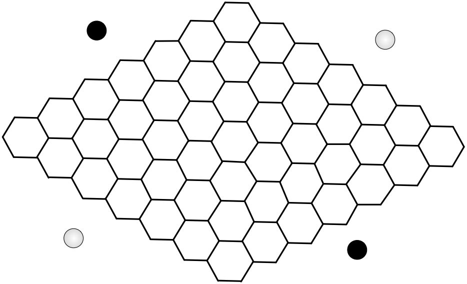
Games and Topology

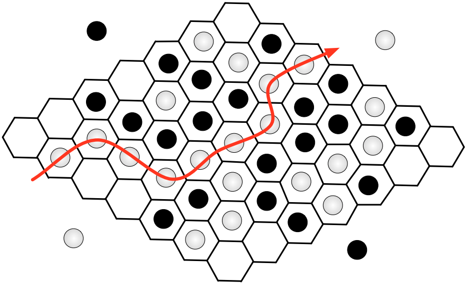
Giovanni Gaiffi
(Università di Pisa)

Scuola Galileiana di Studi Superiori
Padova, 18 marzo 2015

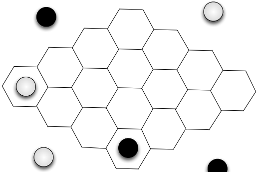
Hex: the game

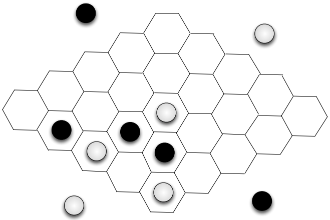
Piet Hein (1905 - 1996), John Nash (1928-)



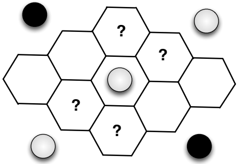


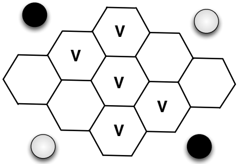
From the Danish newspaper Politiken, 1942:

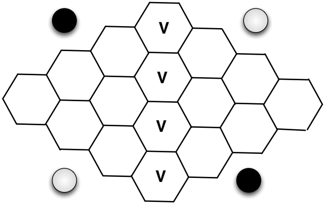


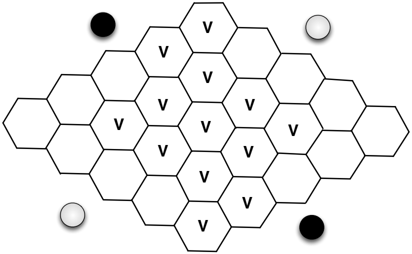


A simple example: the 3×3 case.






V
V
V
V


V
V
V
V
V
V
V
V
V
V
V
V
V

Relevant facts:

1. The game cannot end in a draw;
2. the first player (white) has a winning strategy;
3. the game is actually fun to play since for big boards it doesn't exist a concrete description of the winning strategy.

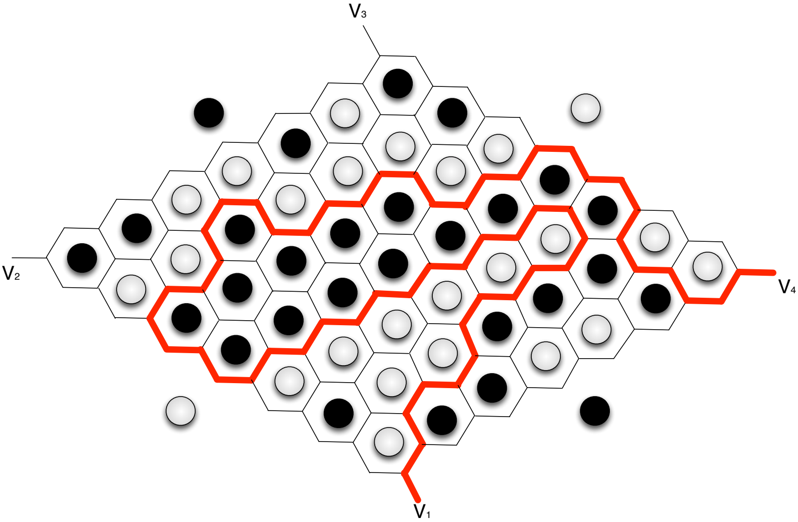
Hex: the topology

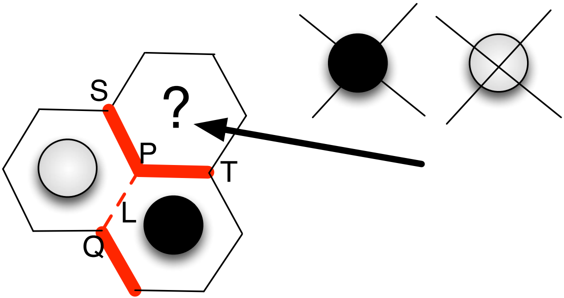
The topological interest of Hex comes from the "no-draw theorem":

Theorem (The Hex theorem)

Let us consider a $n \times n$ Hex board.
If all the tiles of the board are either black or white, then there is either a white path that meets the white boundaries or a black path that meets the black boundaries.

Sketch of a proof:



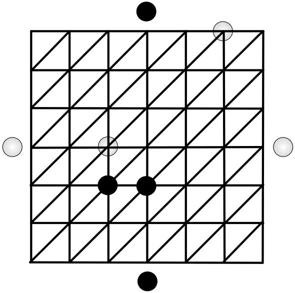

S
?
P
T
L
Q

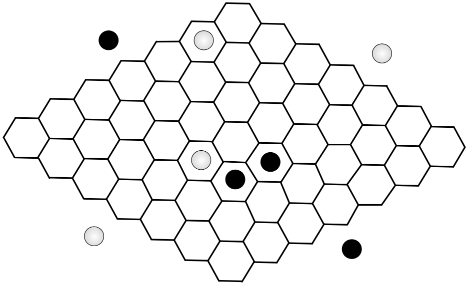
The Hex theorem turns out to be equivalent to the following celebrated theorem:

Theorem (The Brouwer fixed-point theorem)

A continuous mapping $f : Q \to Q$ from the closed unit square into itself has a fixed point,. i.e. there exists a point $x \in Q$ such that $f(x) = x$.

Sketch of a proof that Hex Theorem $\Longrightarrow$ Brouwer Theorem:

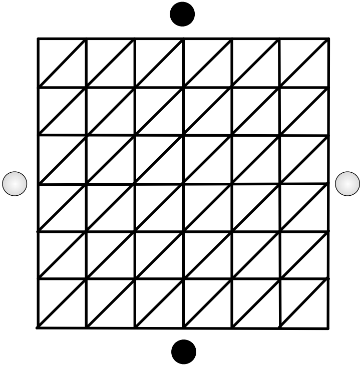




Theorem (equivalent version of the Brouwer fixed-point theorem)

Given a continuous mapping f *from the closed unit square into itself, for any real number* $\epsilon > 0$ *there exists at least a point* x *in the square such that* $|f(x) - x| < \epsilon$*.*

Given $\epsilon > 0$, we can put on the unit square a very thin Hex board so that in particular if x and y are two adjacent vertices then $|x - y| < \frac{\epsilon}{4}$ and moreover, by **uniform continuity**, $|f(x) - f(y)| < \frac{\epsilon}{4}$.
Now we look at the vertices of the board:



If there is a vertex x such that $|f(x) - x| < \epsilon$ our proof is finished.
Otherwise, for every vertex x of the board we have that $f(x)$ “moves away” from x of at least $\epsilon/2$ horizontally or vertically.
We put a white stone on the vertex if the first case happens, otherwise we put a black stone.
So we have filled all the vertices of the board with stones. By the Hex Theorem there is a winning path in the board, say a white path.

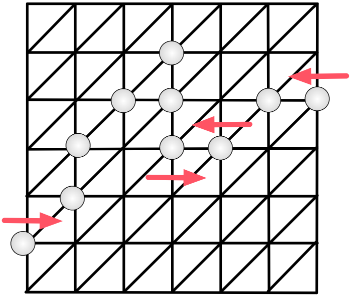
If there is a vertex x such that $|f(x) - x| < \epsilon$ our proof is finished. Otherwise, for every vertex x of the board we have that $f(x)$ “moves away” from x of at least $\epsilon/2$ horizontally or vertically.

We put a white stone on the vertex if the first case happens, otherwise we put a black stone.

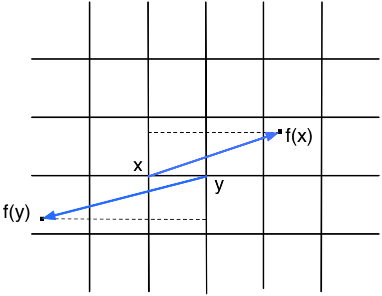
So we have filled all the vertices of the board with stones. By the Hex Theorem there is a winning path in the board, say a white path.

If there is a vertex x such that $|f(x) - x| < \epsilon$ our proof is finished.
Otherwise, for every vertex x of the board we have that $f(x)$ "moves away" from x of at least $\epsilon/2$ horizontally or vertically.
We put a white stone on the vertex if the first case happens, otherwise we put a black stone.
So we have filled all the vertices of the board with stones. By the Hex Theorem there is a winning path in the board, say a white path.

If there is a vertex x such that $|f(x) - x| < \epsilon$ our proof is finished. Otherwise, for every vertex x of the board we have that $f(x)$ "moves away" from x of at least $\epsilon/2$ horizontally or vertically.
We put a white stone on the vertex if the first case happens, otherwise we put a black stone.
So we have filled all the vertices of the board with stones. By the Hex Theorem there is a winning path in the board, say a white path.

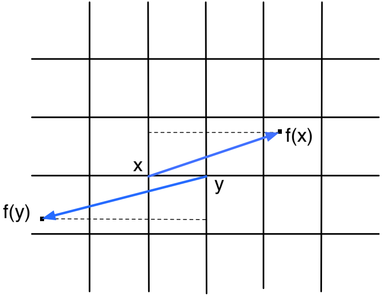


Following the winning path, we must find two adjacent vertices x and y such that $f(x)$ moves away from x horizontally to the right of more than $\frac{\epsilon}{2}$, while $f(y)$ moves away from y horizontally to the left of more than $\frac{\epsilon}{2}$.



It is then immediate to check that $|f(x) - f(y)| > \frac{\epsilon}{4}$ which contradicts uniform continuity and the initial choice of the very thin board.

Following the winning path, we must find two adjacent vertices x and y such that $f(x)$ moves away from x horizontally to the right of more than $\frac{\epsilon}{2}$, while $f(y)$ moves away from y horizontally to the left of more than $\frac{\epsilon}{2}$.



It is then immediate to check that $|f(x) - f(y)| > \frac{\epsilon}{4}$ which contradicts uniform continuity and the initial choice of the very thin board.

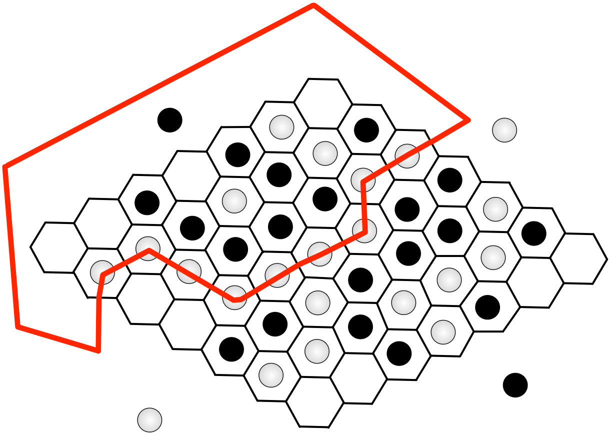
Theorem (The Hex Theorem)

Let us consider a $n \times n$ Hex board.
If all the tiles of the board are either black or white, then there is either a white path that meets the white boundaries or a black path that meets the black boundaries but not both.

The claim but not both is a consequence of another important topological result, the Jordan Curve Theorem.

Theorem (The Jordan Curve Theorem, C. Jordan 1887, O. Veblen 1905.)

Let C be a simple, continuous closed curve in the plane $\mathbb{R}^2$. Then its complement, $\mathbb{R}^2 - C$, consists of exactly two connected components. One of these components is bounded (the interior) and the other is unbounded (the exterior), and the curve C is the boundary of each component. Every path that connects a point P in the interior and a point Q in the exterior intersects C, while if two points are both in the interior (or both in the exterior) there is a path that connects them and does not intersect C.

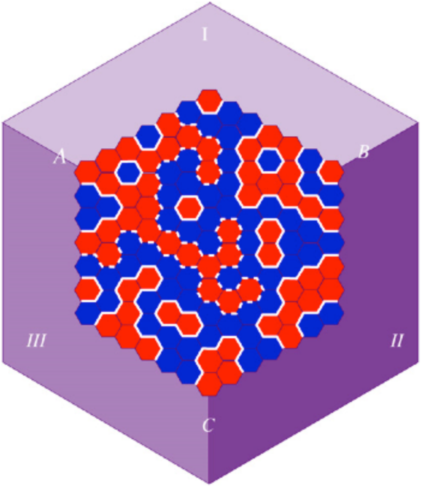


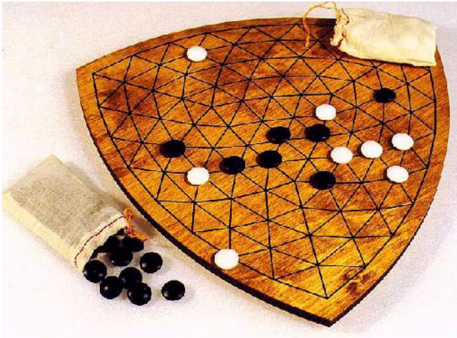
- Good news for game lovers: "Misere" Hex is fun to play, too! Which player has a winning strategy?
- Good news for topology lovers: the Hex Theorem holds also in its n-dimensional version (played on a hypercube, there are n players/colors), and it is equivalent to the n-dimensional Brouwer Fixed Point Theorem.

Remark: in the n-dimensional version there can be winning paths of two or more colors in the same board.

A game inspired by Hex: the Milnor or Y

John Milnor (1931-)



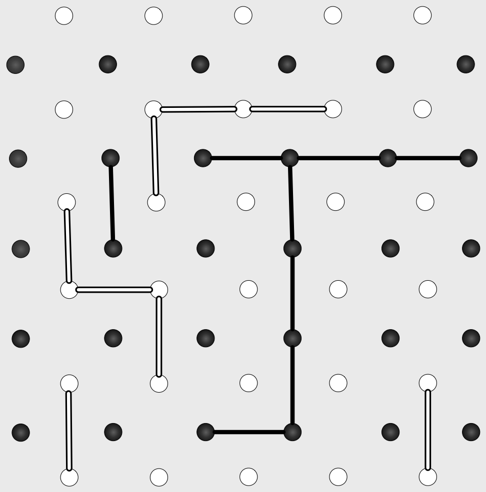


Relevant facts:

1. The game cannot end in a draw (and in a fulfilled board only one of the two colors has a winning “Y”);
2. the first player has a winning strategy.

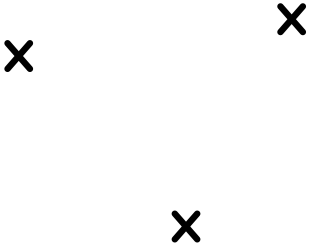
Another game inspired by Hex: the Gale, or Bridge It

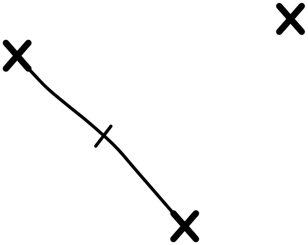
David Gale (1921 - 2008)



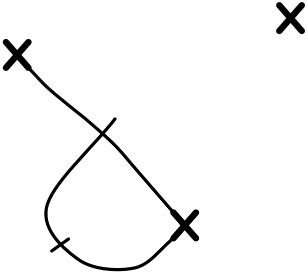
Brussels Sprouts: the game

John Conway (1937 -) and Mike Paterson

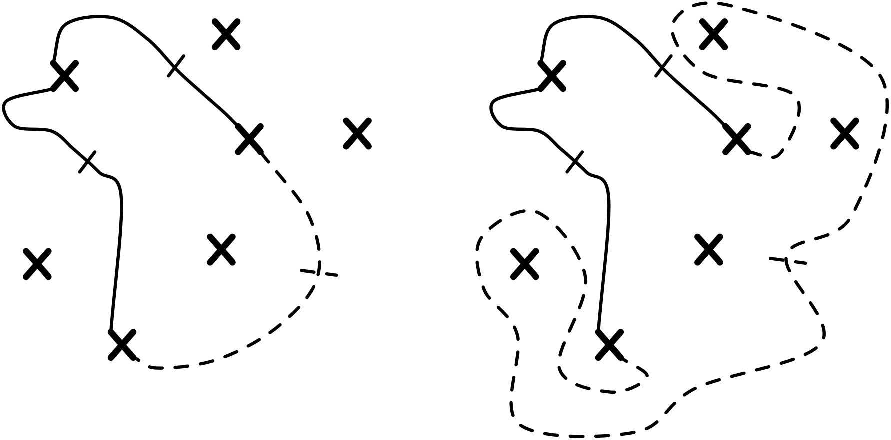




(1)

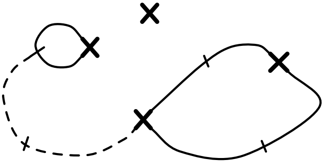


(2)

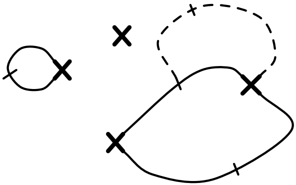


Main question: does this game end in a finite number of moves?

Let r be the number of regions that we see in the picture.
Let i be the number of "islands", i.e., connected components, that we see in the picture.
What happens to $r - i$ after each move?



(a)



(b)

One can prove (using the Jordan Curve Theorem!) that $r - i$ increases by 1 after each move. Therefore, after m moves, we have

$$r - i = 1 - n + m$$

Then, taking into account that one always has $i \geq 1$ and $r \leq 4n$ one finds

$$r - i \leq 4n - 1$$

and therefore

$$m \leq 4n - 1 + n - 1 = 5n - 2$$

The game ends in at most $5n - 2$ moves!

1. Bad news for game lovers: the game ends in exactly $5n - 2$ moves!
2. For topology lovers...

Brussels Sprouts: the topology

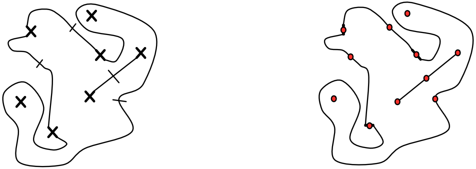
The formula

$$r - i = 1 - n + m$$

can be translated into

$$r - i = 1 + s - p$$

where s and p are respectively the arcs and the points that we see in the picture.



This follows from the equalities $p = n + m$ and $s = 2m$.

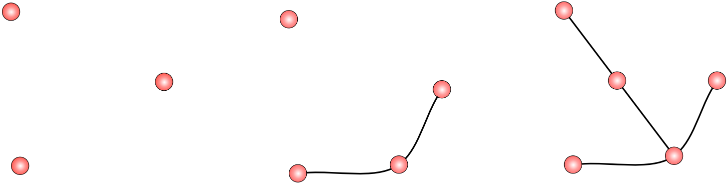
In conclusion:

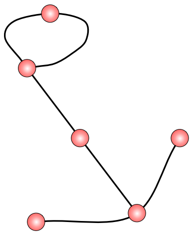
$$r - s + p = 1 + i$$

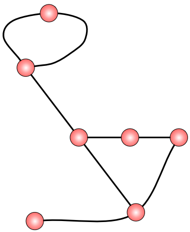
that is the Euler formula for planar graphs, “hidden” in the game.

A variant of Brussels Sprouts: Sprouts

Again invented by Conway and Paterson:







Some facts:

- This game is fun to play: if there are n spots at the beginning, the game ends in m moves, where

$$2n \leq m \leq 3n - 1$$

- There is a conjecture: the first player has a winning strategy if and only if the number n of initial spots divided by 6 leaves a remainder of 3,4, or 5.
- Also the “misere” version of the game is conjectured to have a pattern of length 6...with some initial exceptions.

Some references

- E. Berlekamp, J. Conway, R. Guy, *Winning Ways for your Mathematical Plays*, Vol. I and II, Academic Press, (1982).
- E. Delucchi, G. Gaiffi, L. Pernazza, *Giochi e percorsi matematici*, Springer, 2012.
- D. Gale, *Topological games at Princeton, a mathematical memoir*, Games and Economic Behavior 66 (2009).
- D. Gale, *The game of Hex and the Brouwer Fixed Point Theorem*, Amer. Math. Monthly 86, (1979).
- M. Gardner, *Mathematical Games*, Scientific Amer. 197 (1957).
- M. Gardner, *The second Scientific American Book of Mathematical Puzzles...*, Univ. of Chicago Press, (1987).
- M. Gardner, *New Mathematical Diversions*, MAA (1995).