On the classifying space of an Artin monoid

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Ingredients

- W – a Coxeter group, generated by a finite set S with relations $(st)^{m_{st}} = 1$ for $s, t \in S$ ($m_{ss} = 1$, and $m_{st} \in \{2, 3, \ldots, \infty\}$ if $s \neq t$).
- A – the Artin group generated by $\{\sigma_s \mid s \in S\}$ with relations

$$\underbrace{\sigma_s \sigma_t \sigma_s \cdots}_{m_{st} \text{ times}} = \underbrace{\sigma_t \sigma_s \sigma_t \cdots}_{m_{st} \text{ times}} \quad \text{for } s, t \in S.$$

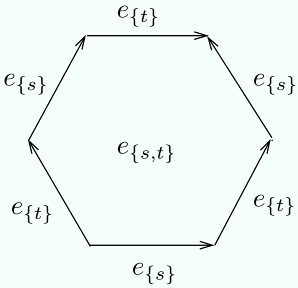
- A^+ – the Artin monoid with the same presentation of A.

Ingredients (2)

- $M(W)$ – the complement in the Tits cone $I \subseteq \mathbb{C}^n$ of the hyperplane arrangement associated to W.
- $\overline{M}(W) = M(W)/W$.
- $\mathrm{Sal}(W)$ – the Salvetti complex, a finite CW model for $\overline{M}(W)$ with n-cells in one-to-one correspondence with the elements of size n in $S^f = \{ T \subseteq S \mid W_T \text{ is finite}\}$.
- BA^+ – the classifying space of A^+ (the geometric realization of the nerve of the monoid, seen as a category with one object). It has the structure of a CW complex having as n-cells the n-tuples $[x_1|x_2|\ldots|x_n]$ of elements $x_1, \ldots, x_n \in A^+ \setminus \{1\}$.

Example

- $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle \cong S_3$.
- $A = \langle \sigma_s, \sigma_t \mid \sigma_s \sigma_t \sigma_s = \sigma_t \sigma_s \sigma_t \rangle$ (group presentation).
- $A^+ = \langle \sigma_s, \sigma_t \mid \sigma_s \sigma_t \sigma_s = \sigma_t \sigma_s \sigma_t \rangle$ (monoid presentation).
- $\overline{M}(W) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_i \neq z_j \text{ for } i \neq j\} \,/\, S_3$.
- $S^f = \{\varnothing, \{s\}, \{t\}, \{s, t\}\}$.
- $\mathrm{Sal}(W) =$



Relations

Theorem (Salvetti 1994)

$Sal(W) \simeq \overline{M}(W)$.

Theorem (Dobrinskaya 2006)

$BA^+ \simeq \overline{M}(W)$.

So it turns out that the three spaces $\overline{M}(W)$, $\mathrm{Sal}(W)$ and BA^+ have all the same homotopy type.

Moreover their fundamental group is isomorphic to A.

Conjecture ($K(\pi, 1)$ conjecture)

These three spaces are classifying spaces for the Artin group A.

Discrete Morse theory on BA^+

Theorem (Ozornova 2013)

Let C_* be the algebraic complex which computes the cellular homology of BA^+. There is an acyclic matching M on C_* such that the (algebraic) Morse complex C_*^M has n-dimensional generators in one-to-one correspondence with the elements of size n in S^f.

The Salvetti complex also gives rise to an algebraic complex which computes the same homology, and with the same number of generators.

Discrete Morse theory on BA^+ (2)

It turns out that the matching on C_* is induced by a topological matching M on BA^+. Moreover, the corresponding Morse complex can be related to the Salvetti complex in the following way.

Theorem (P. 2015)

There exists an acyclic matching M on BA^+ for which the Morse complex $X(W)$ has one n-cell e_T for each element $T \in S^f$ of size n.
Moreover there exists a homotopy equivalence $\psi\colon X(W) \to Sal(W)$ such that, for each subcomplex $X(W)_{\mathcal{F}}$ of $X(W)$ (where $\mathcal{F} \subseteq S^f$), the image of $\psi|_{X(W)_{\mathcal{F}}}$ is contained in $Sal(W)_{\mathcal{F}}$ and

$$\psi|_{X(W)_{\mathcal{F}}}\colon X(W)_{\mathcal{F}} \to Sal(W)_{\mathcal{F}}$$

is also a homotopy equivalence.

Discrete Morse theory on BA^+ (3)

This gives a new proof of Dobrinskaya's theorem:

Corollary

$BA^+ \simeq Sal(W)$.

Moreover it clarifies the relation between Ozornova's Morse complex and the Salvetti complex:

Corollary

Ozornova's algebraic Morse complex coincides with the algebraic complex which computes the cellular homology of the Salvetti complex.

Critical cells

Recall that the n-cells of BA^+ are of the form $[x_1|\dots|x_n]$ with $x_i \in A^+ \setminus \{1\}$. The faces of $[x_1|\dots|x_n]$ are given by:

- $[x_2|\dots|x_n]$;
- $[x_1|\dots|x_i x_{i+1}|\dots|x_n]$ for $i = 1, \dots, n-1$;
- $[x_1|\dots|x_{n-1}]$.

They are all regular faces for $n \geq 2$.

Let $\Delta_T = \operatorname{lcm}\{\sigma_s \mid s \in T\} \in A^+$, for $T \in S^f$. For instance:

$$\begin{aligned} \Delta_\varnothing &= 1 \\ \Delta_{\{s\}} &= \sigma_s \\ \Delta_{\{s,t\}} &= \underbrace{\sigma_s \sigma_t \sigma_s \cdots}_{m_{st} \text{ factors}} = \underbrace{\sigma_t \sigma_s \sigma_t \cdots}_{m_{st} \text{ factors}} \end{aligned}$$

(Δ_T is well defined for $T \in S^f$).

Critical cells (2)

Fix a total order $s_1 < s_2 < \cdots < s_k$ on S.

The critical n-cells are of the form $[x_1|\dots|x_n]$ with

$$x_i = \Delta_{\{t_i,\dots,t_n\}} \Delta^{-1}_{\{t_{i+1},\dots,t_n\}}$$

for some $T = \{t_1 < \cdots < t_n\} \in S^f$.

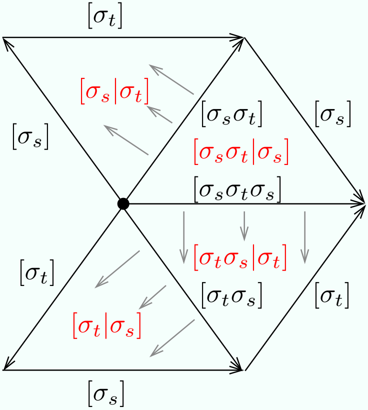
For example:

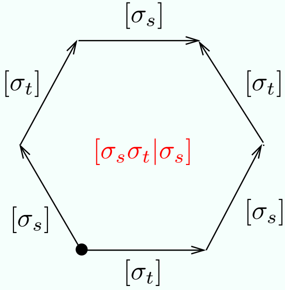
- the only (critical) 0-cell is $[\,]$;
- the critical 1-cells are $[\sigma_s]$ for $s \in S$;
- the critical 2-cells are $[\underbrace{\cdots\sigma_t\sigma_s\sigma_t}_{m_{st}-1 \text{ factors}}|\sigma_s]$ for $t < s$ such that $m_{st} \neq \infty$.

Boundary of the 2-dimensional critical cells

The 2-skeleton of the Morse complex can be determined explicitly.

Let $t < s$ be elements of S with $m_{st} = 3$ (the general case is similar), and consider the critical cell corresponding to $T = \{s, t\} \in S^f$.





The Morse complex of BA^+ and the Salvetti complex

By the previous argument, the 2-skeleton of the Morse complex $X(W)$ of BA^+ coincides with the 2-skeleton of the Salvetti complex $\mathrm{Sal}(W)$.

To prove the main theorem, we start from the 2-skeleton and argue by induction, extending the homotopy equivalence one cell at a time.

Suppose to have constructed a homotopy equivalence ψ up to a certain subcomplex:

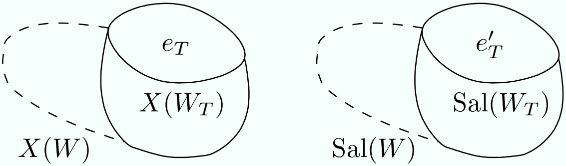
$$\psi\colon X(W)_{\mathcal{F}} \to \mathrm{Sal}(W)_{\mathcal{F}},$$

where $\mathcal{F} \subseteq S^f$. We want to extend ψ to a new cell e_T, for some $T \in S^f$.

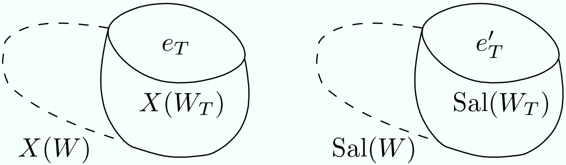
The Morse complex of BA^+ and the Salvetti complex (2)

Let $T \in S^f$. Call e_T and e'_T the corresponding cells in $X(W)$ and $\mathrm{Sal}(W)$, respectively.

- The boundaries of e_T and e'_T lie in subcomplexes isomorphic to $X(W_T)$ and $\mathrm{Sal}(W_T)$, where W_T is the (finite) standard parabolic subgroup of W generated by T.



The Morse complex of BA^+ and the Salvetti complex (3)

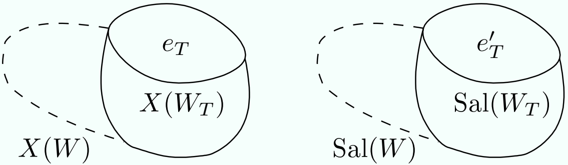


- When W_T is finite, both the spaces $BA^+_T \simeq X(W_T)$ and Sal(W_T) can be proved to be classifying spaces for the Artin group A_T.

  For Sal(W_T), this is the $K(\pi, 1)$ conjecture (proved by Deligne in 1972 for finite Coxeter groups).

  For BA^+_T we proceed as follows: when W_T is finite, the universal cover EA^+_T of BA^+_T is an increasing union of subspaces isomorphic to a certain "positive" contractible subspace $E^+A^+_T$.

The Morse complex of BA^+ and the Salvetti complex (4)



- Since $X(W_T)$ and $\mathrm{Sal}(W_T)$ are both classifying spaces for the Artin group A_T, the homotopy equivalence

$$\psi|_{X(W_T)_{n-1}} \colon X(W_T)_{n-1} \to \mathrm{Sal}(W_T)_{n-1}$$

can be extended to a homotopy equivalence $X(W_T) \to \mathrm{Sal}(W_T)$ ($n = \dim e_T = |T|$).

- Finally we extend ψ to the new cell e_T as above, obtaining a homotopy equivalence.