REAL STRUCTURES OF MODELS OF ARRANGEMENTS

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Abstract. In this paper we will deal with two "hidden" real structures in the theory of models of subspace arrangements. Given a real subspace arrangement $\mathcal{A}$ and its complexification $\mathcal{A}_{\mathbb{C}}$, the first structure is a real De Concini-Procesi model $\widetilde{Y}_{\mathcal{A}}$ that can be seen as the manifold $Re\ Y_{\mathcal{A}_{\mathbb{C}}}$ of (canonical) real points inside the complex De Concini-Procesi model $Y_{\mathcal{A}_{\mathbb{C}}}$. We will study its combinatorial properties by describing it as a quotient of a real model with corners $CY_{\mathcal{A}}$ introduced in [9].

A second structure arises, on the contrary, as an "extension" of $CY_{\mathcal{A}}$, when $\mathcal{A}$ is a Coxeter arrangement. We will "add faces" to $CY_{\mathcal{A}}$ and obtain a convex body (or even a polytope); this gives rise to an interesting new family of "realized" posets which includes for instance Kapranov's permutoassociahedra.

1. Introduction

Let $\mathcal{A}$ be a central subspace arrangement in an euclidean vector space V of dimension n and let us denote its complement by $\mathcal{M}(\mathcal{A})$. In [9] some compactifications for the C^∞ manifold $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ have been described, as a generalization of a construction provided by Kontsevich in his paper on deformation quantization of Poisson manifolds (see [13], and [19] for a related construction).

These compactifications turn out to be C^∞ (non connected) manifolds with corners whose boundary is fully determined by simple combinatorial data. The combinatorics involved in this description has been introduced by De Concini and Procesi in [3] (in which models for complex subspace arrangements have been constructed from the point of view of algebraic geometry), and has been studied from an abstract point of view in [6].

In general we can associate to a real subspace arrangement $\mathcal{A}$ many distinct sets ("building sets") of combinatorial data (see Section 2) which are subspace arrangements with complement $\mathcal{M}(\mathcal{A})$. Given a building set $\mathcal{G}$, the compactification associated to $\mathcal{G}$ is denoted by $CY_{\mathcal{G}}$ and is a "model" for $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ in the following sense:

(1) $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ is embedded in $CY_{\mathcal{G}}$ as an open dense stratum, and all the other strata of $CY_{\mathcal{G}}$ lie in the boundary;
(2) the codimension 1 strata are in a natural bijective correspondence, via a blow-up map, with the elements of $\mathcal{G}$;
(3) combinatorial data encoded by $\mathcal{G}$ allow us to control intersections of closures of strata.

Property (3) above implies that we can fully predict the combinatorics of the boundary of $CY_{\mathcal{G}}$ from the initial data: this can be useful for instance when one applies Stokes' theorem, as in Kontsevich's first example.

In this paper we will single out two real structures which stem from $CY_{\mathcal{G}}$: the first one is a quotient manifold and coincides with the (canonical) real points $Re\ Y_{\mathcal{G}_{\mathbb{C}}}$ of the De Concini - Procesi complex model $Y_{\mathcal{G}_{\mathbb{C}}}$ ($\mathcal{G}_{\mathbb{C}}$ is the complexification of $\mathcal{G}$ -here and from now on the subscript "$\mathbb{C}$" will stand for "complexification of").

The manifold $Re\ \overline{M}_{0,n+1}$, made by the real points of the Mumford - Deligne moduli space of $(n+1)$-pointed stable curves of genus zero, provides us with an interesting example of $Re\ Y_{\mathcal{G}_{\mathbb{C}}}$ (see Section 6), which has been recently studied by Goncharov and Manin ([11]), Ceyhan ([2]), Devadoss ([5]) and Kwon ([15]).

In general, given any building arrangement $\mathcal{G}$, we will describe a differentiable map $\gamma_{\mathcal{G}}$ which goes from $CY_{\mathcal{G}}$ onto $Re\ Y_{\mathcal{G}_{\mathbb{C}}}$ and has fibers whose cardinality depends on the codimension of the boundary strata (see Theorem 5.2). This allows us to control the combinatorial properties of $Re\ Y_{\mathcal{G}_{\mathbb{C}}}$.

Moreover, the map γ_G can be seen as a map of CW- complexes (once $CY_{\mathcal{G}}$ is given the natural CW structure arising from the stratification of the boundary), providing us with an effective method for making homological computations for $Re\ Y_{\mathcal{G}_{\mathbb{C}}}$ (see Section 6).

A second interesting real structure arises in this picture, not as a quotient, but as an "extension" of the models with corners, when we focus on the particular case of a Coxeter arrangement $\mathcal{H}$, i.e., an arrangement in V made by the hyperplanes whose associated reflections are the reflections of a (finite) Coxeter group $G_{\mathcal{H}}$.

Let us consider a building set $\mathcal{G}$ associated to $\mathcal{H}$ and construct the manifold $CY_{\mathcal{G}}$: it has as many connected components as many Coxeter chambers there are, and we can put a diffeomorphic copy of $CY_{\mathcal{G}}$ inside the unit sphere $S(V)$.

As we will show in Section 8, we can add to $CY_{\mathcal{G}}$ some new faces and extend it to the boundary of a convex set. The poset "realized" by the faces of this convex set has a nice combinatorial description (see Section 7.1) controlled by the Coxeter group $G_{\mathcal{H}}$ and by the combinatorial properties of $\mathcal{G}$ and its "nested sets" (see Section 2).

In particular, if $\mathcal{H}$ is the arrangement $\mathcal{A}_{n-1}$ of type A_{n-1}, then $G_{\mathcal{H}}$ is the symmetric group S_n; if we choose the smallest building set associated to it, we recover Kapranov's permutoassociahedron KP_n (see [12]). This allows us to say that we are constructing a family of convex bodies which generalize the combinatorial structure of Kapranov's permutoassociahedra to every Coxeter group and to every building set associated to it (although they are different objects, our posets should be compared with Reiner and Ziegler "Coxeter associahedra" in [18]).

Let us shortly describe the structure of this paper: Sections 2 and 3 are devoted to necessary recallings from De Concini-Procesi papers [3], [4] and from [9]. The smooth manifolds of real points of De Concini-Procesi complex models apper in Section 4 and their quotient relation with the models with corners is investigated in Section 5.

As an application, in Section 6 we compute, via their CW structure, the Euler characteristic of some manifolds of real points (including $Re\ \overline{M}_{0,n+1}$) which arise from Coxeter arrangements.

This introduces us to the last two sections, where the new "Coxeter posets" are defined (Section 7) and realized (Section 8), starting from the models with corners, as convex bodies in an euclidean space.

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2. The combinatorics of building sets and nested sets.

Let us recall some definitions from [3]. We start by a (central) subspace arrangement $\mathcal{A}$ in a real or complex vector space V, and denote by $\mathcal{A}^\perp$ the arrangement formed by the subspaces orthogonal (with respect to the standard scalar product) to the subspaces of $\mathcal{A}$:

$$\mathcal{A}^\perp = \{B^\perp \mid B \in \mathcal{A}\}$$

Then we denote by $\mathcal{C}_{\mathcal{A}}^\perp$ the closure, under the sum, of $\mathcal{A}^\perp$, that is to say, the set of subspaces in V which are sums of subspaces in $\mathcal{A}^\perp$.

Definition 2.1. *Given a subspace $U \in \mathcal{C}_{\mathcal{A}}^\perp$, a decomposition of U is a collection of non zero subspaces $U_1, U_2, \ldots, U_k \in \mathcal{C}_{\mathcal{A}}^\perp$ ($k > 1$) which satisfy the following properties:*

1. $U = U_1 \oplus U_2 \oplus \ldots \oplus U_k$
2. *for every subspace $A \subset U$ in $\mathcal{C}_{\mathcal{A}}^\perp$, we have that $A \cap U_1,\ A \cap U_2, \ldots, A \cap U_k$ lie in $\mathcal{C}_{\mathcal{A}}^\perp$ and $A = (A \cap U_1) \oplus (A \cap U_2) \oplus \ldots \oplus (A \cap U_k)$*

Definition 2.2. *If a subspace in $\mathcal{C}_{\mathcal{A}}^\perp$ does not admit a decomposition, it is called "irreducible". The set of all irreducible subspaces is denoted by $\mathcal{F}_{\mathcal{A}}^\perp$.*

The following proposition essentially says that irreducible subspaces give a decomposition which has the expected "good" properties:

Proposition 2.1. *Every subspace $U \in \mathcal{C}_{\mathcal{A}}^\perp$ has a unique decomposition $U = \oplus_{i=1}^k U_i$ into irreducible subspaces. This is called "the irreducible decomposition" of U. If $A \subset U$ is irreducible, then $A \subset U_i$ for exactly one i.*

In the sequel building sets of subspaces will play a crucial role.

Definition 2.3. *The collection of subspaces $\mathcal{A} \subset V$ is called "building set" if every element C of $\mathcal{C}_{\mathcal{A}}^\perp$ is the direct sum $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ of the set of the maximal elements $G_1, G_2, \ldots, G_k$ of $\mathcal{A}^\perp$ contained in C. We say in this case that $\{G_1, \ldots, G_k\}$ is "the building decomposition of C in $\mathcal{A}^\perp$".*

Remark 2.1. *One can easily see that the "building decomposition of C in $\mathcal{A}^\perp$" is a decomposition in the previous sense.*

Remark 2.2. *The sets $\mathcal{C}_\mathcal{A}$ and $\mathcal{F}_\mathcal{A}$ (according to the notation introduced above, $(\mathcal{C}_\mathcal{A}^\perp)^\perp = \mathcal{C}_\mathcal{A}$ and $(\mathcal{F}_\mathcal{A}^\perp)^\perp = \mathcal{F}_\mathcal{A}$) are building sets. Furthermore, for every building set $\mathcal{A}$, we have $\mathcal{F}_\mathcal{A} \subset \mathcal{A} \subset \mathcal{C}_\mathcal{A}$. Let in fact $A^\perp \in \mathcal{C}_\mathcal{A}^\perp$ be irreducible. Now $A^\perp$ can be decomposed in $\mathcal{A}^\perp$, but then $A^\perp \in \mathcal{A}^\perp$ since $A^\perp$ is irreducible. This proves the first inclusion, the second being trivial. Let now Γ be any subspace arrangement and let $\mathcal{B}$ be a building set such that $\mathcal{C}_\mathcal{B} = \mathcal{C}_\Gamma$. This implies that*

$$\mathcal{F}_\Gamma = \mathcal{F}_\mathcal{B} \subset \mathcal{B} \subset \mathcal{C}_\mathcal{B} = \mathcal{C}_\Gamma$$

Therefore in the family of building sets that have the same intersection lattice as Γ we can always find a minimum and a maximum element with respect to inclusion.

We can now recall the notion of "nested set" (see [3]) which generalizes the one introduced by Fulton and MacPherson in their paper [7] on models of configuration spaces.

Definition 2.4. *Let $\mathcal{K}$ be a building set of subspaces in V. A subset $\mathcal{S} \subset \mathcal{K}$ is called "nested relative to $\mathcal{K}$", or $\mathcal{K}$-nested, if, given any of its subset $\{U_1, \dots, U_k\}$, $k \geq 2$, of pairwise non comparable elements, we have that $\bigcap_{i=1}^k U_i \notin \mathcal{K}$ (or equivalently, $\sum_{i=1}^k U_i^\perp \notin \mathcal{K}^\perp$).*

3. De Concini - Procesi models and real models with corners

3.1. **Definitions.** A model for the complement $\mathcal{M}(\mathcal{G})$ of a complex subspace arrangement $\mathcal{G}$ in $\mathbb{C}^n$, from the point of view of algebraic geometry, is a smooth irreducible variety $Y_\mathcal{G}$ equipped with a proper map $\pi : Y_\mathcal{G} \mapsto \mathbb{C}^n$ which is an isomorphism on the preimage of $\mathcal{M}(\mathcal{G})$ and such that the complement of this preimage is a divisor with normal crossings.

In their paper [3], De Concini and Procesi constructed such models, provided that the set of subspaces $\mathcal{G}$ is building, and computed their cohomology. They also used them to prove that the rational cohomology ring of $\mathcal{M}(\mathcal{G})$ is completely determined by combinatorial data encoded by $\mathcal{G}$.

In [3] arrangements of linear subspaces in $\mathbf{P}(\mathbb{C}^n)$ have also been studied: as a result, a theory has been obtained, which gives compact models and is based on the following construction. Let $\mathcal{G}$ be a building set (we can suppose that it contains $\{0\}$), and let $\mathbf{P}(\mathcal{M}(\mathcal{G}))$ be the complement in $\mathbf{P}(\mathbb{C}^n)$ of the projective subspaces $\mathbf{P}(A)$ ($A \in \mathcal{G}$). Then one considers the map

$$i : \mathbf{P}(\mathcal{M}(\mathcal{G})) \mapsto \mathbf{P}(\mathbb{C}^n) \times \prod_{D \in \mathcal{G} - \{0\}} \mathbf{P}(\mathbb{C}^n/D)$$

where in the first coordinate we have the inclusion and the map from $\mathcal{M}(\mathcal{G})$ to $\mathbf{P}(\mathbb{C}^n/D)$ is the restriction of the canonical projection $\mathbb{C}^n - D \mapsto \mathbf{P}(\mathbb{C}^n/D)$.

Definition 3.1. *The compact model $Y_{\mathcal{G}}$ is obtained by taking the closure of the image of i.*

De Concini and Procesi proved that the complement $\mathcal{D}$ of $\mathbf{P}(\mathcal{M}(\mathcal{G}))$ in $Y_{\mathcal{G}}$ is the union of smooth irreducible divisors $\mathcal{D}_G$ indexed by the elements $G \in \mathcal{G} - \{0\}$. More precisely, if we denote by π the projection onto the first component $\mathbf{P}(\mathbb{C}^n)$, $\mathcal{D}_G$ is equal to the closure of

$$\pi^{-1}\left(\mathbf{P}(G) - \bigcup_{\substack{A \in \mathcal{G} \\ A \cap G \subsetneq G}} \mathbf{P}(A \cap G)\right)$$

It can also be characterized as the unique irreducible component such that $\pi(\mathcal{D}_G) = \mathbf{P}(G)$. A complete characterization of the boundary is provided by the observation that, if we consider a collection $\mathcal{T}$ of subspaces in $\mathcal{G} - \{0\}$, then

$$\mathcal{D}_{\mathcal{T}} \equiv \bigcap_{A \in \mathcal{T}} \mathcal{D}_A$$

is non empty if and only if $\mathcal{T}$ is nested, and in this case $\mathcal{D}_{\mathcal{T}}$ is a smooth irreducible subvariety.

From the point of view of differentiable geometry, the interest in the construction of models of real subspace arrangements has been pointed out by the compact differentiable models of configuration spaces which appear in Kontsevich's paper [13] on deformation quantization of Poisson manifolds.

Kontsevich's compactifications have been shown in [9] to be particular cases of the following more general construction which, starting from a real subspace arrangement $\mathcal{A}$ (we can suppose that it contains $\{0\}$) in $\mathbb{R}^n$, produces C^∞ manifolds with corners.

Let us denote by $S(\mathbb{R}^n)$ the $n-1$-th dimensional unit sphere in $\mathbb{R}^n$, and, for every subspace $A \subset \mathbb{R}^n$, let $S(A) = A \cap S(\mathbb{R}^n)$. Then we can consider the compact manifold

$$K = S(\mathbb{R}^n) \times \prod_{A \in \mathcal{A} - \{0\}} S(A^\perp)$$

and notice that there is an open embedding

$$\phi : \mathcal{M}(\mathcal{A})/\mathbb{R}^+ \longrightarrow K$$

This is obtained as a composition of the section $s : \mathcal{M}(\mathcal{A})/\mathbb{R}^+ \mapsto \mathcal{M}(\mathcal{A})$ provided by

$$s([p]) = \frac{p}{|p|} \in S(\mathbb{R}^n) \cap \mathcal{M}(\mathcal{A})$$

with the map

$$\mathcal{M}(\mathcal{A}) \mapsto S(\mathbb{R}^n) \times \prod_{A \in \mathcal{A} - \{0\}} S(A^\perp)$$

where on each factor we have a well defined projection.

Definition 3.2. *We define $CY_{\mathcal{A}}$ as the closure in K of $\phi(\mathcal{M}(\mathcal{A})/\mathbb{R}^+)$.*

In [9] it has been proven that, when $\mathcal{A}$ is a building set, $CY_{\mathcal{A}}$ is a smooth manifold with corners. It is a differentiable model for $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ in the sense we mentioned in the Introduction: if we denote by $c\pi$ the projection onto the first component $S(\mathbb{R}^n)$, then $c\pi$ is surjective and it is an isomorphism on the preimage of $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$. Furthermore, $c\pi$ establishes a bijective correspondence between the codimension 1 open strata in the boundary of $CY_{\mathcal{A}}$ and the elements of $\mathcal{A} - \{0\}$, as we will see in detail in Section 3.3 .

3.2. **The open charts for $CY_{\mathcal{A}}$.** In this subsection we will recall from [9] the construction of a set of open charts, diffeomorphic to open subsets of $(\mathbb{R}^{\geq 0})^{n-1}$, which cover the manifold with corners $CY_{\mathcal{A}}$.

Remark 3.1. *From now on "nested set" will mean "$\mathcal{A}$-nested set which contains $\{0\}$".*

Let us start by constructing an open covering of $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ by some charts which are associated to the $\mathcal{A}$-nested sets.

Definition 3.3. *Given a subspace $C \subset V$, we define the following two (possibly empty) subspace arrangements.*

(1) $\mathcal{A}_C = \{H \in \mathcal{A} \mid C \subset H\}$
(2) $\mathcal{A}^C = \{B \cap C \mid B \in \mathcal{A} - \mathcal{A}_C\}$

Furthermore, given two subspaces $H, C \subset V$, we will denote by $\mathcal{A}_H^C$ the subspace arrangement $\mathcal{A}_H^C = \{B \cap C \mid B \in \mathcal{A}_H - (\mathcal{A}_C \cap \mathcal{A}_H)\}$.

Let now $\mathcal{S}$ be a nested set in $\mathcal{A}$. We will give a graduation to the elements of $\mathcal{S}^\perp \subset \mathcal{A}^\perp$. Recall that $\mathcal{S}^\perp$ can be represented by a graph, which is an oriented tree, in the following way. The vertices of the tree are labeled by the elements of $\mathcal{S}^\perp$, and the root is $\{0\}^\perp = \mathbb{R}^n$; let then $A^\perp$ and $B^\perp$ be two elements of $\mathcal{S}^\perp$ such that $B^\perp$ "covers" $A^\perp$, i.e., $A^\perp$ is maximal (with respect to inclusion) among the elements of $\mathcal{S}^\perp$ strictly included in $B^\perp$: then we draw an edge which joins the vertices $A^\perp$ and $B^\perp$ and is oriented from $B^\perp$ to $A^\perp$. We say that an element $X^\perp$ of $\mathcal{S}^\perp$ has degree n if it is connected to the root by a n-edges oriented path.

Definition 3.4. *Given a nested set $\mathcal{S}$ and a vertex $A^\perp$ of degree n in the graph associated to $\mathcal{S}^\perp$, we denote by $\mathcal{S}^{A^\perp}$ the (possibly empty) set of the elements "covered" by $A^\perp$. Furthermore, we denote by $\mathcal{S}_\cap^{A^\perp}$ the common intersection of $A^\perp$ and of all the subspaces which are orthogonal to the subspaces in $\mathcal{S}^{A^\perp}$ (we put $\mathcal{S}_\cap^{A^\perp} = A^\perp$ if $\mathcal{S}^{A^\perp}$ is empty).*

We can now associate to $\mathcal{S}$ an open set $\widehat{U}_{\mathcal{S}}$. It is constructed as a product of open sets, according to the following algorithm. We provide an open manifold in correspondence with every element in $\mathcal{S}^\perp$. The open manifold which corresponds to the root V is :

$$N_V = \mathcal{M}_{\mathcal{S}_\cap^V}(\mathcal{A}^{\mathcal{S}_\cap^V}) \cap S(V).$$

Remark 3.2. *Here and from now on we use the following notation: if $\mathcal{A}$ is a subspace arrangement whose elements are contained in a subspace F of V, $\mathcal{M}_F(\mathcal{A})$ will denote the complement of $\mathcal{A}$ in F.*

Notice that, if $\mathcal{S}^\perp = \{V\}$, we have $N_V = \mathcal{M}(\mathcal{A}) \cap S(V)$.
Now, given an element $A^\perp \in \mathcal{S}^\perp$ we construct

$$N_{A^\perp} = \mathcal{M}_{\mathcal{S}_\cap^{A^\perp}}(\mathcal{A}_A^{\mathcal{S}_\cap^{A^\perp}}) \cap S(V)$$

Then, for any $A^\perp$ in $\mathcal{S}^\perp$, we consider a "small" positive real number $\varepsilon_{A^\perp}$, and we can define $\widehat{U}_{\mathcal{S}}$ as

$$\widehat{U}_{\mathcal{S}} = N_V \times \prod_{A^\perp \in \mathcal{S}^\perp - \{V\}} N_{A^\perp} \times (0, \varepsilon_{A^\perp})$$

Choosing in every space N_V or $N_{A^\perp}$ a ball $\rho(N_V)$ or $\rho(N_{A^\perp})$ we obtain an open subset $\widehat{U}_{\mathcal{S}}(\rho)$ of $\widehat{U}_{\mathcal{S}}$. We can embed $\widehat{U}_{\mathcal{S}}(\rho)$ in $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ as a chart using the following map $\widehat{\tau}$:

$$(p_V, \ldots, p_{A^\perp}, t_{A^\perp}, \ldots) \overset{\widehat{\tau}}{\mapsto} [p_V + \text{the point in } (\mathcal{S}_\cap^V)^\perp \text{ such that}, \forall\, A^\perp \in \mathcal{S}^\perp - \{V\},$$
$$\text{its orthogonal projection to } \mathcal{S}_\cap^{A^\perp} \text{ is } t_{T_1^\perp} t_{T_2^\perp} \cdots t_{A^\perp} p_{A^\perp}]$$

where $T_1^\perp, T_2^\perp \ldots$ are all the internal vertices in the path which connects V to $A^\perp$.

The map $\widehat{\tau}$ is a well defined embedding provided that the balls $\rho(N_V)$, $\rho(N_{A^\perp})$ and the numbers $\varepsilon_{A^\perp}$ are sufficiently small. Therefore we have an open atlas $\widehat{\mathcal{U}} = \bigcup_{\mathcal{S}} \widehat{U}_{\mathcal{S}}(\rho)$ which covers $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ (we recall that $\mathcal{S}$ ranges over all the nested sets in $\mathcal{A}$ which contain $\{0\}$ and ρ over all possible suitable collections of balls $\rho(N_V)$, $\rho(N_{A^\perp})$).

If we allow the real numbers $\varepsilon_{A^\perp}$ to be 0, we have the corresponding new space

$$U_{\mathcal{S}}(\rho) = \rho(N_V) \times \prod_{A^\perp \in \mathcal{S}^\perp - \{V\}} \rho(N_{A^\perp}) \times [0, \varepsilon_{A^\perp})$$

which is diffeomorphic to an open set of a simplicial cone $(\mathbb{R}^{\geq 0})^{n-1}$.

Remark 3.3. *In the sequel we will often write $\widehat{U}_{\mathcal{S}}$ and $U_{\mathcal{S}}$ instead of $\widehat{U}_{\mathcal{S}}(\rho)$ and $U_{\mathcal{S}}(\rho)$, the choice of a collection of balls $\rho(N_V)$, $\rho(N_{A^\perp})$ being implicit.*

It turns out (see [9]) that the open embedding $\widehat{\tau} : \widehat{U}_{\mathcal{S}} \mapsto \mathcal{M}(\mathcal{A})/\mathbb{R}^+$ can be extended by continuity to a map $\tau : U_{\mathcal{S}} \mapsto CY_{\mathcal{A}}$.

Moreover, we have that the charts $U_{\mathcal{S}}$ give rise to a C^∞ atlas which gives $CY_{\mathcal{A}}$ the structure of a C^∞ manifold with corners.

In the sequel we will refer to the following algorithm, which was used in [9] to show that the charts $U_{\mathcal{S}}$ cover $CY_{\mathcal{A}}$.

Let us view a point p in the boundary of $\mathcal{M}(\mathcal{A})/\mathbb{R}^+ \subset K = S(\mathbb{R}^n) \times \prod_{A \in \mathcal{A} - \{0\}} S(A^\perp)$ as the limit of a path $\delta = \delta(t) : [0,1) \mapsto \mathcal{M}(\mathcal{A})/\mathbb{R}^+$. We will associate to δ a nested set $\mathcal{S}$ such that $U_{\mathcal{S}}$ contains p. If we look at this path in $S(V)$, we can choose the minimal subspace B in the intersection lattice of $\mathcal{A}$ such that δ converges to a point in B. Let $B^\perp = B_1^\perp \oplus \cdots \oplus B_\varrho^\perp$ be the direct sum of $B^\perp$ in terms of the maximal elements of $\mathcal{A}^\perp$ which are included in $B^\perp$ (this is possible since $\mathcal{A}$ is building). Then $B = B_1 \cap \cdots \cap B_\varrho$ and our first step in the construction of $\mathcal{S}$ consists in putting $\mathcal{S} = \{\{0\}, B_1, \ldots, B_\varrho\}$. Now, for every $1 \leq i \leq \varrho$ let us consider the projection $\delta_{B_i^\perp}$ of δ to $S(B_i^\perp)$ (it is well defined since $\delta \subset \mathcal{M}(\mathcal{A})/\mathbb{R}^+$). Let v_i be the limit of the vector $\delta_{B_i^\perp}(t)$ as $t \to 1$. If v_i does not lie in any subspace of the intersection lattice of $\mathcal{A}_{B_i}$, we will not add any element to $\mathcal{S}$. Otherwise, let C_i be the minimal subspace in the intersection lattice of $\mathcal{A}_{B_i}$ such that v_i belongs to C_i. Then we can decompose $C_i^\perp$ as $C_i^\perp = C_{i1}^\perp \oplus \cdots \oplus C_{i\mu_i}^\perp$ (notice that $\mathcal{A}_{B_i}$ is building). After doing this for every i $(1 \leq i \leq \varrho)$, our second step in the construction of $\mathcal{S}$ is to put $\mathcal{S} = \{\{0\}, B_1, \ldots, B_\varrho, \ldots, C_{i1}, \ldots, C_{i\mu_i}, \ldots\}$. We can now project $\delta_{B_i^\perp}$ to $C_{ij}^\perp$ for every i and j and continue. It turns out that $p \in U_\mathcal{S}(\rho)$ with $\mathcal{S}$ as above and obvious ρ; therefore

$$\mathcal{U} = \bigcup_{\mathcal{S},\rho} U_\mathcal{S}(\rho) = CY_\mathcal{A}.$$

3.3. **Recalls of results on the boundary of $CY_\mathcal{A}$.** As we recalled in Section 3.1, given a building arrangement $\mathcal{A}$ (which contains $\{0\}$) in $\mathbb{R}^n$, the boundary $C\mathcal{D} = c\pi^{-1}\left(\bigcup_{A \in \mathcal{A}} S(A)\right)$ of $CY_\mathcal{A}$ is the union of some codimension 1 manifolds with corners which correspond to the elements of $\mathcal{A} - \{0\}$. More precisely, if $A \in \mathcal{A} - \{0\}$, its associated boundary component is

$$C\mathcal{D}_A = \overline{c\pi^{-1}\left(S(A) - \bigcup_{B \in \mathcal{A}^A} S(B)\right)}$$

Let us now state some results from [9] on the combinatorial structure of the boundary.

Theorem 3.1. *The boundary component $C\mathcal{D}_A$ is equal to the closure in K of*

$$\mathcal{U} = \bigcup_\rho (U_{\{\{0\},A\}}(\rho) \cap \{t_{A^\perp} = 0\}).$$

Moreover $C\mathcal{D}_A$ is a manifold with corners of the following type

$$C\mathcal{D}_A \cong CY_{\mathcal{A}_A^{A^\perp}} \times CY_{\mathcal{A}^A}$$

and its internal points coincide with $\mathcal{U}$.

This theorem can be generalized to a result on the intersection of any number of boundary components $C\mathcal{D}_A$:

Theorem 3.2. *Let $\mathcal{T}$ be a subset of $\mathcal{A}$ which includes $\{0\}$; then*

$$C\mathcal{D}_{\mathcal{T}} = \bigcap_{B \in \mathcal{T}-\{0\}} C\mathcal{D}_B$$

is non empty if and only if $\mathcal{T}$ is nested in $\mathcal{A}$. Moreover we have that $C\mathcal{D}_{\mathcal{T}}$ is a manifold with corners whose internal points are described by the local equations

$$\bigcup_{\rho} \left(U_{\mathcal{T}}(\rho) \cap \bigcap_{B \in \mathcal{T}-\{0\}} \{t_{B^\perp} = 0\} \right).$$

We can also describe $C\mathcal{D}_{\mathcal{T}}$ as a product of real models since, for $A \in \mathcal{T}$, the sets $\mathcal{A}_A^{\mathcal{T}_\cap^{A^\perp}}$ are building and

$$C\mathcal{D}_{\mathcal{T}} \cong \prod_{A \in \mathcal{T}} CY_{\mathcal{A}_A^{\mathcal{T}_\cap^{A^\perp}}}$$

4. Real points of De Concini -Procesi models

In this section we will deal with complexified arrangements and we will focus on the manifold provided by the real points of a complex De Concini-Procesi model.

Let us start by considering the natural embedding of $\mathbf{P}(\mathbb{R}^n)$ into $\mathbf{P}(\mathbb{C}^n)$. Given a subspace $A \subset \mathbb{R}^n$, this embedding induces a natural embedding of $\mathbf{P}(\mathbb{R}^n/A)$ into $\mathbf{P}(\mathbb{C}^n/A_{\mathbb{C}})$:

$$[(a_1, a_2, \ldots, a_n) + A] \mapsto [(a_1, a_2, \ldots, a_n) + A_{\mathbb{C}}]$$

where the expression inside brackets $[(a_1, a_2, \ldots, a_n)+A]$ (resp. $[(a_1, a_2, \ldots, a_n)+A_{\mathbb{C}}]$) represents, a basis of $\mathbb{R}^n$ being fixed, the projective coordinates of a point in $\mathbf{P}(\mathbb{R}^n/A)$ (resp. in $\mathbf{P}(\mathbb{C}^n/A_{\mathbb{C}})$).

Therefore, in our hypothesis of a real subspace arrangement $\mathcal{A}$ which is building and contains $\{0\}$, we have an embedding

$$\theta \, : \, \mathbf{P}(\mathbb{R}^n) \times \prod_{D \in \mathcal{A}-\{0\}} \mathbf{P}(\mathbb{R}^n/D) \mapsto \mathbf{P}(\mathbb{C}^n) \times \prod_{D_{\mathbb{C}} \in \mathcal{A}_{\mathbb{C}}-\{0\}} \mathbf{P}(\mathbb{C}^n/D_{\mathbb{C}})$$

There is an interesting real De Concini-Procesi model embedded in

$$\mathbf{P}(\mathbb{R}^n) \times \prod_{D \in \mathcal{A}-\{0\}} \mathbf{P}(\mathbb{R}^n/D)$$

In fact the De Concini - Procesi construction can be performed also in the real case (see [3]): one considers the embedding

$$\mathbf{P}_{\mathbb{R}}(\mathcal{M}(\mathcal{A})) \mapsto \mathbf{P}(\mathbb{R}^n) \times \prod_{D \in \mathcal{A}-\{0\}} \mathbf{P}(\mathbb{R}^n/D)$$

and obtains (by taking the closure) a smooth real manifold $\widetilde{Y}_{\mathcal{A}}$ whose boundary is characterized in the same way as in the complex case (see Section 3.1).

The map θ allows us to compare this real model with the real points $Re\ Y_{\mathcal{A}_\mathbb{C}}$ of $Y_{\mathcal{A}_\mathbb{C}}$.

We notice that the real points in $\mathbf{P}(\mathbb{C}^n) \times \prod_{D_\mathbb{C} \in \mathcal{A}_\mathbb{C} - \{0\}} \mathbf{P}(\mathbb{C}^n/D_\mathbb{C})$ are those points that have in each factor a real representative for their projective coordinates. In other words, let x be a real point and let us denote by π and by $\pi_{D_\mathbb{C}}$ the projection onto the first factor and onto $\mathbf{P}(\mathbb{C}^n/D_\mathbb{C})$ respectively. Then we have that, after choosing a $\mathbb{C}$-basis of $\mathbb{C}^n$ made by real vectors, a real point x satisfies $\pi(x) = [(x_1, x_2, \ldots, x_n)]$ with $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n - \{0\}$ and, for every $D \in \mathcal{A} - \{0\}$, $\pi_{D_\mathbb{C}}(x) = [(a_1, a_2, \ldots, a_n) + D_\mathbb{C}]$ with $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n - D$.

This corresponds to considering the natural complex structure in $\mathbf{P}(\mathbb{C}^n)$ and in $\mathbf{P}(\mathbb{C}^n/D_\mathbb{C})$ and taking the points which are fixed by complex conjugation.

Theorem 4.1. *The real points $Re\ Y_{\mathcal{A}_\mathbb{C}}$ of $Y_{\mathcal{A}_\mathbb{C}}$ coincide with $\theta(\widetilde{Y}_\mathcal{A})$.*

Proof.
We know that $\mathbf{P}_\mathbb{R}(\mathcal{M}(\mathcal{A}))$ is embedded as an open set in $\widetilde{Y}_\mathcal{A}$. As a first step, let us check that $\theta(\mathbf{P}_\mathbb{R}(\mathcal{M}(\mathcal{A})))$ coincides with the real points in $\mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C})) \subset Y_{\mathcal{A}_\mathbb{C}}$. In fact, let x be a real point in $\mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C})) \subset Y_{\mathcal{A}_\mathbb{C}}$. This means that, using the notation introduced above, x is completely determined by its projection $\pi(x)$ to the first factor and that, in a basis given by real vectors, $\pi(x)$ can be expressed in projective coordinates as $[(x_1, \ldots, x_n)]$ with $(x_1, x_2, \ldots, x_n) \in \mathcal{M}(\mathcal{A}) \subset \mathbb{R}^n$.

Now the point $(x_1, x_2, \ldots, x_n) \in \mathcal{M}(\mathcal{A}) \subset \mathbb{R}^n$ projects to a point $x' \in \mathbf{P}_\mathbb{R}(\mathcal{M}(\mathcal{A})) \subset \widetilde{Y}_\mathcal{A}$ and, by definition of θ, $\theta(x') = x$.

Since $\widetilde{Y}_\mathcal{A}$ is compact, so is $\theta(\widetilde{Y}_\mathcal{A})$ which in particular is closed. This implies that, being $\widetilde{Y}_\mathcal{A} = \overline{\mathbf{P}_\mathbb{R}(\mathcal{M}(\mathcal{A}))}$, we have $\theta(\widetilde{Y}_\mathcal{A}) = \overline{\theta(\mathbf{P}_\mathbb{R}(\mathcal{M}(\mathcal{A})))}$.

As we have seen, $\overline{\theta(\mathbf{P}_\mathbb{R}(\mathcal{M}(\mathcal{A})))} = \overline{Re\ \mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))}$, therefore it only remains to prove that $Re\ Y_{\mathcal{A}_\mathbb{C}} = \overline{Re\ \mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))}$. One inclusion is trivial: in fact $Re\ Y_{\mathcal{A}_\mathbb{C}}$ is a closed subset of $Y_{\mathcal{A}_\mathbb{C}}$ (it is the intersection with the locus of points fixed by complex conjugation); since it includes $Re\ \mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))$ this implies $\overline{Re\ \mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))} \subset Re\ Y_{\mathcal{A}_\mathbb{C}}$.

Now, if p is a point in $Re\ Y_{\mathcal{A}_\mathbb{C}}$, then p is the limit of a sequence of points $\{p_n\} \subset \mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))$. We will show that it is possible to modify $\{p_n\}$ to obtain a sequence of real points in $\mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))$ that converges to p.

We start by associating to $\{p_n\}$ a nested set $\mathcal{S} \subset \mathcal{A}$ ($\mathcal{S}$ contains $\{0\}$), according to the algorithm described at the end of Section 3.2 (with straightforward modifications since this is the complex case and we are dealing with a sequence). We observe that, by construction of $\mathcal{S}$, if we know the projection $\pi_{B_\mathbb{C}}(p)$ of p to $\mathbf{P}(\mathbb{C}^n/B_\mathbb{C})$ for every $B \in \mathcal{S}$, we know the projection $\pi_{A_\mathbb{C}}(p)$ to $\mathbf{P}(\mathbb{C}^n/A_\mathbb{C})$ for every $A \in \mathcal{A}$.

Therefore, it is sufficient to find a sequence $\{x_n\} \subset Re\ \mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))$ such that, for every $B \in \mathcal{S}$, $\pi_{B_\mathbb{C}}(x_n) \to \pi_{B_\mathbb{C}}(p)$.

Then we can find a basis $b = \{b_1, b_2, \ldots, b_n\}$ of $\mathbb{R}^n$ adapted to $\mathcal{S}$, according to the following definition.

Definition 4.1. *We say that a basis $b = \{b_1, b_2, \ldots, b_n\}$ of $\mathbb{R}^n$ is adapted to a nested set $\mathcal{S}$ if:*
1) Given any $B^\perp \in \mathcal{S}^\perp$, the elements $b_{i_1}, b_{i_2}, \ldots b_{i_k}$ of b which belong to $B^\perp$ form a basis of $B^\perp$.
2) Let $B^\perp \in \mathcal{S}^\perp$ and let $C_1^\perp, \ldots, C_\nu^\perp$ be the elements of $\mathcal{S}^\perp$ which are covered by $B^\perp$. Then the elements $b_{i_1}, b_{i_2}, \ldots b_{i_s}$ of b which belong to $B^\perp$ but do not belong to any $C_i^\perp$'s are orthogonal to the sum $\sum_{i=1}^{\nu} C_i^\perp$ (that is to say, they belong to $\bigcap_{i=1}^{\nu} C_i$).

Now, let $A_1^\perp, \ldots, A_s^\perp$ be the smallest (with respect to inclusion) elements in $\mathcal{S}^\perp$: we will start our construction of $\{x_n\}$ by determining the projections $\pi_{A_{1\mathbb{C}}}(x_n), \ldots, \pi_{A_{s\mathbb{C}}}(x_n)$.

Remark 4.1. *Here and in similar expressions where complexification is involved we may omit a parenthesis, that is to say, we may write $\pi_{A_{1\mathbb{C}}}$ instead that $\pi_{(A_1)_\mathbb{C}}$, $\mathcal{A}_{A_j\mathbb{C}}$ instead that $(\mathcal{A}_{A_j})_\mathbb{C}$ and so on..*

For every $1 \le j \le s$ we know, by construction of $\mathcal{S}$ and by minimality of $A_1^\perp, \ldots, A_s^\perp$, that $\pi_{A_{j\mathbb{C}}}(p_n) \to \pi_{A_{j\mathbb{C}}}(p)$ and $\pi_{A_{j\mathbb{C}}}(p)$ belongs to the complement of the arrangement $\pi_{A_{j\mathbb{C}}}(\mathcal{A}_{A_j\mathbb{C}})$.

Now we can consider in $\mathbf{P}(\mathbb{C}^n/A_{j\mathbb{C}})$ the real basis of $\mathbb{C}^n/A_{j\mathbb{C}}$ made by the projections of the vectors b_t in $b \cap A_j^\perp$.

With respect to the associated projective coordinates $[(a_1, \ldots, a_{dim\ A_j^\perp})]$, we can cover $\mathbf{P}(\mathbb{C}^n/A_{j\mathbb{C}})$ with the canonical affine charts $U_i \equiv \{a_i = 1\}$ $(i = 1, \ldots, dim\ A_j^\perp)$.

There is a chart, say U_1, which $\pi_{A_{j\mathbb{C}}}(p)$ belongs to, and, since $p \in Re\ Y_{\mathcal{A}_\mathbb{C}}$, in U_1 $\pi_{A_{j\mathbb{C}}}(p)$ has real coordinates. Therefore definitively the projections $\pi_{A_{j\mathbb{C}}}(p_n)$ belong to U_1 and in U_1 we have that $Re\ \pi_{A_{j\mathbb{C}}}(p_n) \to \pi_{A_{j\mathbb{C}}}(p)$.

Then we start our construction of $\{x_n\}$ by requesting that, for every $1 \le j \le s$, $\pi_{A_{j\mathbb{C}}}(x_n) = Re\ \pi_{A_{j\mathbb{C}}}(p_n)$.

Remark 4.2. *Here we may need to extract a subsequence from p_n. In fact, since we want that $\{x_n\} \subset \mathbf{P}(\mathcal{M}(\mathcal{A}_\mathbb{C}))$, we ask that the projections $\pi_{A_{j\mathbb{C}}}(x_n) = Re\ \pi_{A_{j\mathbb{C}}}(p_n)$ do not belong to any subspace of the arrangement $\pi_{A_{j\mathbb{C}}}(\mathcal{A}_{A_j\mathbb{C}})$. But this condition is definitely satisfied otherwise their limit $\pi_{A_{j\mathbb{C}}}(p)$ would belong to a subspace of the arrangement $\pi_{A_{j\mathbb{C}}}(\mathcal{A}_{A_j\mathbb{C}})$, which contradicts the construction of $\mathcal{S}$.*

Now, let us suppose that there is an element D in $\mathcal{S}$ such that $D^\perp$ is not minimal in $\mathcal{S}^\perp$, but it would be if we deleted $A_1^\perp, \ldots, A_s^\perp$. To fix notations, let us suppose that $A_1^\perp, A_2^\perp \ldots, A_m^\perp \subset D^\perp$.

We can use in $\mathbf{P}(\mathbb{C}^n/D_\mathbb{C})$ the projective coordinates associated to the real basis of $\mathbb{C}^n/D_\mathbb{C}$ made by the projections of the vectors b_t in $b \cap D^\perp$.

We know that, by construction of $\mathcal{S}$, $\pi_{D_{\mathbb{C}}}(p)$ belongs to

$$\mathbf{P}(A_{1\mathbb{C}} \cap A_{2\mathbb{C}} \cap \cdots \cap A_{m\mathbb{C}}/D_{\mathbb{C}})$$

Therefore, by definition of adapted basis, the projective coordinates of $\pi_{D_{\mathbb{C}}}(p)$ which are related to the vectors b_r in $\left(A_1^{\perp} + A_2^{\perp} + \cdots + A_m^{\perp}\right) \cap b$ are equal to 0.

As before we can consider an affine chart of type U_i (one which $\pi_{D_{\mathbb{C}}}(p)$ belongs to) and focus on the points $Re\ \pi_{D_{\mathbb{C}}}(p_n)$ in U_i.

Then we substitute in $Re\ \pi_{D_{\mathbb{C}}}(p_n)$ the coordinates associated to the vectors in $\left(A_1^{\perp} + A_2^{\perp} + \cdots + A_m^{\perp}\right) \cap b$ (which are equal to zero), with the coordinates of a point such that, for every $i = 1, \dots, m$, its projection to $\mathbb{C}^n/A_{i\mathbb{C}}$ is $\frac{1}{n}||\pi_{A_{i\mathbb{C}}}(x_n)||$ (here "$||\ ||$" means that we are considering the real unit representative for $\pi_{A_{i\mathbb{C}}}(x_n)$) .

These modified points provides us our second step in the construction of the sequence $\{x_n\}$: we request that they coincide with $\pi_{D_{\mathbb{C}}}(x_n)$ and therefore by construction $\pi_{D_{\mathbb{C}}}(x_n) \to \pi_{D_{\mathbb{C}}}(p)$.

Remark 4.3. *Also in this second step we may need to extract a subsequence, to avoid that* $\pi_{D_{\mathbb{C}}}(x_n) \in \pi_{D_{\mathbb{C}}}(\mathcal{A}_{D\mathbb{C}})$.

If by absurd it was not possible to extract this subsequence, then there would be at least a subsequence included in a subspace $\pi_{D_{\mathbb{C}}}(L_{\mathbb{C}})$ *of* $\pi_{D_{\mathbb{C}}}(\mathcal{A}_{D\mathbb{C}})$. *Therefore, also* $\pi_{D_{\mathbb{C}}}(p)$ *would belong to* $\pi_{D_{\mathbb{C}}}(L_{\mathbb{C}})$. *But, by construction of* $\mathcal{S}$, *this implies that we would have* $L^{\perp} \subset A_m^{\perp}$ *for a certain* m.

Therefore $\pi_{A_{m\mathbb{C}}}(\pi_{D_{\mathbb{C}}}(x_n))$ *would have a subsequence in* $\pi_{A_{m\mathbb{C}}}(L_{\mathbb{C}})$. *But, by construction of* $\pi_{D_{\mathbb{C}}}(x_n)$, *for every* n, $\pi_{A_{m\mathbb{C}}}(\pi_{D_{\mathbb{C}}}(x_n))$ *is equal to* $\pi_{A_{m\mathbb{C}}}(x_n)$; *this, in its turn belongs by construction to the complement of* $\pi_{A_m}(\mathcal{A}_{A_m\mathbb{C}})$ *and we have a contradiction.*

In this way we can construct, step by step, our sequence $\{x_n\} \subset Re\ \mathbf{P}(\mathcal{M}(\mathcal{A}_{\mathbb{C}}))$ which converges to p.

■

Corollary 4.2. *Let* $\mathcal{S}$ *be a nested set which contains* $\{0\}$ *and* $\mathcal{S}_{\mathbb{C}}$ *be its complexification. Then, if* $\widetilde{\mathcal{D}}_{\mathcal{S}}$ *and* $\mathcal{D}_{\mathcal{S}_{\mathbb{C}}}$ *are the boundary components associated to* $\mathcal{S}$ *in* $\widetilde{Y}_{\mathcal{A}}$ *and* $Y_{\mathcal{A}_{\mathbb{C}}}$ *respectively, we have* $Re\ \mathcal{D}_{\mathcal{S}_{\mathbb{C}}} = \theta(\widetilde{\mathcal{D}}_{\mathcal{S}})$

5. Combinatorics of a map from $CY_{\mathcal{A}}$ to $Re\ Y_{\mathcal{A}_{\mathbb{C}}}$

Let us consider as above a real building arrangement $\mathcal{A}$ (containing $\{0\}$) in $\mathbb{R}^n$. In this section we want to compare the real compact model with corners $CY_{\mathcal{A}}$ with the real points of $Y_{\mathcal{A}_{\mathbb{C}}}$.

We start by focussing on the combinatorial properties of a surjective map $\Gamma\ :\ CY_{\mathcal{A}} \mapsto \widetilde{Y}_{\mathcal{A}}$.

The model CY_A is embedded in

$$K = S(\mathbb{R}^n) \times \prod_{A \in \mathcal{A} - \{0\}} S(A^{\perp})$$

while $\widetilde{Y}_{\mathcal{A}}$ is embedded inside

$$K' = \mathbf{P}(\mathbb{R}^n) \times \prod_{D \in \mathcal{A} - \{0\}} \mathbf{P}(\mathbb{R}^n/D)$$

Now, given any $A \in \mathcal{A}$, we can consider the natural isomorphism between $A^{\perp}$ and $\mathbb{R}^n/A$ given by the projection.

As a consequence of this identification, there is a map Γ' from K to K' which coincides on each factor $S(A^{\perp})$ with the $2 \mapsto 1$ projection $S(A^{\perp}) \mapsto \mathbf{P}(\mathbb{R}^n/A)$ (in particular this means that on the first factor we are considering the projection $S(\mathbb{R}^n) \mapsto \mathbf{P}(\mathbb{R}^n)$).

Proposition 5.1. *If we restrict Γ' to $CY_{\mathcal{A}}$ we obtain a surjective map*

$$\Gamma \,:\, CY_{\mathcal{A}} \mapsto \widetilde{Y}_{\mathcal{A}}$$

Proof. This follows immediately from the observation that a point in $\mathcal{M}(\mathcal{A})/\mathbb{R}^+ \subset CY_{\mathcal{A}}$ (resp. $\mathbf{P}_{\mathbb{R}}(\mathcal{M}(\mathcal{A})) \subset \widetilde{Y}_{\mathcal{A}}$) is completely determined by its projection to the first factor $S(\mathbb{R}^n)$ (resp. $\mathbf{P}(\mathbb{R}^n)$). Therefore the map Γ' is $2 \to 1$ from $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$ to $\mathbf{P}_{\mathbb{R}}(\mathcal{M}(\mathcal{A}))$. This implies that $\Gamma(CY_{\mathcal{A}}) = \widetilde{Y}_{\mathcal{A}}$ since $CY_{\mathcal{A}}$ is compact and $\widetilde{Y}_{\mathcal{A}}$ is the closure of $\mathbf{P}_{\mathbb{R}}(\mathcal{M}(\mathcal{A}))$. ■

If we compose Γ with the map θ described in the preceding section, we have a smooth map from $CY_{\mathcal{A}}$ to *Re* $Y_{\mathcal{A}_{\mathbb{C}}}$, which we will denote by γ. We can now state our main theorem concerning these maps:

Theorem 5.2. *Let $\mathcal{S}$ be a nested set which contains 0. Then Γ (resp. γ) restricted to the internal points of $C\mathcal{D}_{\mathcal{S}}$ is a $2^{|\mathcal{S}|}$-sheeted covering of the open part of the boundary component $\widetilde{\mathcal{D}}_{\mathcal{S}}$ in $\widetilde{Y}_{\mathcal{A}}$ (resp Re $\mathcal{D}_{\mathcal{S}_{\mathbb{C}}}$ in Re $Y_{\mathcal{A}_{\mathbb{C}}}$).*

Remark 5.1. *In particular (considering $\mathcal{S} = \{0\}$), this statement reduces to the observation (see Proposition 5.1) that Γ and γ, restricted to $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$, are 2-sheeted coverings of $\mathbf{P}_{\mathbb{R}}(\mathcal{M}(\mathcal{A}))$ and Re $\mathbf{P}_{\mathbb{C}}(\mathcal{M}(\mathcal{A}_{\mathbb{C}}))$ respectively.*

Proof. We will prove the statement for Γ. Let us consider $A \in \mathcal{A} - \{0\}$ and the boundary component $C\mathcal{D}_A$. A point $p \in C\mathcal{D}_A$ is the limit of a succession $\{p_n\}$ in $c\pi^{-1}\left(S(A) - \bigcup_{B \in \mathcal{A}^A} S(B)\right)$. The points $\Gamma(p_n) = \bar{p}_n \in \widetilde{Y}_{\mathcal{A}}$ belong to

$$\pi^{-1}\left(\mathbf{P}(A) - \bigcup_{B \in \mathcal{A}^A} \mathbf{P}(B)\right)$$

and the limit of this succession is $\bar{p} = \Gamma(p)$ which by definition belongs to $\widetilde{\mathcal{D}}_A$. Therefore $\Gamma(C\mathcal{D}_A) \subset \widetilde{\mathcal{D}}_A$. Now a point $q \in \widetilde{\mathcal{D}}_A$ is the limit of a sequence $\{q_n\}$ in $\pi^{-1}\left(\mathbf{P}(A) - \bigcup_{B \in \mathcal{A}^A} \mathbf{P}(B)\right)$. Since Γ is surjective we can find a sequence $\{\check{q}_n\}$ in $CY_{\mathcal{A}}$ of points such that $\Gamma(\check{q}_n) = q_n$.

Let $\check{q}$ be the limit in the compact manifold $CY_{\mathcal{A}}$ (of a subsequence) of $\{\check{q}_n\}$. Then $\Gamma(\check{q}) = q$ and, since, for every n, $\check{q}_n \in c\pi^{-1}\left(S(A) - \bigcup_{B \in \mathcal{A}^A} S(B)\right)$ we have that $\check{q} \in C\mathcal{D}_A$: therefore $\Gamma(C\mathcal{D}_A) = \widetilde{\mathcal{D}}_A$.

Now, according to Theorem 3.1, for every $A \in \mathcal{A}$ the internal points of the manifold with corners $C\mathcal{D}_A$ coincide with

$$\mathcal{U} = \bigcup_{\rho} (U_{\{\{0\},A\}}(\rho) \cap \{t_{A^\perp} = 0\}).$$

Furthermore, since, for every $B \in \mathcal{A}$, we have that $\Gamma(C\mathcal{D}_B) = \widetilde{\mathcal{D}}_B$, then the inverse image of the set of all the internal points of $\widetilde{\mathcal{D}}_A$ is exactly $\mathcal{U}$.

It remains to compute the cardinality of the fibers of $\Gamma|_{\mathcal{U}}$. A point $p \in \mathcal{U}$ is uniquely determined by its projection to $S(\mathbb{R}^n)$ which lies in $S(A) \cap \mathcal{M}_A(\mathcal{A}^A)$ and by its projection $c\pi_A(p)$ in $S(A^\perp) \cap \mathcal{M}_{A^\perp}(\mathcal{A}_A^{A^\perp})$. Therefore the point $\Gamma(p)$ is determined by its projections to $\mathbf{P}(\mathbb{R}^n)$ and to $\mathbf{P}(\mathbb{R}^n/A)$. This implies that $\Gamma|_{\mathcal{U}}$ is $4 \to 1$ which is our claim for the codimension 1 open boundary strata.

The general case of Γ restricted to the internal points of $C\mathcal{D}_{\mathcal{S}}$ ($\mathcal{S}$ nested set) can be proven in a similar way even if the notation is more complicated: one refers to Theorem 3.2 which states that the internal points we are interested in coincide with

$$\bigcup_{\rho} \left(U_{\mathcal{T}}(\rho) \cap \bigcap_{B \in \mathcal{T} - \{0\}} \{t_{B^\perp} = 0\} \right).$$

■

As an immediate consequence, we have the following algebraic-topological corollary, which will be applied in the next section to compute the Euler characteristic of some real models of Coxeter arrangements.

Corollary 5.3. *Let us equip $CY_{\mathcal{A}}$ with the CW structure provided by the connected components of the open boundary strata. Then $\widetilde{\mathcal{D}}_{\mathcal{S}}$ (resp. $\mathcal{D}_{\mathcal{S}_{\mathbb{C}}}$), with the structure given by the images of these components, is a CW complex and Γ (resp. γ) is a map of CW complexes.*

6. Examples: Coxeter arrangements

A finite Coxeter arrangement $\mathcal{H}$ in an euclidean space V is an arrangement in V made by the hyperplanes whose associated reflections are the reflections of a finite Coxeter group $G_{\mathcal{H}}$). Let us consider a building set $\mathcal{G} \subset \mathcal{C}_{\mathcal{H}}$ containing $\mathcal{H}$ and $\{0\}$.

Now we fix a Coxeter chamber W and call by $CY_{\mathcal{G}}(W)$ the component of $CY_{\mathcal{G}}$ which is a compactification of $S(V) \cap W$. The following definition singles out the elements of $\mathcal{G}$ which control the boundary of $CY_{\mathcal{G}}(W)$.

Definition 6.1. *We will say that a non zero subspace $A \in \mathcal{G}$ is (W)-fundamental if dim $(A \cap \overline{W})$ = dim A. A (W)-fundamental nested set is a ($\mathcal{G}$)-nested set which contains $\{0\}$ and such that the other subspaces in it are (W)-fundamental subspaces.*

We are now going to study in detail the examples of Coxeter groups of types A_n, B_n, D_n.

6.1. **The braid arrangement.**

Let us focus on the essential braid arrangement $\mathcal{A}_{n-1}$: it consists of the hyperplanes $\{x_i = x_j\}$ $(1 \leq i < j \leq n)$ in $V = \mathbb{R}^n/\mathbb{R}\begin{pmatrix} 1 \\ 1 \\ . \\ 1 \end{pmatrix}$. These hyperplanes are orthogonal to the roots of a root system $\Phi_{\mathcal{A}_{n-1}}$ of type A_{n-1}. In [12] it is proven that $\mathbf{P}(\mathcal{M}(\mathcal{A}_{n-1,\mathbb{C}}))$ is isomorphic to the moduli space $M_{0,n+1}$ of complex $n+1$- pointed genus 0 curves. The minimal compact De Concini - Procesi model for $\mathbf{P}(\mathcal{M}(\mathcal{A}_{n-1,\mathbb{C}}))$, that is to say, the compact model $Y_{\mathcal{F}_{\mathcal{A}_{n-1,\mathbb{C}}}}$ associated to the set $\mathcal{F}^{\perp}_{\mathcal{A}_{n-1,\mathbb{C}}}$ of irreducibles (see Section 2), turns out to be isomorphic to the Mumford- Deligne compactification $\overline{M}_{0,n+1}$ (see [8]).

From the moduli point of view, a stable curve which lives over a point of $Re\ \overline{M}_{0,n+1}$ is "real" in the following sense: it is equipped with a conjugation involution fixing all labelled points and all singular points, i.e., each of its irreducible components has its labelled and singular points lying on a copy of $\mathbf{P}(\mathbb{R})$. The space $Re\ \overline{M}_{0,n+1}$ has been recently studied by Goncharov and Manin ([11]), Ceyhan ([2]), Devadoss ([5]) and Kwon ([15]).

In this section we will show how the combinatorial properties of γ described by Theorem 5.2 allow us to point out the relations between $Re\ \overline{M}_{0,n+1}$ and some well known polytopes, i.e. Stasheff associahedra and Kapranov permutoassociahedra. We will recover a tessellation of $Re\ \overline{M}_{0,n+1}$ which was described by Kapranov (see [12]) and Devadoss (see [5]).

The Stasheff associahedron for n letters K_n was defined (see [20]) as the poset whose elements are all the possible bracketings of the "product" of n formal symbols $A_1 A_2 \cdots A_n$ (the empty bracketing is allowed), with the following order relation: a bracketing β is said to be greater than another bracketing β' if β' can be obtained from β by adding some new pairs of brackets.

Thus the empty bracketing is the maximal element of K_n and the complete bracketings are the minimal elements.

This poset was first realized by Stasheff as a convex body in $\mathbb{R}^{n-2}$ and then in [10] and [16] it was given a realization as a convex polytope.

In his paper [12], Kapranov introduced another poset KP_n (the permutoassociahedron for n letters) whose elements correspond to all bracketed and permuted products of n formal symbols. This poset was realized as a convex polytope in $\mathbb{R}^{n-1}$ by Reiner and Ziegler (see [18]): in particular, they embedded a copy of K_n in the intersection of a Weyl chamber W of $\mathbb{R}^n$ with an S_n-invariant hyperplane and obtained KP_n as the convex hull of $S_n \cdot K_n$.

These polytopes play an important role in several fields of mathematics: for instance they encode the product rules for commutative (the associahedron K_n) and general (the permutoassociahedron KP_n) monoidal categories.

Theorem 6.1. *The model $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is diffeomorphic to the disjoint union of $n!$ copies of the Stasheff associahedron K_n.*

Remark 6.1. *This implies that γ, via Theorem 5.2, tells how to glue these $n!$ copies of K_n in order to obtain Re $\overline{M}_{0,n+1}$ (compare with* [5], [12]*).*

Proof.

Let us first recall (see [21], [8]) that the elements of $\mathcal{F}^{\perp}_{\mathcal{A}_{n-1}}$ are the subspaces of $V = \mathbb{R}^n / \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{pmatrix}$ spanned by the irreducible root subsystems of $\Phi_{\mathcal{A}_{n-1}}$. We can thus describe $\mathcal{F}_{\mathcal{A}_{n-1}}$ by means of a collection of subsets of $\{1, 2, \ldots, n\}$. In fact, given a subset $\overline{\Delta} = \{i_1, i_2, \ldots, i_p\} \subset \{1, 2, \ldots, n\}$ with $|\overline{\Delta}| \geq 2$, the subspace $\Delta \subset V$,

$$\Delta = \{x_{i_1} = x_{i_2} = \cdots = x_{i_p}\}$$

is in $\mathcal{F}_{\mathcal{A}_{n-1}}$. This correspondence is also bijective, therefore every subspace in $\mathcal{F}_{\mathcal{A}_{n-1}}$ can be determined by a subset of $\{1, 2, \ldots, n\}$ with at least two elements.

The following algorithm constructs a nested set $\mathcal{S}$ in $\mathcal{F}_{\mathcal{A}_{n-1}}$:
a) choose an ordering $a_1 > a_2 > \cdots > a_n$ of the numbers $\{1, 2, \ldots, n\}$ and form the "product" $a_1 a_2 \cdots a_n$;
b) consider a (partial) parenthesization of $a_1 a_2 \cdots a_n$;
c) consider a couple of parentheses $(a_i, a_{i+1}, \ldots, a_j)$ which is minimal (that is to say, there are not parentheses inside $(a_i, a_{i+1}, \ldots, a_j)$), and add to $\mathcal{S}$ the subspace Δ associated to $\overline{\Delta} = \{a_i, a_{i+1}, \ldots, a_j\}$;
d) delete these parentheses and continue. At the end add $\{0\}$ to $\mathcal{S}$.

Choosing all the possible orderings of $\{1, 2, \ldots, n\}$ and all the possible parenthesizations we obtain all the nested sets containing $\{0\}$.

Now, the manifold with corners $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ is made by $n!$ connected components, which are compactifications of the $n!$ connected components of $\mathcal{M}(\mathcal{A}_{n-1}) \cap S(V)$. We notice that the Weyl chambers are in bijective correspondence with the orderings of $\{1, 2, \ldots, n\}$: given a ordering $a_1 > a_2 > \cdots > a_n$ we associate to it the Weyl chamber $W(a_1 > a_2 > \cdots > a_n)$ made by the points of V whose representatives in $\mathbb{R}^n$ have coordinates $x_{a_1} > x_{a_2} > \cdots > x_{a_n}$. Thus, given a Weyl chamber $W(a_1 > a_2 > \cdots > a_n)$, we will denote by $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}(W(a_1 > a_2 > \cdots > a_n))$ the associated connected component of $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$.

We also notice that, once a ordering $a_1 > a_2 > \cdots > a_n$ of $\{1,2,\ldots,n\}$ is fixed, the collection $C(a_1 > a_2 > \cdots > a_n)$ of $\mathcal{F}_{\mathcal{A}_{n-1}}$ - nested sets obtained according to the algorithm is exactly the collection of $W(a_1 > a_2 > \cdots > a_n)$-fundamental nested sets and is in bijective correspondence with the elements of K_n. Moreover, if we order the elements of $C(a_1 > a_2 > \cdots > a_n)$ by reverse inclusion, we have that $C(a_1 > a_2 > \cdots > a_n)$ is isomorphic to K_n as a poset.

Thus the description of the boundary of $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ (see Theorem 3.2) implies that $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}(W(a_1 > a_2 > \cdots > a_n))$ is a realization of the poset $C(a_1 > a_2 > \cdots > a_n) \cong K_n$ as a connected $(n-2)$-dimensional manifold with corners and is therefore diffeomorphic to the convex $(n-2)$-dimensional polytope K_n.

■

One can show that all the information concerning the face lattice of KP_n is encoded by $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and by its projection $c\pi : CY_{\mathcal{F}_{\mathcal{A}_{n-1}}} \to S(V)$.

Now, the preceding theorem establishes a diffeomorphism $\psi : CY_{\mathcal{F}_{\mathcal{A}_{n-1}}} \to S_n \cdot K_n$ (here we are referring to the construction of Reiner and Ziegler in [18]); therefore we can also express the combinatorial relation between $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ and KP_n in the following way:

Corollary 6.2. *The manifold $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ has a diffeomorphic copy in $\mathbb{R}^{n-1}$ whose convex hull is a polytope which realizes Kapranov's permutoassociahedron KP_n.*

From a topological point of view, Corollary 5.3 allows us to compute the Euler characteristic of $Re\ Y_{\mathcal{F}_{\mathcal{A}_{n-1}},\mathbb{C}} = Re\ \overline{M}_{0,n+1}$, in accordance with [5].

Theorem 6.3. *We have that*

$$\chi\left(Re\ \overline{M}_{0,n+1}\right) = \sum_{k=0}^{n-2}(-1)^{n-2-k}\,\frac{n!}{2^{k+1}}\left[\frac{1}{k+1}\binom{n-2}{k}\binom{n+k}{k}\right]$$

Remark 6.2. *We notice that, for compactness,*

$$\chi\left(Re\ \overline{M}_{0,2r}\right) = 0 \quad \forall\ r$$

The first even dimensional cases are: $\chi\left(Re\ \overline{M}_{0,3}\right) = 1$, $\chi\left(Re\ \overline{M}_{0,5}\right) = -3$, $\chi\left(Re\ \overline{M}_{0,7}\right) = 45$, $\chi\left(Re\ \overline{M}_{0,9}\right) = -1575$, $\chi\left(Re\ \overline{M}_{0,11}\right) = 99225$.

Proof.

Since $Re\ \overline{M}_{0,n+1}$ is a $(n-2)$- dimensional CW-complex, the addendum

$$(-1)^{n-2-k}\,\frac{n!}{2^{k+1}}\left[\frac{1}{k+1}\binom{n-2}{k}\binom{n+k}{k}\right]$$

in the above formula represents the contribution to the Euler characteristic of the codimension k cells.

Their number has been computed using Theorem 6.1: the expression inside brackets gives the number of codimension k cells in a copy of K_n. This computation goes back to Cayley (see [1]), since this number coincides with the number of possible distinct k-parenthesizations (i.e., parenthesizations obtained with k couples of parentheses) of a product of n variables, which in its turn coincides with all the possible partitions of a $(n+1)$-gon with k non intersecting diagonals.

Then we take into account that in $CY_{\mathcal{F}_{\mathcal{A}_{n-1}}}$ there are $n!$ copies of K_n and that, by Theorem 5.2, the map γ restricted to codimension k cells is 2^{k+1} to 1.

■

Remark 6.3. *The CW-complex structure described in Corollary 5.3 could be used to compute the integer cohomology of Re* $\overline{M}_{0,n+1}$*. Anyway it involves computations on large matrices; for instance, in the case of Re* $\overline{M}_{0,5}$ *one has 12 two-dimensional faces, 30 edges and 15 vertices. For the homology we obtain:* $H_0(Re\ \overline{M}_{0,5},\mathbb{Z}) = \mathbb{Z}$, $H_1(Re\ \overline{M}_{0,5},\mathbb{Z}) = \mathbb{Z}^4 \times \mathbb{Z}/2\mathbb{Z}$, $H_2(Re\ \overline{M}_{0,5},\mathbb{Z}) = 0$ *(we point out that 2-torsion appears).*

6.2. **The Coxeter arrangement $\mathcal{B}_n$.** Let us consider the Coxeter arrangement $\mathcal{B}_n$ of type B_n $(n \geq 2)$, i.e., the arrangement provided in $\mathbb{R}^n$ by the hyperplanes $\{x_i - x_j = 0\}$, $\{x_i + x_j = 0\}$ and $\{x_r = 0\}$ $(i, j, r \in \{1, 2, \ldots, n\}$ $, i < j)$, and let us fix a Weyl chamber W. For instance we can suppose that W is the chamber described by $x_1 > x_2 > \cdots > x_n > 0$.

As for all root arrangements (see [21], [8]), if A belongs to the minimum building set $\mathcal{F}_{\mathcal{B}_n}$, then $A^\perp$ is spanned by an irreducible root subsystem.

Therefore a W- fundamental subspace is provided in this case by a (non zero) subspace of type

$$A = \{x_i = x_{i+1} = x_{i+2} = \cdots = x_{i+k}\}$$

where $k > 1, \quad i + k \leq n$, or of type

$$B = \{x_s = x_{s+1} = \cdots = x_n = 0\}$$

with $s \leq n$.

Proposition 6.4. *The poset of W-fundamental nested sets (containing 0), ordered by reverse inclusion, is isomorphic to the Stasheff associahedron* K_{n+1}*.*

Proof.

The following algorithm provides us with a bijective correspondence between these two sets.

Let us consider a (partial) parenthesization of the product $a_1 a_2 a_3 \cdots a_n a_{n+1}$. We will associate to it a nested set $\mathcal{S}$:

a) Let $(a_i a_{i+1} \cdots a_{i+j})$ be a minimal couple of parentheses (that is to say, there aren't any other parentheses inside $(a_i a_{i+1} \cdots a_{i+j})$), and add to $\mathcal{S}$ the subspace

$$\{x_i = x_{i+1} = x_{i+2} = \cdots = x_{i+j}\}$$

if $i+j<n+1$. Otherwise, if $i+j=n+1$, we add the subspace

$$\{x_i = x_{i+1} = x_{i+2} = \cdots = x_{i+j-1} = 0\}$$

b) delete these parentheses and continue. At the end add $\{0\}$ to $\mathcal{S}$.

It is easy to check that this correspondence is in fact a poset isomorphism. ■

In view of the above proposition we can compute the Euler characteristic of the manifold $Re\ Y_{\mathcal{F}_{\mathcal{B}_n,\mathbb{C}}}$ applying the same method used for the braid arrangement.

Theorem 6.5. *We have that*

$$\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{B}_n,\mathbb{C}}}\right) = \sum_{k=0}^{n-1}(-1)^{n-1-k}\ \frac{n!\ 2^n}{2^{k+1}}\left[\frac{1}{k+1}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\left(\begin{array}{c} n+k+1 \\ k \end{array}\right)\right] =$$

$$= \frac{2^n}{n+1}\chi\left(Re\ \overline{M}_{0,n+2}\right)$$

Remark 6.4. *We notice that, for compactness,*

$$\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{B}_{2r}}}\right) = 0 \quad \forall\ r$$

Proof.
Since $Re\ Y_{\mathcal{F}_{\mathcal{B}_n,\mathbb{C}}}$ is a $(n-1)$- dimensional CW-complex, the addendum

$$(-1)^{n-1-k}\ \frac{n!\ 2^n}{2^{k+1}}\left[\frac{1}{k+1}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\left(\begin{array}{c} n+k+1 \\ k \end{array}\right)\right]$$

in the above formula represents the contribution to the Euler characteristic of the codimension k cells. The expression inside brackets gives the number of codimension k cells in a copy of K_{n+1} and has been obtained using the same Cayley's formula as in the case of the braid arrangement.

Then we consider that in $CY_{\mathcal{F}_{\mathcal{B}_n}}$ there are $2^n n!$ copies of K_{n+1} and, by Theorem 5.2, the map γ restricted to codimension k cells is 2^{k+1} to 1. ■

6.3. **The Coxeter arrangement $\mathcal{D}_n$.** The arrangement is in this case of type D_n and is provided in $\mathbb{R}^n$ by the hyperplanes $\{x_i - x_j = 0\}$ and $\{x_i + x_j = 0\}$ $(i, j \in \{1, 2, \ldots, n\}\ ,\ i<j)$.

As a Weyl chamber W we can choose the one described by $x_1 > x_2 > \cdots > |x_n|$. We know that the minimum building set $\mathcal{F}_{\mathcal{D}_n}$ is formed by the orthogonals to the subspaces spanned by irreducible root subsystems.

This implies that a W- fundamental subspace is provided by a (non zero) subspace of type

$$A = \{x_i = x_{i+1} = x_{i+2} = \cdots = x_{i+k}\}$$

where $k > 1, \quad i + k \leq n$, or of type

$$B = \{x_s = x_{s+1} = \cdots = x_{s+r} = 0\}$$

where it is important to note that $s + r = n$ and $r \geq 2$, or of type

$$C = \{x_i = x_{i+1} = x_{i+2} = \cdots = x_{n-1} = -x_n\}$$

with $n - i \geq 1$.

The Euler characteristic can be expressed by a recursive formula in which the already computed Euler characteristics for cases A_r and B_j also appear.

Theorem 6.6. *We have that, for $n \geq 4$,*

$$\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{D}_n},\mathbb{C}}\right) = -\frac{3}{2}2^{n-1}\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{A}_{n-1}},\mathbb{C}}\right) - \frac{1}{4}2^{n-1}n\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{A}_{n-2}},\mathbb{C}}\right) +$$
$$-\frac{1}{8}2^{n-1}n(n-1)\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{A}_{n-3}},\mathbb{C}}\right) +$$
$$+\sum_{l=2}^{n-2}(-1)^{l+1}2^{l-2}\frac{n!}{(n-l+1)!}\sum_{r=0}^{l-2}\frac{(-1)^r}{2^r}\left[\frac{1}{r+1}\binom{l-1}{r}\binom{l+r+1}{r}\right]\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{D}_{n-l+1}},\mathbb{C}}\right)$$

Remark 6.5. *As before we notice that, for compactness,*

$$\chi\left(Re\ Y_{\mathcal{F}_{\mathcal{D}_{2r}}}\right) = 0 \quad \forall\ r$$

7. An extended real structure for Coxeter arrangements

Let us consider a finite Coxeter arrangement $\mathcal{H}$, with group $G_{\mathcal{H}}$, in an euclidean space V, and a building set $\mathcal{G} \subset L_{\mathcal{H}}$ containing $\mathcal{H}$ and $\{0\}$.

Starting from these data we are going to define a new family of posets (the "Coxeter posets") which includes Kapranov's permutoassociahedra. The combinatorial properties of a poset $Cox\ (\mathcal{H}, \mathcal{G})$ of the family will turn out to be controlled by $G_{\mathcal{H}}$ and $\mathcal{G}$.

In the next section we will extend $CY_{\mathcal{G}}$ to a convex body $\mathcal{C}(\mathcal{H}, \mathcal{G})$ in V such that the poset determined by its boundary is $Cox(\mathcal{H}, \mathcal{G})$: therefore the combinatorics of $Cox(\mathcal{H}, \mathcal{G})$ points out a further real structure which is implicit ("hidden") in the real model $CY_{\mathcal{G}}$.

7.1. **Definition of Coxeter posets.** Let us fix a Coxeter chamber W.

The elements of $Cox\ (\mathcal{H}, \mathcal{G})$ are all the couples $(wH, \mathcal{S})$ where

- $\mathcal{S}$ is a W-fundamental nested set (containing $\{0\}$) with labels attached to maximal elements: if an element A in $\mathcal{S}$ is maximal then it is labelled either by the subgroup G_A of $G_{\mathcal{H}}$ which stabilizes A or by the trivial subgroup $\{e\}$;
- H is the subgroup of $G_{\mathcal{H}}$ given by the product of all the labels in $\mathcal{S}$;
- w is an element in $G_{\mathcal{H}}$.

The order relation is the following: given two elements $(w'H', \mathcal{S}')$ and $(wH, \mathcal{S})$ then

$$(w'H', \mathcal{S}') < (wH, \mathcal{S})$$

if and only if

(1) $w'H' \subset wH$
(2) $\mathcal{S}'$ is obtained by $\mathcal{S}$ by a composition of the following moves:
  - adding a non maximal element;
  - adding a maximal element labelled by $\{e\}$ if the other maximal elements remain maximal after this move;
  - adding a maximal element B labelled by G_B, when B contains a maximal element $A \in \mathcal{S}$ labelled by G_A. After adding B, A is no more maximal and loses its label;
  - changing a label G_A of a maximal element A to the label $\{e\}$.

A motivation for the definition we provided above is that, when we specialize to the case of the braid arrangement $\mathcal{A}_{n-1}$, if we choose, among all the possible building sets, the minimal one, i.e. $\mathcal{F}_{\mathcal{A}_{n-1}}$, we find that $Cox\ (\mathcal{A}_{n-1}, \mathcal{F}_{\mathcal{A}_{n-1}})$ is the poset of Kapranov's permutoassociahedron KP_n, as the following section explains.

7.2. **Example: Kapranov's permutoassociahedron.** Let us assume the realization of KP_n in [18] and describe KP_n as a poset by describing all the faces of its realization. We can restrict to describe the faces which intersect W, since all the others are obtained by the S_n action.

Let us take a nested set $\mathcal{S}$ and suppose that it has r non maximal elements and k maximal elements A_1, A_2, $\dots A_k$. Then we consider the vertices of $K_n \subset W$ whose bracketing is associated to a nested set which contains $\mathcal{S}$. They determine a face $F_{\mathcal{S}}$ of K_n of dimension $n-r-k-1$. As a specialization of the notation introduced for Coxeter groups, let S_A be the subgroup of S_n which stabilizes a subspace A. Then, according to Reiner and Ziegler realization, if we let the group $S_{A_{i_1}} \times \cdots \times S_{A_{i_l}}$ act on $F_{\mathcal{S}}$ and take the convex hull of the image we obtain a face of KP_n of dimension $n-r-k+l-1$.

In this way we obtain all the faces of KP_n. This points out that the faces of KP_n which intersect W are determined by two data: the nested set $\mathcal{S}$ and the subset of its maximal elements whose stabilizers act on $F_{\mathcal{S}}$. It turns out that these faces form a subposet of KP_n which is isomorphic to the subposet of $Cox\ (\mathcal{A}_{n-1}, \mathcal{F}_{\mathcal{A}_{n-1}})$ given by couples $(eH, \mathcal{S})$ (where e is the identity). Taking into account the action of S_n we obtain an isomorphism between KP_n and $Cox\ (\mathcal{A}_{n-1}, \mathcal{F}_{\mathcal{A}_{n-1}})$.

8. Geometric construction of Coxeter posets

Let us consider, as in the preceding section, a Coxeter arrangement $\mathcal{H}$ in an euclidean space V of dimension n, a building set $\mathcal{G} \subset L_{\mathcal{H}}$ which contains $\mathcal{H} \cup \{0\}$ and the "Coxeter poset" $Cox(\mathcal{H}, \mathcal{G})$ associated to these data.

We will construct a convex body $\mathcal{C}(\mathcal{H}, \mathcal{G})$ in V such that the poset determined by its boundary is $Cox(\mathcal{H}, \mathcal{G})$.

The basis of our construction is provided by the real model $CY_{\mathcal{G}}$. The number of its connected components is equal to the number of Coxeter chambers, i.e., to the cardinality of $G_{\mathcal{H}}$. Let us choose a chamber W; we already noticed in the examples of the preceding section that, as a consequence of

Theorem 3.2, the set of all the boundary components of $CY_{\mathcal{G}}(W)$ is in bijection with the set of all (W)-fundamental nested sets (in particular, vertices of $CY_{\mathcal{G}}(W)$ are determined by the maximal (W)-fundamental nested sets).

Our firt step consists in embedding a diffeomorphic copy of $CY_{\mathcal{G}}(W)$ inside $S(V) \cap W$.

This can be obtained by referring to a different construction of $CY_{\mathcal{G}}$ as a result of a series of real blowups (see [9]). It turns out that a diffeomorphic copy of $CY_{\mathcal{G}}$ is the complement in $W \cap S(V)$ to a union of tubolar neighbouroods of fundamental subspaces. Let us state this more in detail:

Definition 8.1. *Given a fundamental subspace A and a positive real number γ_A, we denote by $T_{\gamma_A}(A)$ the open tubolar neighbourood of A in $S(V)$ which is given by all the points whose distance from A is lesser then γ_A.*

We will say that a choice of the numbers γ_A (for every A W-fundamental) is "admissible" if the numbers are small (say lesser then 10^{-n}) and if, whenever $dim\ A > dim\ B$, then $\gamma_B/\gamma_A > 10^{dim\ A - dim\ B}$.

It turns out (it can also be proven directly by using the explicit charts for $CY_{\mathcal{G}}(W)$ described in Section 3.2) that $CY_{\mathcal{G}}(W)$ is diffeomorphic to the complement in $W \cap S(V)$ to the union

$$\bigcup_{A \text{ fundamental}} T_{\gamma_A}(A)$$

Theorem 8.1. *The manifold with corners $CY_{\mathcal{G}}(W)$ has a diffeomorphic copy in $W \cap S(V)$ which can be extended to a convex set $\mathcal{C}(\mathcal{H}, \mathcal{G})$ whose face lattice is a realization of the poset $Cox(\mathcal{H}, \mathcal{G})$.*
If $(wH, \mathcal{S})$ is an element of $Cox(\mathcal{H}, \mathcal{G})$, the dimension of the face associated to it is $n - |\mathcal{S}| + k$, where k is the number of non trivial labels in $\mathcal{S}$.

Proof.
Let us first construct a diffeomorphic copy of $CY_{\mathcal{G}}(W)$ by fixing an admissible choice of numbers γ_A (for every A W-fundamental in $\mathcal{G}$) with the further stronger condition that, whenever $dim\ A > dim\ B$, then $\gamma_B/\gamma_A > 10^{n(dim\ A - dim\ B)}$.

Then we can consider the Coxeter group action $G_{\mathcal{H}} \cdot CY_{\mathcal{G}}(W)$ and obtain a copy of $CY_{\mathcal{G}}(W)$ in every chamber.

We want that $G_{\mathcal{H}} \cdot CY_{\mathcal{G}}(W) \subset \mathcal{C}(\mathcal{H}, \mathcal{G})$; this means that we have already constructed part of the boundary of $\mathcal{C}(\mathcal{H}, \mathcal{G})$.

Our next step consists of adding to $G_{\mathcal{H}} \cdot CY_{\mathcal{G}}(W)$ some 1-dimensional faces. If v_τ is a vertex of $CY_{\mathcal{G}}(W)$ (τ is a maximal fundamental nested set) and $A \in \tau$ is maximal (i.e. $(n-1)$-dimensional), we connect with a linear edge v_τ with the other element in $S_A \cdot v_\tau$ (i.e., with its reflection with respect to A).

Then we obtain all the other edges in $\mathcal{C}(\mathcal{H}, \mathcal{G})$ which are not in $G_{\mathcal{H}} \cdot CY_{\mathcal{G}}(W)$ by considering the $G_{\mathcal{H}}$ action on the above constructed linear edges.

Now let us consider an element $(eH, \mathcal{S})$ of $Cox(\mathcal{H}, \mathcal{G})$, where $\mathcal{S}$ is a W-fundamental labelled nested set with k non trivial labels $S_{R_1}, \ldots, S_{R_k}$. Let

$\mathcal{D}_{\mathcal{S}}$ be the boundary component of $CY_{\mathcal{G}}(W)$ associated to $\mathcal{S}$. Then the elements in $V_{\mathcal{S}} = \{S_{R_1} \times S_{R_2} \times \cdots \times S_{R_k} \cdot v_\tau \mid v_\tau \in \mathcal{D}_{\mathcal{S}}\}$ will determine a face of dimension $n - |\mathcal{S}| + k$.

Lemma 8.2. *If $A^\perp$ and $B^\perp$ are two non comparable (with respect to inclusion) elements in $\mathcal{S}^\perp$, then we have $A^\perp \perp B^\perp$.*

Proof of the Lemma.
Since $\mathcal{S}$ is nested, $A^\perp$ and $B^\perp$ are in direct sum and their sum $C^\perp$ is not in $\mathcal{G}^\perp$. Moreover, $C^\perp = A^\perp \oplus B^\perp$ is the "building decomposition" of $C^\perp$ in $\mathcal{G}^\perp$.

Now, $A^\perp$ and $B^\perp$ are generated by some subsets of roots of the Coxeter root system. We claim that if α is a root which belongs to $A^\perp$ and β is a root that belongs to $B^\perp$, then $\alpha \perp \beta$ (this will imply our thesis).

In fact in [8] it has been proven that the elements of $\mathcal{F}_{\mathcal{H}}^\perp = \mathcal{F}_{\mathcal{G}}^\perp$ are all the subspaces spanned by the irreducible root subsystems of the Coxeter root system. Remark 2.2 implies that $\mathcal{F}_{\mathcal{H}}^\perp \subset \mathcal{G}^\perp$.

Now, if α is not orthogonal to β, we can consider the two dimensional root subsystem generated by α and β. It contains a root of type $\alpha + a\beta$ with $a \neq 0$. But $\alpha + a\beta$ does not belong to $A^\perp$ or to $B^\perp$, otherwise they don't give a direct sum, and spans an irreducible one dimensional subspace. This contradicts the fact that $C^\perp = A^\perp \oplus B^\perp$ is the "building decomposition" of $C^\perp$ in $\mathcal{G}^\perp$, i.e. that $A^\perp$ and $B^\perp$ are the maximal elements of $\mathcal{G}^\perp$ contained in $C^\perp$. ■

Proposition 8.3. *Let $A \in \mathcal{S}$, and let us consider the orthogonal projections $\pi_{\mathcal{S}_\cap^{A^\perp}}(v_\tau)$ of $v_\tau \in \mathcal{D}_{\mathcal{S}}$ to $\mathcal{S}_\cap^{A^\perp}$. These projections determine on a sphere of $\mathcal{S}_\cap^{A^\perp}$ a copy of the manifold $CY_{\mathcal{G}_A^{\mathcal{S}_\cap^{A^\perp}}}$.*

Proof of the Proposition.
Let $B_1^\perp, \ldots, B_r^\perp$ be the elements in $\mathcal{S}^\perp$ which are covered by $A^\perp$. The sphere we are dealing with has then, by the above lemma, radius equal to

$$\sqrt{\gamma_A^2 - \sum_{i=1}^r \gamma_{B_i}^2}$$

On this sphere we have a picture which is again a complement of tubolar neighbouroods yielding a copy of $CY_{\mathcal{G}_A^{\mathcal{S}_\cap^{A^\perp}}}$. In fact, let us soppose that $\mathcal{S} \cup B$ is still nested, where $B \supset A$, and no one of the B_i's is included in B; the distance of the boundary component $\mathcal{D}_B$ from B is γ_B, and the distance of $\pi_{\mathcal{S}_\cap^{A^\perp}}(\mathcal{D}_{\mathcal{S}} \cap \mathcal{D}_B)$ from $B \cap \mathcal{S}_\cap^{A^\perp}$ is

$$\sqrt{\gamma_B^2 - \sum_{i \text{ s.t. } B \subsetneq B_i} \gamma_{B_i}^2}$$

Because of our special choice of admissible numbers γ_D we then have that the image via $\pi_{\mathcal{S}_\cap^{A^\perp}}$ of the boundary component $\mathcal{D}_\mathcal{S} \subset CY_\mathcal{G}$ is the complement of the tubolar neighbouroods $T_{\gamma'_C}(C)$ ($C \in \mathcal{G}_A^{\mathcal{S}_\cap^{A^\perp}}$) where the numbers γ'_C are still admissible. This implies that $\pi_{\mathcal{S}_\cap^{A^\perp}}(\mathcal{D}_\mathcal{S})$ is a diffeomorphic copy of $CY_{\mathcal{G}_A^{\mathcal{S}_\cap^{A^\perp}}}$. ■

Now we let the group $S_{R_1} \times \cdots \times S_{R_k}$ act on the vertices v_τ in $\mathcal{D}_\mathcal{S}$ and we obtain $V_\mathcal{S}$, which is a subset of the manifold

$$\widehat{\mathcal{D}_\mathcal{S}} = \mathcal{D}_\mathcal{S} + R_1^\perp + R_2^\perp + \cdots + R_k^\perp$$

Since $\pi_{\mathcal{S}^{R_j^\perp}}(\mathcal{D}_\mathcal{S})$ is a codimension 1 manifold in $R_j^\perp$ we have that $\widehat{\mathcal{D}_\mathcal{S}}$ has dimension equal to $(dim\ \mathcal{D}_\mathcal{S}) + k$, i.e. $n - |\mathcal{S}| + k$.

A portion of $\widehat{\mathcal{D}_\mathcal{S}} = e \cdot \widehat{\mathcal{D}_\mathcal{S}}$ will give us a face of $\mathcal{C}(\mathcal{H}, \mathcal{G})$. The portion we are interested in is cut on $\widehat{\mathcal{D}_\mathcal{S}}$ by the submanifolds $w \cdot \widehat{\mathcal{D}_{\mathcal{S}'}}$ of codimension 1, where $\mathcal{S}'$ is obtained by $\mathcal{S}$ using the "moves" described in Section 7.1 and w belongs to the product of the labels of $\mathcal{S}$.

Let for instance $\mathcal{S}' = \mathcal{S} \cup K$ where K is maximal with trivial label and the products of labels of $\mathcal{S}$ and $\mathcal{S}'$ coincide.

Then the manifold $e \cdot \widehat{\mathcal{D}_{\mathcal{S}'}}$ is determined by the subset

$$V_{\mathcal{S}'} = \{S_{R_1} \times S_{R_2} \times \cdots \times S_{R_k} \cdot v_\tau \mid v_\tau \in \mathcal{D}_{\mathcal{S}'}\}$$

It can also be seen as the subset of $V_\mathcal{S}$ made by the elements which satisfy the following extra condition: let $B^\perp$ be the element in $\mathcal{S}^\perp$ which covers $K^\perp$; then, when we project the points of $V_\mathcal{S}$ to $\mathcal{S}_\cap^{B^\perp}$, the points in $V_{\mathcal{S}'}$ are the ones whose projections lie at a minimal distance from K. This is therefore a "boundary" condition and all the possible boundary conditions are expressed by the moves of Section 7.1.

If we now denote by $F(e, \mathcal{D}_\mathcal{S})$ the face of $\mathcal{C}(\mathcal{H}, \mathcal{G})$ that is cut in $e \cdot \widehat{\mathcal{D}_\mathcal{S}}$ by the above mentioned submanifolds, we have that the set of all faces of $\mathcal{C}(\mathcal{H}, \mathcal{G})$ is given by

$$\{F(w, \mathcal{D}_\mathcal{S}) \simeq w \cdot F(e, \mathcal{D}_\mathcal{S}) \mid w \in G_\mathcal{H}\}$$

It remains to prove that $\mathcal{C}(\mathcal{H}, \mathcal{G})$ is convex. In fact $\mathcal{C}(\mathcal{H}, \mathcal{G})$ can be viewed as the intersection of convex spaces.

One is the sphere $S(V)$. The others are associated to the facets of $\mathcal{C}(\mathcal{H}, \mathcal{G})$ which are not in $G_\mathcal{H} \cdot CY_\mathcal{G}(W)$. We note that these facets are the $F(w, \mathcal{D}_\mathcal{S})$ where $\mathcal{S}$ satisfies the following properties:

(1) it has only one unlabelled element (i.e., $\{0\}$ is the only non maximal element);
(2) all the labels are non trivial.

Let $\mathcal{S} = \{\{0\}, A_1, A_2, \dots, A_k\}$ be as above, and let $w = e$ for simplicity. Then we associate to $F(e, \mathcal{D}_\mathcal{S})$ the codimension 1 submanifold of V

$$\Gamma(\mathcal{S}) + A_1^\perp + A_2^\perp + \cdots + A_k^\perp$$

where $\Gamma(\mathcal{S})$ is the intersection of a closed ball of radius $\sqrt{1 - \sum_{i=1}^k \gamma_{A_i}^2}$ with $A_1 \cap \cdots \cap A_k$.

Let $Conv(\mathcal{H}, \mathcal{G})$ be the common intersection of all these convex spaces. The inclusion $\mathcal{C}(\mathcal{H}, \mathcal{G}) \subset Conv(\mathcal{H}, \mathcal{G})$ is provided by the observation that every point $c \in \mathcal{C}(\mathcal{H}, \mathcal{G})$, when projected to $\mathcal{S}_\cap^V$, lies inside the ball of radius $\sqrt{1 - \sum_{i=1}^k \gamma_{A_i}^2}$.

In fact, on one hand, if $\mathcal{S} = \{\{0\}, A_1, A_2, \dots, A_k\}$ $(k > 0)$ and $c \in F(w, \mathcal{D}_\mathcal{S})$ $(w \in S_{A_1} \times S_{A_2} \times \cdots \times S_{A_k})$, then its projection belongs exactly to the sphere of radius $\sqrt{1 - \sum_{i=1}^k \gamma_{A_i}^2}$.

On the other hand, when $\mathcal{S} = \{\{0\}\}$ and $c \in F(w, \mathcal{D}_\mathcal{S})$ $(w \in G_\mathcal{H})$ the projection of c lies inside the ball of radius $\sqrt{1 - \sum_{i=1}^k \gamma_{A_i}^2}$ since, by the construction via tubolar neighbouroods, the distance of c from A_j $(j = 1, 2, \dots, k)$ is greater then γ_{A_j}.

Let now $x \in Conv(\mathcal{H}, \mathcal{G})$. A scalar multiple λx of x $(\lambda > 0)$ belongs to $\mathcal{C}(\mathcal{H}, \mathcal{G})$ (which by construction contains a neighbourood of the origin). Let $\overline{\lambda}$ be the maximum value such that $\overline{\lambda} x \in \mathcal{C}(\mathcal{H}, \mathcal{G})$. If $\overline{\lambda} \geq 1$, then $x \in \mathcal{C}(\mathcal{H}, \mathcal{G})$ since $\mathcal{C}(\mathcal{H}, \mathcal{G})$ is a star with respect to the origin.

Let us suppose $\overline{\lambda} < 1$. Now, $\overline{\lambda} x$ belongs to at least one of the closed facets of $\mathcal{C}(\mathcal{H}, \mathcal{G})$.

If this facet is determined by a nested set $\mathcal{S} = \{\{0\}, A_1, A_2, \dots, A_r\}$ ($r \geq 1$, all the labels are non trivial), then we have that $\pi_{\mathcal{S}_\cap^V}(\overline{\lambda} x)$ lies on a sphere of radius $\sqrt{1 - \sum_{i=1}^r \gamma_{A_i}^2}$. This implies that $\pi_{\mathcal{S}_\cap^V}(x)$ has norm strictly greater then $\sqrt{1 - \sum_{i=1}^r \gamma_{A_i}^2}$, which contradicts $x \in Conv(\mathcal{H}, \mathcal{G})$.

If instead $\overline{\lambda} x$ belongs to a facet determined by $\mathcal{S} = \{\{0\}\}$, this means that $|\overline{\lambda} x| = 1$, and therefore $|x| > 1$ which again gives a contradiction. ■

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