

Problem 1. *(The Hermitian eigenvalue problem) Given two n-tuples of non-increasing real numbers: $\boldsymbol{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1 \geq \cdots \geq \mu_n)$, determine all possible $\boldsymbol{\nu} = (\nu_1 \geq \cdots \geq \nu_n)$ such that there exist Hermitian matrices A, B, C with $\lambda_A = \boldsymbol{\lambda}, \lambda_B = \boldsymbol{\mu}, \lambda_C = \boldsymbol{\nu}$ and $A + B + C = 0$.*

Even though this problem goes back to the nineteenth century, the first significant result was obtained by H. Weyl in 1912. With contributions from several mathematicians over the century, this problem was finally solved by combining the works of Klyachko (1998), Knutson-Tao (1999), Belkale (2001) and Knutson-Tao-Woodward (2004).

This problem can be generalized for an arbitrary reductive algebraic group as follows: Let G be a connected algebraic group with a maximal compact subgroup K and let $\mathfrak{g}$ and $\mathfrak{k}$ be their Lie algebras. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Choose a maximal subalgebra (which is necessarily abelian) $\mathfrak{a} \subset \mathfrak{p}$ and let $\mathfrak{a}_+$ be a dominant chamber in $\mathfrak{a}$. Then, any $K-$orbit in $\mathfrak{p}$ intersects $\mathfrak{a}_+$ in a unique point.

Then the analogue of the above Hermitian eigenvalue problem is the determination of the following subset Δ_n (for any $n \geq 2$) of $\mathfrak{a}_+^n$:

$$\Delta_n := \{(a_1, \dots, a_n) \in \mathfrak{a}_+^n : \exists (x_1, \dots, x_n) \in \mathfrak{p}^n \text{ with} \sum x_i = 0 \text{ and } x_i \in AdK.a_i\}.$$

By works of several mathematicians including Berenstein-Sjamaar (2000), Kapovich-Leeb-Millson (2005), Belkale-Kumar (2006) and Ressayre (2008), Δ_n has been determined in terms of an irredundant set of inequalities.

This series of talks will give a complete solution of the problem. The main tools used are: Geometric Invariant Theory and Topology. We will assume familiarity with basic algebraic geometry and topology. Otherwise, the lectures will be self-contained.

We will also discuss the following parallel 'saturated' tensor product decomposition problem.

Problem 2. *Determine the set* $\hat{\Gamma}(s)$ *of* s*-tuples* $(\lambda_1, \ldots, \lambda_s)$ *of dominant rational weights such that the tensor product* $V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s)$ *has a nonzero* $\mathfrak{g}$*-invariant subspace for some positive integer* N*.*