From approximating to interpolatory non-stationary subdivision schemes with the same reproduction properties

Costanza Conti · Luca Gemignani · Lucia Romani

Received: date / Accepted: date

Abstract In this paper we describe a general, computationally feasible strategy to deduce a family of interpolatory non-stationary subdivision schemes from a symmetric non-stationary, non-interpolatory one satisfying quite mild assumptions. It is shown that the interpolatory schemes are (mostly) capable of generating the same functional space as the approximating one. Moreover, the interplay between structured matrices and polynomials provides an effective tool for designing efficient numeric and/or numeric-symbolic methods for their construction and analysis.

Keywords Subdivision schemes · Structured matrices · Polynomials

Mathematics Subject Classification (2000) MSC 65F05 · MSC 65D05

1 Introduction

This paper is the generalization of our recent work [2] to the non-stationary situation. In fact, many important subdivision schemes are of non-stationary nature like those able to reproduce conic sections, spirals or classical trigonometric curves which are important analytical shapes in geometric modeling. In particular we discuss how it is possible to move from a non-stationary approximating subdivision scheme to a non-stationary interpolatory one.
Interpolatory subdivision schemes are efficient iterative procedures for the generation of interpolatory curves: Starting with the set of points to be interpolated, at each recursion step a new point is inserted in between any two given points so that the limit curve, whenever exists, not only interpolates the initial set of points but also all the points generated through the process. Interpolatory subdivision schemes play a crucial role in CAGD as well as in wavelets construction (see for example [5] and [8], respectively). Indeed, from a positive symmetric interpolatory symbol, via its spectral factorization, we can construct an orthogonal refinable function that is the building block of orthonormal wavelets.

In this paper we provide a general strategy to deduce a family of interpolatory non-stationary subdivision schemes from a symmetric non-stationary, non-interpolatory one satisfying quite mild assumptions. In most cases the resulting interpolatory schemes have the remarkable property of generating the same functional space as the approximating one. The approximating symbols we start with, say \( \{ a_k(z), k \geq 0 \} \), are required to be symmetric polynomials such that \( a_k(z), a_k(-z) \) are relatively prime for any \( k \geq 0 \). Under this assumption we show that a family of interpolatory masks associated with the symbol \( a_k(z) \) can be generated by solving a family of Bezout-like polynomial equations or, equivalently, by inverting a certain Hurwitz matrix constructed from the coefficients of \( a_k(z) \).

From a computational viewpoint, the interplay between the polynomial and the structured matrix formulation turns out to be an effective tool for the construction and the analysis of the interpolatory schemes. If \( a_k(z) \) is a Hurwitz polynomial, then the associated Hurwitz matrix is totally positive and, therefore, can be factorized and inverted stably by using techniques based on Neville’s elimination [7]. A root-based polynomial equation solver leads to an efficient strategy for the computation of the coefficients of the corresponding interpolatory symbols in several important cases where the zeros of \( a_k(z) \) are known a-priori. Furthermore, a specific mixed strategy based on polynomial and matrix computations can be especially efficient in situations where the polynomial \( a_k(z) \) admits a “short” representation in some Bernstein-type polynomial basis. All these approaches together allow us to efficiently address the case of B-splines and their “shifted” affine combinations as well as the case of exponential reproducing splines (ERS).

The paper is organized as follows. In Section 2 the needed background on non-stationary subdivision schemes is given. In Section 3 we review and generalize to some extent the basic strategy proposed in [2] for the construction of an interpolatory subdivision mask from a given approximating one. Effective computational procedures for implementing this strategy are discussed in Section 4. These procedures are the key ingredients of our algorithm, named Appint, to move from a non-stationary approximating subdivision scheme to a family of non-stationary interpolatory ones. The algorithm is sketched in Section 5 by showing the reproduction property of the generated schemes. Then, Section 6 discusses the application of the Appint Algorithm to (non-stationary) B-spline symbols and their “shifted” affine combinations, whereas in Section 7 the application of the algorithm to several instances of non-stationary approximating exponential reproducing subdivision schemes is considered. Finally, the conclusion and further work are drawn in Section 8.
2 Background

Any \((\text{non-stationary})\) subdivision scheme is defined by an infinite sequence of refinement masks \(\{a^k, k \geq 0\}\). We assume that any sequence \(a^k := \left( a^k_i, i \in \mathbb{Z} \right) \) is of real numbers and has finite support for all \(k \geq 0\) i.e. \(a^k_i = 0\) for \(i \not\in [-n(k), n(k)]\) for suitable \(n(k) \geq 0\). The \(k\)-level subdivision operator associated with the \(k\)-level mask \(a^k\) is

\[
S_{a^k} : \ell(\mathbb{Z}) \rightarrow \ell(\mathbb{Z}),
\]

\[
(S_{a^k} q)_i := \sum_{j \in \mathbb{Z}} a^k_{i-2j} q_j, \quad i \in \mathbb{Z},
\]  

(2.1)

where \(\ell(\mathbb{Z})\) denotes the linear space of real sequences indexed by \(\mathbb{Z}\) whose elements will be denoted by boldface letter, \(q = (q_i \in \mathbb{R}, i \in \mathbb{Z})\).

The non-stationary subdivision scheme consists of the subsequent application of \(S_{a^0}, \ldots, S_{a^k}\) from a given starting sequence, say \(q\), generating the scalar sequences \(q^0 := q\), \(q^{k+1} := S_{a^k} q^k\) for \(k \geq 0\).

A subdivision scheme is termed \(L_\infty\)-convergent if, for any \(q \in \ell_\infty(\mathbb{Z})\), the linear subspace of bounded scalar sequences, there exists a continuous function \(f_q\) (depending on the starting sequence \(q\)) satisfying

\[
\lim_{k \rightarrow \infty} \| f_q \left( \frac{z}{2^k} \right) - a^k \|_\infty = 0
\]

and \(f_q \neq 0\) for at least some initial data \(q\). Here, the symbol \(f_q \left( \frac{z}{2^k} \right)\) abbreviates the scalar sequence \(\{f_q \left( \frac{z}{2^i} \right) \}_{i \in \mathbb{Z}}\) and \(\|q\|_\infty := \sup_{i \in \mathbb{Z}} |q_i|\).

The limit of the subdivision process when starting with \(q = \delta := (\delta_i, 0 : i \in \mathbb{Z})\), where \(\delta_{i,j}\) is the Kronecker delta symbol, is called its basic limit function and is denoted by \(\phi\).

Useful tools for the subdivision analysis are the \emph{symbols}

\[
a^k(z) = \sum_{i \in \mathbb{Z}} a^k_i z^i, \quad k \geq 0, \quad z \in \mathbb{C} \setminus \{0\}
\]

and the corresponding \emph{sub–symbols}

\[
a^k_{\text{even}}(z) = \sum_{i \in \mathbb{Z}} a^k_{2i} z^i, \quad a^k_{\text{odd}}(z) = \sum_{i \in \mathbb{Z}} a^k_{2i+1} z^i, \quad z \in \mathbb{C} \setminus \{0\},
\]

with

\[
a^k_{\text{even}}(z^2) + z \cdot a^k_{\text{odd}}(z^2) = a^k(z),
\]

associated to the masks \(\{a^k, k \geq 0\}\). Since the masks are always supposed to be finitely supported, all symbols are Laurent polynomials. We observe that the conditions

\[
a^k(1) = 2, \quad a^k(-1) = 0, \quad k \geq 0
\]

(2.2)

ensure reproduction of constant sequences, a minimal request for a scheme to be effective. In the stationary case, where the masks \(\{a^k, k \geq 0\}\) are kept fixed over the iterations, that is \(a^k = a\) for all \(k \geq 0\), the conditions (2.2) become a necessary condition for convergence, i.e.

\[
a(1) = 2, \quad a(-1) = 0.
\]
A well known class of stationary subdivision schemes is given by degree-

\[ n \] B-spline subdivision schemes, whose symbol is

\[ a(z) = a_n(z) = \frac{(1 + z)^{n+1}}{2^n}. \quad (2.3) \]

The non-stationary counterpart of (2.3) is the symbol of the so-called L-spline schemes [12], a large family of smoothing splines defined in terms of a linear differential operator, that we recall in the next definition. Since they are able to generate exponential polynomials, they turn out to be of great interest in curve design for reproducing important analytical shapes like conic sections, spirals and classical trigonometric curves.

**Definition 1** (Space of Exponential Polynomials) Let \( T \in \mathbb{Z}_+ \) and \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_T) \) with \( \gamma_T \neq 0 \) a finite set of real or imaginary numbers and let \( D^n \) the \( n \)-th order differentiation operator. The space of exponential polynomials \( V_{T, \gamma} \) is the subspace

\[ V_{T, \gamma} = \{ f : \mathbb{R} \to \mathbb{C}, f \in C^T(\mathbb{R}) : \sum_{j=0}^{T} \gamma_j D^j f = 0 \}. \quad (2.4) \]

A characterization of the space \( V_{T, \gamma} \) is provided by the following:

**Lemma 1** [9, 13] Let \( \gamma(z) = \sum_{j=0}^{T} \gamma_j z^j \) and denote by \( \{\theta_\ell, \tau_\ell\}_{\ell=1,\ldots,N} \) the set of zeros with multiplicity of \( \gamma(z) \) satisfying

\[ \gamma^{(h_\ell)}(\theta_\ell) = 0, \quad h_\ell = 0, \ldots, \tau_\ell - 1, \quad \ell = 1, \ldots, N. \]

It results

\[ T = \sum_{\ell=1}^{N} \tau_\ell, \quad V_{T, \gamma} := \text{Span}\{x^{h_\ell} e^{\theta_\ell x}, h_\ell = 0, \ldots, \tau_\ell - 1, \quad \ell = 1, \ldots, N\}. \]

As proved in [13], any non-stationary subdivision scheme with a symbol of the form

\[ a_n^k(z) = 2 \prod_{\ell=1}^{N} \prod_{h_\ell=0}^{\tau_\ell-1} \frac{e^{\theta_\ell x}}{e^{\theta_\ell x} + 1}, \quad k \geq 0, \quad (2.5) \]

generates limit functions belonging to the subclass of \( C^{T-2} \) degree-\( n \) L-splines (with \( n = T - 1 \)) whose pieces are exponentials of the space \( V_{T, \gamma} \). These functions are called exponential reproducing splines (ERS). Notice that, when \( \theta_\ell = 0 \) for all \( \ell = 1, \ldots, N \), then \( a_n^k(z) \) in (2.5) does not depend on \( k \) being the symbol of a degree-\( n \) B-spline given in (2.3).

We conclude by recalling that a subdivision scheme is said to be interpolatory if the refinement masks \( \{a^k, k \geq 0\} \) satisfy

\[ a^k_{2i} = \delta_{i,0}, \quad k \geq 0, \]

or equivalently

\[ a^k_{\text{even}}(z) = 1 \quad (2.6) \]

meaning that all points generated by the subdivision process at a given level \( k \) will be kept in the next level \( k + 1 \). In the latter case, any basic limit function is cardinal since it satisfies \( \phi(i) = \delta_{i,0} \). We conclude by mentioning that from (2.6) it follows that a mask \( a^k \) is interpolatory if and only if all its symbols \( a^k(z) \) satisfy the condition

\[ a^k(z) + a^k(-z) = 2. \quad (2.7) \]
3 From approximating to interpolatory subdivision schemes: The general approach

In this section we review and generalize to some extent the strategy proposed by the authors in [2] to construct a family of interpolatory subdivision symbols from a non-interpolatory one \(a^k(z)\) for \(k\) fixed, \(k \in \mathbb{Z}, k \geq 0\). For the sake of notational simplicity we omit the superscript \(k\) by denoting \(a^k(z) = a(z) = \sum_{i \in \mathbb{Z}} a_i z^i, z \in \mathbb{C} \setminus \{0\}\).

It is worth noting that in a matrix setting the linear operator \(S_a\) defined in (2.1) and associated with \(a(z)\) is represented by a bi-infinite Toeplitz-like matrix \(S_a = (a_{i-j})\), \(i, j \in \mathbb{Z}\). Since \(a(z)\) is a Laurent polynomial, say \(a(z) = \sum_{j=-\kappa}^{\kappa} a_j z^j\), \(\max\{|a-\kappa|, |a+\kappa|\} > 0\), it follows that \(S_a\) is banded with bandwidth \([\frac{\kappa}{2}]\) at most. Let \(p(z) = \sum_{j=-h}^{h} p_j z^j, \max\{|p_{-h}|, |p_{h}|\} > 0\), be another Laurent polynomial and denote by \(P\) the bi-infinite Toeplitz matrix associated with \(p(z)\), namely, \(P = (p_{i-j})\). Observe that \(P\) is again banded with bandwidth \(h\). For the product operator \(S : = P \cdot S_a = (s_{i,j}), \quad i,j \in \mathbb{Z}\), we have

\[
s_{i,j} = \sum_{r=i-h}^{i+h} p_{i-r} a_{r-2j} = \sum_{\ell=-h}^{h} p_{\ell} a_{i-2j-\ell} = s_{i+2,j+1}, \quad i, j \in \mathbb{Z}.
\]

This means that the product operator \(S\) is a bi-infinite Toeplitz-like matrix of the same form as the subdivision operator \(S_a\) with entries \(s_{i,j} = s_{i-2j}, \quad i, j \in \mathbb{Z}\). By setting \(q(z) = a(z) \cdot p(z) = \sum_{j=-h-k}^{h+k} q_j z^j\), \((q_j = 0\) if \(|j| > h + \kappa)\), we find that

\[
q_j = \sum_{i=-h}^{h} p_i a_{j-i}, \quad -(h + \kappa) \leq j \leq h + \kappa,
\]

and, therefore,

\[
s_{i-2j} = s_{i,j} = s_{i-2j}, \quad i,j \in \mathbb{Z}.
\]

There follows that the product operator \(S\) can be seen as the subdivision operator associated with the Laurent polynomial \(q(z)\), i.e.,

\[
S = S_q, \quad q(z) = a(z) \cdot p(z),
\]

where \(a(z)\) is the symbol of \(S_a\) and \(p(z)\) can be suitably chosen in such a way to satisfy the interpolation condition. By expressing \(q(z)\) in terms of its sub–symbols

\[
q(z) = q_{\text{even}}(z^2) + z \cdot q_{\text{odd}}(z^2), \quad z \in \mathbb{C} \setminus \{0\},
\]

we find that

\[
q(z) + q(-z) = 2 \cdot q_{\text{even}}(z^2).
\]

Then by imposing the interpolation condition (2.6), i.e., \(q_{\text{even}}(z) = 1\), we arrive at the relation

\[
a(z) \cdot p(z) + a(-z)p(-z) = 2 \quad (3.8)
\]
which is a generalized Bezout equation providing necessary and sufficient conditions for a Laurent polynomial \( p(z) \) to convert the subdivision operator associated with \( a(z) \) into the interpolating subdivision operator generated by \( q(z) = a(z) \cdot p(z) \).

Suitable coefficient-wise representations of \( p(z) \) are introduced to investigate conditions under which the (generalized) Bezout equation is solvable as well as to develop effective computational methods for its solution. In the sequel of this section the sought polynomial \( p(z) \) is of the form

\[
p(z) = p_1(z) = p_u z^\kappa + p_{\kappa+1} z^{\kappa+1} + \cdots + p_{\kappa+m} z^{\kappa+m}, \tag{3.9}
\]

or

\[
p(z) = p_2(z) = z \cdot p_1(z). \tag{3.10}
\]

with \( 2\kappa = m + 1 \). For such given polynomials \( p(z) \) the product Laurent polynomial \( q(z) = a(z) \cdot p(z) \) can be expressed as

\[
q(z) = a(z) \cdot p_1(z) = \sum_{\ell=0}^{2\kappa+m} q_{\ell} z^\ell = \sum_{\ell=0}^{2m+1} q_{\ell} z^\ell,
\]

and

\[
q(z) = a(z) \cdot p_2(z) = \sum_{\ell=1}^{2\kappa+m+1} q_{\ell} z^\ell = \sum_{\ell=1}^{2m+2} q_{\ell} z^\ell.
\]

Therefore we can try to determine the \( m + 1 \) unknown coefficients of \( p(z) \) to satisfy the following modified forms of (3.8)

\[
a(z) \cdot p_1(z) + a(-z) \cdot p_1(-z) = 2z^{2s}, \quad 0 \leq s \leq m, \tag{3.11}
\]

and

\[
a(z) \cdot p_2(z) + a(-z) \cdot p_2(-z) = 2z^{2s}, \quad 1 \leq s \leq m + 1, \tag{3.12}
\]

or, equivalently,

\[
a(z) \cdot p_1(z) - a(-z) \cdot p_1(-z) = 2z^{2s-1}, \quad 1 \leq s \leq m + 1. \tag{3.13}
\]

Indeed, it is clear that a solution of (3.11) or (3.13) provides a solution of (3.8) obtained by simply shifting the coefficient-wise representation of \( p(z) \).

In a matrix setting the solution of (3.11) and (3.13) reduces to solving a structured linear system whose coefficient matrix is Sylvester-like. Let \( a_0 = [a_{-\kappa}, \ldots, a_0, \ldots, a_\kappa]^T \in \mathbb{R}^{2\kappa+1} \) denote the coefficient vector of the Laurent polynomial \( a(z) \). The associated extended coefficient vector \( \tilde{a}_+ \in \mathbb{R}^{2\kappa+m+1} \) is defined by \( \tilde{a}_0 = [a_0^T, 0, \ldots, 0]^T \).

Similarly let us introduce the extended coefficient vector \( \tilde{a}_- \in \mathbb{R}^{2\kappa+m+1} \) associated with the polynomial \( a(-z) \). Moreover let \( Z = (z_{i,j}) \in \mathbb{R}^{2(m+1)\times 2(m+1)} \) be the down-shift matrix given by \( z_{i,j} = \delta_{i-1,j} \), where \( \delta_{i,j} \) is the Kronecker delta symbol. Set \( \mathcal{R}_+ \in \mathbb{R}^{2(m+1)\times 2(m+1)} \) the striped Toeplitz matrix

\[
\mathcal{R}_+ = [\tilde{a}_+ | Z \tilde{a}_+ | \ldots | Z^m \tilde{a}_+ ],
\]

and, similarly, define

\[
\mathcal{R}_- = [\tilde{a}_- | Z \tilde{a}_- | \ldots | Z^m \tilde{a}_- ].
\]
The coefficient matrix of the linear system (3.11) and (3.13) is \( R \in \mathbb{R}^{2(m+1)\times 2(m+1)} \). It is well known that \( R \) is invertible if and only if \( a(z) \) and \( a(-z) \) are relatively prime polynomials.

A different, alternative reduction which is useful for the computation of \( p(z) \) is the following. From the interpolation condition \( q_{\text{even}}(z) = z^{2s} \) for a suitable \( s \), we obtain that

\[
\sum_{\ell=0}^{m} q_{2\ell} z^{\ell} = z^s, \quad 0 \leq s \leq m, \quad (q(z) = a(z) \cdot p_1(z)),
\]

or

\[
\sum_{\ell=1}^{m+1} q_{2\ell} z^{\ell-1} = z^s, \quad 1 \leq s \leq m+1, \quad (q(z) = a(z) \cdot p_2(z)).
\]

The equations (3.14) translate in a linear system whose coefficient matrix \( H_1 \in \mathbb{R}^{(m+1)\times (m+1)} \) is of Hurwitz type, namely,

\[
H_1 = \begin{bmatrix}
  a_{-\kappa} & 0 & \ldots & \ldots & \ldots \\
  a_{-\kappa+2} & a_{-\kappa+1} & a_{-\kappa} & 0 & \ldots \\
  a_{-\kappa+4} & a_{-\kappa+3} & a_{-\kappa+2} & \ldots & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ldots \\
  a_{-\kappa+2m} & a_{-\kappa+2m-1} & a_{-\kappa+2m-2} & \ldots & \ldots
\end{bmatrix}.
\]

Similarly, the equations (3.15) reduce to a linear system whose coefficient matrix \( H_2 \in \mathbb{R}^{(m+1)\times (m+1)} \) as follows

\[
H_2 = \begin{bmatrix}
  a_{-\kappa+1} & a_{-\kappa} & 0 & \ldots & \ldots & \ldots \\
  a_{-\kappa+3} & a_{-\kappa+2} & a_{-\kappa+1} & a_{-\kappa} & 0 & \ldots \\
  a_{-\kappa+5} & a_{-\kappa+4} & a_{-\kappa+3} & a_{-\kappa+2} & \ldots & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\
  a_{-\kappa+2m+1} & a_{-\kappa+2m} & a_{-\kappa+2m-1} & a_{-\kappa+2m-2} & \ldots & \ldots
\end{bmatrix}.
\]

Observe that the last row of \( H_2 \) is

\[
[a_{-\kappa+2m+1}, \ldots, a_{-\kappa+m+1}] = [0, \ldots, 0, a_\kappa],
\]

and, therefore, both linear systems can be further reduced to smaller systems of size \( m \).

The transformation from the Sylvester resultant matrix \( R \) and the Hurwitz-type matrices \( H_1 \) and \( H_2 \) can be accomplished by matrix manipulations. Specifically, we can find a suitable matrix \( S \) such that \( R \cdot S \) is a direct sum of \( H_1 \) and \( H_2 \). This property is used in [2] in order to obtain conditions for the invertibility of \( H_1 \) and \( H_2 \) as well as to characterize the inverses in the case where \( a(z) \) is a symmetric Laurent polynomial. Under this condition the matrices \( H_1 \) and \( H_2 \) coincide up to a suitable permutation of rows and columns.

Summing up, we obtain a full correspondence among the solutions of (3.11), (3.13) and (3.14) and (3.15), respectively. For the sake of notational convenience we express this correspondence for the modified “shifted” polynomials \( a(z): = z^\kappa \cdot a(z) \) and
\[ p(z) = z^{-\kappa} \cdot p(z) \text{ that are polynomials expressed in the standard power basis of degree } m + 1 := n \text{ and } m = n - 1, \text{ respectively. Moreover, set} \]
\[ a(z) = a_0 + a_1 z + \ldots + a_n z^n, \quad a_j = a_{n-j}, \quad 0 \leq j \leq n, \quad a_n \neq 0, \]
and \( \mathcal{H}_n \) the associated Hurwitz type matrix
\[
\mathcal{H}_n = \begin{bmatrix}
    a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\
    a_0 & a_2 & a_4 & a_6 & \cdots & 0 \\
    0 & a_1 & a_3 & a_5 & \cdots & 0 \\
    0 & a_0 & a_2 & a_4 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & 0 & \cdots & a_n
\end{bmatrix}.
\] (3.16)

Observe that \( \mathcal{H}_n \) is the transpose of \( \mathcal{H}_2 \). Denote by \( A_n \) the leading principal submatrix of order \( n - 1 \). Whenever \( \mathcal{H}_n \) is non singular, \( \mathcal{H}_n \) and \( A_n \) are related by
\[
\mathcal{H}_n^{-1} = \begin{bmatrix}
    A_n^{-1} & \mathbf{0} \\
    \mathbf{0} & a_n^+ 
\end{bmatrix}.
\]

The following results are proved in [2].

**Proposition 1** Let \( a(z) \) be a symmetric degree-\( n \) polynomial relatively prime with \( a(-z) \) and \( \mathcal{H}_n \) its associated matrix of order \( n \). Then \( \mathcal{H}_n \) is invertible and, moreover, the polynomial \( p_i(z) \) with coefficients given by the entries of the \( i \)-th row of \( A_n^{-1} \), \( 1 \leq i \leq n - 1 \), is the unique polynomial of degree less than \( n \) such that
\[
a(z)p_i(z) - a(-z)p_i(-z) = 2z^{2i-1}, \quad 1 \leq i \leq n - 1. \] (3.17)

This means that the polynomial solutions of the Bezout-like equation (3.13) are determined by the entries in the rows of \( \mathcal{H}_n^{-1} \) and, equivalently, in the columns of \( \mathcal{H}_2^{-1} \).

**Remark 1** The proof of Proposition 1 established in [2] used matrix manipulations to relate \( \mathcal{H}_n \) with the Sylvester resultant matrix \( \mathcal{R} \). The derivation presented here starts with the functional problem by enlightening its possible reductions in a linear algebra setting. In this way the relationships among polynomial and matrix formulations of the problem are clarified and, moreover, the set of the solutions of the problem is enriched. A result analogous to the above proposition also holds for the solutions of (3.11) that are obtained from the entries located in the columns of \( \mathcal{H}_1^{-1} \). For a symmetric polynomial \( a(z) \) these solutions are related to the ones of (3.17) by the reversion of the coefficients. In passing, it is worth noting that the approach can further be extended to deal with subdivision masks of arity greater than 2. This extension will be the subject of a forthcoming paper.

As an immediate consequence of Proposition 1 we obtain the following.

**Proposition 2** Given a symmetric degree-\( n \) polynomial \( a(z) \) relatively prime with \( a(-z) \) and such that \( a(1) = 2, a(-1) = 0 \), the Laurent polynomials
\[
m_i(z) := \frac{a(z)p_i(z)}{z^{2i-1}}, \quad 1 \leq i \leq n - 1, \] (3.18)
are interpolatory symbols and satisfy
\[
m_i(1) = 2, \quad m_i(-1) = 0, \quad 1 \leq i \leq n - 1.
\]
The result provides a practical way to construct a family of interpolatory masks from a given approximating one consisting in computing the matrix $A^{-1}$ and reading its entries. This approach seems to be especially tailored for symmetric Hurwitz subdivision symbols which result into totally positive (TP) Hurwitz matrices. The procedures described in [10] can be adjusted for the efficient and stable computations of the coefficients of the interpolatory masks generated in the B-spline case and “shifted” affine combinations of them. Different solution methods can rely upon the equivalent polynomial formulations (3.11), (3.12) and (3.13). Although these latter methods can perform poorly in a numerical environment, they have at least two advantages for applications in a numeric-symbolic environment. First, in several interesting cases they yield semi-explicit representations of the associated interpolatory symbols which can be useful for the convergence analysis. Secondly, these polynomial methods are particularly suited to exploit additional features of the symbol $a(z)$. In the next section we present two computational procedures for solving (3.17) and its variations (3.11), (3.12) and (3.13) that are effective in contexts of applicative value.

4 Polynomial equation solvers

The condition $a(-1) = 0$ implies that $a(z) = (z + 1)^{\kappa} \cdot \tilde{a}(z)$, $\kappa \geq 1$. Effective computational procedures for computing the interpolatory masks associated with the symmetric polynomial $a(z)$ of degree $n$ can be developed in two important situations. The first one is the case where $n - \kappa$ is very small compared with $n$, which covers many generalizations of B-splines. In the second interesting case the value of $\kappa$ can be small but the knowledge of the zeros of $a(z)$ permits the representation of the symbol in factored form. This is the case treated in Definition 1. In the next two subsections we describe computational methods for dealing with these two cases. Specifically, in Subsection 4.1 we address the first case by representing $a(z)$ in a convenient polynomial basis and then computing the coefficients of the solution $p_i(z)$ represented in such basis. The resulting algorithm is described in [2]. The case of factored symbols is the subject of Subsection 4.2 where a novel polynomial solver is designed which is based on the partial fraction decomposition of $\frac{p_i(z)}{a(-z)}$.

4.1 Polynomial equation solver for symbols in Bernstein-type polynomial bases

Let us consider the task of solving a polynomial equation of the more general form

$$a(z)p(-z) + a(-z)p(z) = b(z),$$

with

$$\deg(a(z)), \deg(p(z)) \leq n, \ \deg(b(z)) \leq 2n, \ b(z) = b(-z).$$

The solution method we propose in [2] relies upon the representation of the polynomials in (4.19) by using the Bernstein type polynomial basis

$$(1 - z)^n, \ (1 - z)^{n-1}(1 + z), \ldots, (1 - z)(1 + z)^{n-1}, \ (1 + z)^n$$
of the vector space of real polynomials of degree less than or equal to $n$. That is,
\[
a(z) = \sum_{j=0}^{n} \hat{a}_j (1 - z)^{n-j} (1 + z)^j, \quad p(z) = \sum_{j=0}^{n} \hat{p}_j (1 - z)^{n-j} (1 + z)^j.\]

Moreover let us assume that $b(z)$ is also suitably represented by using the polynomial basis $(1 - z)^{2n}, (1 - z)^{2n-1}(1 + z), \ldots, (1 - z)^{2n-1}(1 + z)^{2n}$ of the vector space of real polynomials of degree less than or equal to $2n$, namely
\[
b(z) = \sum_{j=0}^{2n} \hat{b}_j (1 - z)^{2n-j} (1 + z)^j.\]

The Möbius transformation $z \to w = \frac{1 + z}{1 - z}$ is employed to transform the polynomial equation (4.19) in the equivalent form
\[
\hat{a}(w)\hat{p} \left( \frac{1}{w} \right) + \hat{a} \left( \frac{1}{w} \right) \hat{p}(w) = \hat{b}(w) + \hat{b} \left( \frac{1}{w} \right),
\]
where
\[
\hat{a}(w) := \sum_{j=0}^{n} \hat{a}_j w^j, \quad \hat{p}(w) := \sum_{j=0}^{n} \hat{p}_j w^j,
\]
and
\[
\hat{b}(w) := \frac{\hat{b}_n}{2} + \sum_{j=1}^{n} \hat{b}_{n+j} w^j.
\]
The equation (4.21) reduces to a Toeplitz–plus–Hankel linear system whose coefficient matrix of order $n + 1$ is the Toeplitz–plus–Hankel matrix given by
\[
\mathcal{J}(\hat{a}_0, \ldots, \hat{a}_n) = \begin{bmatrix} \hat{a}_0 & \cdots & \hat{a}_n \\ \vdots & \ddots & \vdots \\ \hat{a}_0 & & \hat{a}_n \end{bmatrix} + \begin{bmatrix} \hat{a}_0 & \cdots & \hat{a}_n \\ \vdots & \ddots & \vdots \\ \hat{a}_0 & & \hat{a}_n \end{bmatrix}.
\]

Under the assumption that $a(z)$ and $a(-z)$ are relatively prime the matrix $\mathcal{J}$ turns out to be invertible [4] and, therefore, the computation of the coefficients of $\hat{p}(w)$ amounts to compute the coefficients of $\hat{b}(w)$ forming the known vector and then to solve the corresponding linear system. The case $b(z) = 2z^{2s}, 1 \leq s \leq n$, is considered in the following proposition essentially given in [2].

**Proposition 3** Let $a(z) = \sum_{j=0}^{n} \hat{a}_j (1 - z)^{n-j} (1 + z)^j$, be a symmetric polynomial of degree $n$ with $a(z)$ and $a(-z)$ relatively prime. Then, the unique polynomial solution $p_i(z)$ of (3.17) satisfies
\[
zp_i(z) = (1 + z)^n \hat{p}_i \left( \frac{1}{w} \right), \quad \hat{p}_i(w) := \sum_{j=0}^{n} \hat{p}_j^{(i)} w^j, \quad w = \frac{1 + z}{1 - z}.
\]
where the coefficients $\rho_s^{(i)}$ of $\tilde{p}_i(w)$ are the entries of the solution of the linear system

$$F(a_0, \ldots, a_n)
\begin{bmatrix}
\rho_0^{(i)} \\
\vdots \\
\rho_{2n}^{(i)}
\end{bmatrix}_T = 2^{1-2n}
\begin{bmatrix}
\rho_0^{(i)} \\
\vdots \\
\rho_{2n}^{(i)}
\end{bmatrix}_T,$$

with

$$\rho_s^{(i)} = \sum_{j=2i-2n+s}^{2i} (-1)^j \binom{2i}{j} \left(\frac{2n - 2i}{s - j}\right), \quad n \leq s \leq 2n. \quad (4.22)$$

The computational interest of this result stems from the observation that for symbols obtained as generalization of the one of B-splines the representation in the Bernstein-type polynomial basis is extremely sparse and, moreover, in many important cases the resulting matrix $F(a_0, \ldots, a_n)$ has some special structure so that it can be explicitly inverted. In this way we are able to find explicit representations of the polynomial solutions of (3.17). Applications of these techniques are given in Section 6.

4.2 A root-based polynomial equation solver

A different strategy for solving (3.17) can be pursued whenever the factorization of the symbol is assumed to be known. To be specific, let us suppose that

$$a(z) = a_0 + a_1 z + \ldots + a_n z^n = a_n \prod_{j=0}^{m} (z - z_j)^{k_j},$$

with $z_i \neq z_j$ if $i \neq j$ and $k_0 + \ldots + k_m = n$. Then it is shown that the unique solution $p_i(z)$ of (3.17) can be obtained by imposing certain interpolation conditions at the zeros of $a(z)$ and $a(-z)$.

Let us start by recalling the concept of Hermite-Lagrange interpolation polynomial of a given differentiable function $f(z)$ on the set of nodes $\eta_0, \ldots, \eta_{\ell}$ with multiplicities $b_0, \ldots, b_{\ell}$, $b_0 + \ldots + b_{\ell} = r + 1$, respectively. Suppose that the function $f(z)$ possesses derivatives $f^{(j)}(\eta_i)$, $0 \leq j \leq h_i - 1$, $0 \leq i \leq \ell$. Then there exists a unique polynomial $H_f(z)$ of degree at most $r$ satisfying the interpolation conditions

$$H_f^{(j)}(\eta_i) = f^{(j)}(\eta_i), \quad 0 \leq j \leq h_i - 1, \quad 0 \leq i \leq \ell.$$ 

This polynomial is generally referred to as the Hermite-Lagrange interpolation polynomial of $f(z)$ on the prescribed set of nodes. By setting $\omega(z) := (z - \eta_0)^{h_0} \cdots (z - \eta_{\ell})^{h_{\ell}}$ we find the Lagrange-type representation

$$H_f(z) = \sum_{i=0}^{\ell} \sum_{j=0}^{h_i - 1} \sum_{h=0}^{h_i - j - 1} f^{(j)}(\eta_i) \frac{1}{h!j!} \left(\frac{(z - \eta_i)^h}{\omega(z)}\right)_{z=\eta_i} \frac{\omega(z)}{(z - \eta_{h_i-j-h})}$$

and, equivalently, the partial-fraction representation

$$H_f(z) = \omega(z) \sum_{i=0}^{\ell} \sum_{s=1}^{h_i} \frac{1}{(z - \eta_i)^s} \left(\sum_{j=0}^{h_i-s} S(h_i - j - s, j, i)\right),$$

where

$$S(h, j, i) = f^{(j)}(\eta_i) \frac{1}{h!j!} \left(\frac{(z - \eta_i)^h}{\omega(z)}\right)_{z=\eta_i}^{(h)}, \quad 0 \leq h, j \leq h_i - 1, \quad 0 \leq i \leq \ell. \quad (4.23)$$
Let \( \ell = 2m + 1 \) and \( \eta_0 = \ldots = \eta_{(\ell-1)/2} = z_m, \eta_{(\ell+1)/2} = -z_m, \eta_\ell = -z_m \) with multiplicities \( h_0 = h_{(\ell+1)/2} = k_0, \ldots, h_\ell = h_{(\ell-1)/2} = k_m \). Observe that

\[ r + 1 = h_0 + \ldots + h_\ell = 2k_0 + \ldots + 2k_m = 2n \]

and

\[ \omega(z) = \prod_{i=0}^{m} (z - z_i)^{k_i} \prod_{i=0}^{m} (z + z_i)^{k_i} = a_n^{-2} (-1)^n a(z) a(-z). \]

Moreover, let \( f(z) = 2z^{2t-1} \) for a given fixed integer \( t \) with \( 1 \leq t \leq n \). By replacing \( f(z) \) with its Hermite-Lagrange form in (3.17) we find that

\[ (-1)^n a_n^2 \left( \frac{p_t(z)}{a(z)} - \frac{p_t(-z)}{a(z)} \right) = \sum_{i=0}^{k_0} \sum_{s=1}^{k_i} \frac{1}{(z - z_i)^s} \left( \sum_{j=0}^{h_i - s} S(h_i - j - s, j, i) \right). \]

Since \( a(z) \) and \( a(-z) \) are relatively prime we can separate the partial fraction decompositions of the two rational functions on the left-hand side. This gives

\[ (-1)^n a_n^2 \frac{p_t(-z)}{a(z)} = \sum_{i=0}^{m} \prod_{j=0}^{k_i} (z - z_j)^{k_j} \sum_{i=0}^{m} \sum_{s=1}^{k_i} \frac{1}{(z - z_i)^s} \left( \sum_{j=0}^{k_i - s} S(k_i - j - s, j, i) \right), \]

and, therefore,

\[ p_t(-z) = (-1)^n a_n^{-1} \prod_{j=0}^{m} (z - z_j)^{k_j} \sum_{i=0}^{m} \sum_{s=1}^{k_i} \frac{1}{(z - z_i)^s} \left( \sum_{j=0}^{k_i - s} S(k_i - j - s, j, i) \right), \]

and

\[ p_t(z) = -a_n^{-1} \prod_{j=0}^{m} (z + z_j)^{k_j} \sum_{i=0}^{m} \sum_{s=1}^{k_i} \frac{(-1)^s}{(z + z_i)^s} \left( \sum_{j=0}^{k_i - s} S(k_i - j - s, j, i) \right). \]  \hspace{1cm} (4.24)

Summing up we arrive at the following

**Proposition 4** Let \( a(z) = a_n \prod_{j=0}^{m} (z - z_j)^{k_j} \) be a polynomial of degree \( n \), where \( z_i \neq z_j \) if \( i \neq j \), \( k_0 + \ldots + k_m = n \) and \( a(z) \) and \( a(-z) \) are relatively prime. Then, the unique polynomial solution \( p_t(z) \), \( 1 \leq t \leq n \), of (3.17) satisfies (4.24) with

\[ S(h, j, i) = f^{(j)}(\eta_i) \frac{1}{h! j!} \left( \frac{(z - z_i)^{k_i}}{\omega(z)} \right)^{(h)} \bigg|_{z = z_i}, \quad 0 \leq h, j \leq k_i - 1, \quad 0 \leq i \leq m, \]

where \( f(z) = 2z^{2t-1} \) and \( \omega(z) \) is the monic polynomial associated with \( a(z) a(-z) \).

From a computationally viewpoint the computation of this form of \( p(z) \) essentially reduces to evaluating the coefficients \( S(h, j, i) \) defined in (4.23). The derivatives of \( w_i(z) = (z - \eta_i)^{k_i} \omega(z) \) evaluated at \( z = \eta_i = z_i \) can be obtained iteratively by successive differentiations of the relation \( w_i(z) \cdot \gamma_i(z) = 1 \), where \( \gamma_i(z) \) is the reciprocal of \( w_i(z) \), namely,

\[ \gamma_i(z) := \prod_{j=0, j \neq i}^{m} (z - z_j)^{k_j} \prod_{j=0}^{m} (z + z_j)^{k_j}. \]

The process involves the derivatives of \( \gamma_i(z) \) at \( z = \eta_i = z_i \) which are determined from those of \( a(z) \cdot a(-z) \) obtained by using the Ruffini-Horner scheme.
5 An algorithm for deriving non-stationary interpolatory subdivision schemes

So far we have introduced a quite general strategy for computing a family of non-stationary interpolatory subdivision schemes associated with a non-stationary approximating subdivision scheme via the solution of equations (3.17) at each recursion step. The basic procedure turns out to be as follows: Assuming \( \{ a_k(z), k \geq 0 \} \) are the symmetric degree-\( n(k) \) symbols of an approximating non-stationary scheme with \( a_k(z) \) and \( a_k(-z) \) relatively prime for all \( k \geq 0 \), we construct the non-stationary interpolatory subdivision scheme based on the symbols \( \{ m^k_i(k)(z), k \geq 0 \} \) where, for each \( k \), \( m^k_i(k)(z), 1 \leq i(k) \leq n(k) - 1 \), is one of the interpolatory symbols satisfying (3.17). Certainly, the performance of the non-stationary subdivision scheme will depend on the selection of the sequence \( i(k), k \geq 0 \). For clarity we describe the procedure in algorithmic form.

```
Appint Algorithm

Input: \( \{ a_k(z), k \geq 0 \} \), symmetric degree-\( n(k) \) symbols;
      \( \{ i(k), k \geq 0 \} \), with \( 1 \leq i(k) \leq n(k) - 1 \)

For \( k = 0, 1, \ldots \)
  Check whether \( a_k(z) \) is relatively prime with \( a_k(-z) \)
  From \( a_k \) construct the matrix \( A_{n(k)} \)
  Determine \( A_{n(k)}^{-1}(i(k),:) \), the \( i(k) \)th row of \( A_{n(k)}^{-1} \)
  Set \( p^k_i(k):= A_{n(k)}^{-1}(i(k),:) \)
  Construct the interpolatory symbol \( m^k_i(k)(z):= a_k(z)p^k_i(k)(z) \)

Output: \( \{ m^k_i(k)(z), k \geq 0 \} \)
```

The computation of \( A_{n(k)}^{-1} \) can be performed by matrix inversion methods applied to \( H_{n(k)} \) or, in the view of the polynomial reductions shown in the previous sections, by polynomial methods applied for the solution of (3.17). For the non-stationary subdivision scheme generated by masks \( m^k_i(k)(z), 1 \leq i(k) \leq n(k) - 1, k \geq 0 \), we can prove an important reproduction property: The function space reproduced by the interpolatory schemes is the same reproduced by the approximating one, at least for all relevant approximating schemes. First we need a preliminary result given in [6].

**Proposition 5** Let \( \{ q_k(z), k \geq 0 \} \) a sequence of interpolatory symbols. The subdivision scheme associated with such a sequence reproduces \( V_{T, \gamma} \) if and only if for each \( k \geq 0 \)

\[
q^k(e^{-\frac{\theta}{2\tau_{\ell+1}}}) = 2, \quad q^k(-e^{-\frac{\theta}{2\tau_{\ell+1}}}) = 0, \quad \ell = 1, \ldots, N
\]

\[
\frac{d^{h\ell}}{dx^{h\ell}} q^k(\pm e^{-\frac{\theta}{2\tau_{\ell+1}}}) = 0, \quad h\ell = 1, \ldots, \tau\ell - 1, \quad \ell = 1, \ldots, N.
\]

(5.25)

We are now in a position to state the reproduction result.
Proposition 6 Let \(a_n^k(z), \quad k \geq 0\), symmetric symbols of the form
\[
a_n^k(z) = 2 \prod_{\ell=1}^{N} \prod_{h=0}^{\tau_{\ell}-1} \frac{\theta_{\ell}^h z}{\xi^{2h+1}} + 1, \quad k \geq 0,
\]
with \(a_n^k(z), a_n^k(-z)\) relatively prime. \hspace{1cm} (5.26)

If the non-stationary subdivision scheme based on the symbols \(\{a_n^k(z), \quad k \geq 0\}\) is convergent, the interpolatory subdivision scheme based on the symbols
\[
m_i^k(z) = \frac{a_n^k(z)p_i^k(z)}{\xi^{2i-1}}, \quad 1 \leq i \leq T - 1,
\]
whenever convergent, reproduces functions from the space
\[
V_{T\gamma} = \text{Span}\{x^{h_\ell}e^{\theta_\ell x}, \quad h_\ell = 0, \cdots, \tau_\ell - 1, \quad \ell = 1, \cdots, N\}, \quad \text{with} \quad T = \sum_{\ell=1}^{N} \tau_\ell.
\]

Proof: Due to the special form of the polynomials \(a_n^k(z)\), explicitly given in (5.26), it results
\[
a_n^k(-e^{-\theta_\ell}) = 0, \quad \frac{d^{h_\ell}}{dz^{h_\ell}} a_n^k(-e^{-\theta_\ell}) = 0, \quad h_\ell = 1, \cdots, \tau_\ell - 1, \quad \ell = 1, \cdots, N.
\]

By the Leibnitz’s differentiation rule, we easily get an analogous relation to be satisfied by all \(m_i^k(z)\) that is
\[
m_i^k(-e^{-\theta_\ell}) = 0, \quad \frac{d^{h_\ell}}{dz^{h_\ell}} m_i^k(-e^{-\theta_\ell}) = 0, \quad h_\ell = 1, \cdots, \tau_\ell - 1, \quad \ell = 1, \cdots, N.
\]

It remains to consider the behavior of \(m_i^k(z)\) and its derivatives at the points \(e^{-\theta_\ell}\). Now, since for each \(k\)
\[
m_i^k(z) + m_i^k(-z) = 2, \quad 1 \leq i \leq T - 1,
\]
it follows that
\[
m_i^k(e^{-\theta_\ell}) = 2
\]
as well as
\[
\frac{d^{h_\ell}}{dz^{h_\ell}} m_i^k(e^{-\theta_\ell}) = (-1)^{h_\ell+1} \frac{d^{h_\ell}}{dz^{h_\ell}} m_i^k(-e^{-\theta_\ell}) = 0, \quad h_\ell = 1, \cdots, \tau_\ell - 1, \quad \ell = 1, \cdots, N.
\]

The use of Proposition 5 concludes the proof. \square

Examples of interpolatory subdivision schemes generated by the Appint Algorithm and their properties will be investigated in the next sections.
6 The interpolatory non-stationary B-spline case and its “generalizations”

This section is devoted to the application of the Appint Algorithm to non-stationary B-spline subdivision schemes and their generalizations, i.e. symbols being “shifted” affine combinations of B-spline symbols. In these cases, at each iteration, the solution of the polynomial equation can be addressed by using the techniques discussed in subsection 4.1 and this leads to an efficient procedure for the computation of the coefficients of the corresponding family of interpolatory masks. A non-stationary B-spline subdivision scheme is associated to \( \{a^k(z), \; k \geq 0\} \), the set of degree-\( k \) symmetric polynomials \( a^k(z) = \frac{(1 + z)^k}{2^{k-1}} \) trivially relatively prime with \( a^k(-z) = \frac{(1 - z)^k}{2^{k-1}} \). Thus, equation (3.17) reads as

\[
(1 + z)^k (z p_i^k(z)) + (1 - z)^k (-z p_i^k(-z)) = 2^k z^{2i}, \quad 1 \leq i \leq k - 1,
\] (6.27)

which gives us a simple way to compute the coefficients of the polynomial \( p_i^k(z) \). For the sake of simplicity we refer to the polynomial \( -z p_i^k(-z) \) as to \( p_i^k(z) \). Since \( a^k(z) = \frac{(1 + z)^k}{2^{k-1}} \) we find that \( \hat{a}^k(w) := \frac{w^k}{2^{k-1}} \), the matrix \( \mathcal{J}(\hat{a}_0, \ldots, \hat{a}_n) \) is antidiagonal and, therefore, the solution \( \hat{p}^k(w) \) of (4.21) can immediately be reconstructed from the coefficients of \( \hat{b}^k(w) \) in Proposition 3. Indeed, we obtain that

\[
w^k \sum_{j=0}^k \hat{p}_j w^{-j} + \frac{1}{w^k} \sum_{j=0}^k \hat{p}_j w^j = 2^{-k} \left( \rho_0^{(i)} w^{-k} + \cdots + \rho_k^{(i)} w^{-1} + \rho_k^{(i)} + \rho_{k+1}^{(i)} w + \cdots + \rho_{2k}^{(i)} w^k \right),
\] (6.28)

which gives

\[
\hat{p}_j = 2^{-k} \rho_j^{(i)}, \quad 0 \leq j \leq k - 1, \quad \hat{p}_k = 2^{-k-1} \rho_k^{(i)}.
\] (6.29)

As a generalization of the previous situation we can consider a five term “shifted” affine combination of a B-spline symbol, i.e. we consider the symbol

\[
c_{\alpha}^k(z) = \frac{(1 + z)^{n-4}}{2^{n-k}} \left( a^k + \beta^k z + \left(1 - 2a^k - 2\beta^k\right)z^2 + \beta^k z^3 + \alpha^k z^4 \right),
\] (6.30)

with \( b^k(z) := \left( a^k + \beta^k z + \left(1 - 2a^k - 2\beta^k\right)z^2 + \beta^k z^3 + \alpha^k z^4 \right) \) and \( b^k(-z) \) relatively prime for all \( k \). As shown in [3], this symbol allows us to get a unified representation of several of the well known (stationary and non-stationary) subdivision schemes corresponding to specific settings of the combination parameters \( \alpha^k, \beta^k \in \mathbb{R} \) and to different values of \( n \in \mathbb{N}, \; n > 4 \). For example, we can range from degree-(\( n - 5 \)) to degree-(\( n - 1 \)) B-splines. In fact, when \( \alpha^k = \beta^k = 0, \; c_{\alpha}^k(z) \) coincides exactly with the symbol of the degree-(\( n - 5 \)) B-spline, while for \( \alpha^k = 0, \; \beta^k = \frac{1}{4} \) and \( \alpha^k = \frac{1}{16}, \; \beta^k = \frac{1}{4} \) we have the degree-(\( n - 3 \)) and degree-(\( n - 1 \)) B-splines, respectively. In view of (6.27), it is convenient to write the quartic polynomial \( b^k(z) \), as

\[
\frac{1}{16} \left( 1 - 4\beta^k \right) (1 - z)^4 + (16\alpha^k + 4\beta^k - 2) (1 - z)^2 (1 + z)^2 + (1 + z)^4,
\]

and derive the polynomial \( \hat{a}^k(w) = \hat{a}_{n-4}^k w^{-n-4} + \hat{a}_{n-2}^k w^{-n-2} + \hat{a}_n^k w^n \) in (6.27) explicitly as

\[
\hat{a}^k(w) = \frac{1 - 43\beta^k}{16} w^{-n-4} 16\alpha^k + 4\beta^k - 2 \frac{16}{w^{-n-2} + 16} w^n.
\]
The above expression of $\hat{a}(w)$ allows us to provide an efficient strategy for the computation of $\hat{p}(w) = \sum_{j=0}^{n} \hat{p}_j w^j$ which reduces to the solution of a $5 \times 5$ linear system. In fact, we can observe that in this case the matrix $J(\hat{a}_0, \ldots, \hat{a}_n)$ is upper triangular (with respect to to the main antidiagonal) except that for a bulge of order 5 in the up–right corner. A direct comparison of the polynomial coefficients in the left hand side of (6.27), i.e.,

$$
\begin{pmatrix}
\hat{a}_{n-4}w^{-4} + \hat{a}_{n-2}w^{-2} + \hat{a}_n w
\end{pmatrix}
(\hat{p}_0 + \cdots + \hat{p}_n w^{-n}) +
$$

with those in the right hand side of (6.27), i.e.,

$$
(\rho_0^{(i)} w^{-n} + \cdots + \rho_{n-1}^{(i)} w^{-1} + \rho_n^{(i)} w + \cdots + \rho_{2n}^{(i)} w^{n}),
$$

leads to the following relations for the coefficients

$$
\hat{a}_n\hat{p}_0 = 2^{4-n} \rho_0^{(i)},
$$

$$
\hat{a}_n\hat{p}_1 = 2^{4-n} \rho_1^{(i)},
$$

$$
\hat{a}_n\hat{p}_2 = 2^{4-n} \rho_2^{(i)} - \hat{a}_{n-2}\hat{p}_3,
$$

$$
\hat{a}_n\hat{p}_3 = 2^{4-n} \rho_3^{(i)} - \hat{a}_{n-2}\hat{p}_4,
$$

$$
\hat{a}_n\hat{p}_j = 2^{4-n} \rho_j^{(i)} + \hat{a}_{n-4}\hat{p}_{j-4} - \hat{a}_{n-2}\hat{p}_{j-2}, \quad j = 4, \cdots, n - 5,
$$

and the linear system

$$
\begin{cases}
\hat{a}_n\hat{p}_{n-4} + \hat{a}_n\hat{p}_{n} &= 2^{4-n} \rho_{n-4}^{(i)} - \hat{a}_{n-2}\hat{p}_{n-6} - \hat{a}_{n-4}\hat{p}_{n-8} \\
\hat{a}_n\hat{p}_{n-3} + \hat{a}_n\hat{p}_{n-1} &= 2^{4-n} \rho_{n-3}^{(i)} - \hat{a}_{n-2}\hat{p}_{n-5} - \hat{a}_{n-4}\hat{p}_{n-7} \\
(\hat{a}_n + \hat{a}_n)\hat{p}_{n-2} + \hat{a}_{n-2}\hat{p}_{n-4} + \hat{a}_{n-2}\hat{p}_{n} &= 2^{4-n} \rho_{n-2}^{(i)} - \hat{a}_{n-4}\hat{p}_{n-6} \\
(\hat{a}_n + \hat{a}_n)\hat{p}_{n-1} + (\hat{a}_n + \hat{a}_n)\hat{p}_{n-3} &= 2^{4-n} \rho_{n-1}^{(i)} - \hat{a}_{n-4}\hat{p}_{n-5} \\
2\hat{a}_n\hat{p}_{n-4} + 2\hat{a}_n\hat{p}_{n-2} + 2\hat{a}_n\hat{p}_{n} &= 2^{3-n} \rho_n^{(i)},
\end{cases}
$$

to be solved for getting the remaining coefficients.

A further generalization of the B-spline case is given by the non-stationary affine combination (as in (6.30)) of exponential reproducing spline symbols defined in (2.5). This generalization will be considered in several examples in the next section.
7 Interpolatory exponential reproducing non-stationary subdivision schemes

In the following we consider several examples of non-stationary subdivision symbols depending on a parameter $v^k \in (0, +\infty)$, $k \geq 0$, defined through the expression

$$v^k = \frac{1}{2}(e^{\theta/2^k+1} + e^{-\theta/2^k+1})$$

with $\theta = \theta_{\ell}$, $\ell = 1, ..., N$, as in Lemma 1. As shown in [1] this means that, once assigned the starting value $v^{-1} \in (-1, +\infty)$, the parameter $v^k$ can be recursively updated at each successive iteration through the formula $v^k = \sqrt{\frac{v^{k-1}+1}{2}}$, $k \geq 0$. For each non-stationary subdivision scheme associated to these symbols we construct the corresponding family of interpolatory symbols maintaining the same reproduction properties. The interpolatory schemes can be obtained directly by inverting the subdivision matrix or, differently, by using the strategy discussed in subsection 4.2.

Example 1: We consider the $C^2$ approximating scheme by Morin et al. [9] whose $k$-level symbol is

$$a^k_3(z) = \frac{1}{2}(z + 1)^2 \frac{z^2 + 2v^k z + 1}{2(v^k + 1)}$$

and the related mask

$$a^k_3 = \begin{bmatrix} \frac{1}{4(1+v^k)} & \frac{1}{2} & \frac{1 + 2v^k}{2(1+v^k)} & \frac{1}{2} & \frac{1}{4(1+v^k)} \end{bmatrix}.$$ 

Since the polynomials $a^k_3(z)$, $a^k_3(-z)$ are relatively prime, we can apply the Appint Algorithm involving at each step the matrix

$$A^k_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2(1+v^k)} & \frac{1 + 2v^k}{2(1+v^k)} & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix},$$

from which we obtain the following inverse

$$(A^k_3)^{-1} = \frac{1}{2v^k} \begin{pmatrix} 4v^k + 1 & -2(1 + v^k) & 1 \\ -1 & 2(1 + v^k) & -1 \\ 1 & -2(1 + v^k) & 4v^k + 1 \end{pmatrix}.$$ 

The rows of $(A^k_3)^{-1}$ define three interpolatory subdivision masks

$$m^k_{3,1} = \begin{bmatrix} 1 + 4v^k \\ 8v^k(v^k+1) \\ 8v^k(v^k+1) \\ 8v^k(v^k+1) \end{bmatrix},$$

$$m^k_{3,2} = \begin{bmatrix} (2v^k+1)^2 \\ 8v^k(v^k+1) \\ 8v^k(v^k+1) \\ 8v^k(v^k+1) \end{bmatrix},$$

$$m^k_{3,3} = \begin{bmatrix} 1 \\ 8v^k(v^k+1) \\ 8v^k(v^k+1) \\ 8v^k(v^k+1) \end{bmatrix}.$$ (7.32)
associated with the symbols

\[ m_{3,1}^k(z) = 4v^{k+1}a_k(z) - \frac{v^{k+1}}{v^k}za_k^4(z) + \frac{1}{2v^k}z^2a_k^5(z), \]

\[ m_{3,2}^k(z) = -\frac{1}{2v^k}a_k(z) + \frac{v^{k+1}}{v^k}za_k^4(z) - \frac{1}{2v^k}z^2a_k^5(z), \]

\[ m_{3,3}^k(z) = \frac{1}{2v^k}a_k^5(z) - \frac{v^{k+1}}{v^k}za_k^4(z) + \frac{4v^{k+1}+1}{2v^k}z^2a_k^5(z). \] (7.33)

Notice that the interpolatory masks \( \{ m_{3,2}^k, k \geq 0 \} \) corresponds to the \( C^1 \) four-point scheme introduced in [1], which possesses the special feature of reproducing functions from the space \( \langle 1, x, e^{\theta x}, e^{-\theta x} \rangle \) for any non-negative real or imaginary constant \( \theta \), as well as the approximating non-stationary subdivision scheme correspondent to the masks \( \{ a_k^3, k \geq 0 \} \). More precisely, whenever we apply this interpolatory subdivision mask to an initial polyline made of points lying equidistant with uniform parameter spacing \( u > 0 \) on a given curve from that function space, by setting

\[ v^{-1} = \frac{1}{2}(e^{\theta u} + e^{-\theta u}) \] (7.34)

we are able to reproduce the curve from which those points are sampled. Fundamental shapes in CAGD that turn out to be reproducible by this subdivision scheme are all conic sections (Figure 1).

![Fig. 1](image)

Observe that the pictures in Figure 1 correspond also to a reproduction capability of the subdivision schemes discussed in the sequel.

In order to derive the interpolatory symbols associated to \( \{ a_k^3(z), k \geq 0 \} \), one can alternatively employ the techniques introduced in subsection 4.2 based on the computation of the partial fraction decompositions of certain rational functions associated with the given symbol. For instance, we find

\[
\frac{2z^3}{a_k^3(z)a_k^5(-z)} = \frac{2(v^k + 1)}{(v^k - 1)(z + 1)^2} - \frac{2(v^k + 1)}{(v^k - 1)(z - 1)^2} + \frac{2(v^k + 1)}{v^k(v^k - 1)(z^2 - 2v^kz + 1)} - \frac{2(v^k + 1)}{v^k(v^k - 1)(z^2 + 2v^kz + 1)}
\]
which gives
\[
\frac{p_k^2(z)}{a_k^2(z)} = \frac{2(v^k + 1)}{v^k(v^k - 1)(z^2 + 2v^kz + 1)} - \frac{2(v^k + 1)}{(v^k - 1)(z + 1)^2}
\]
and, hence,
\[
p_k^2(z) = \frac{(1 - z)^2}{2v^k(v^k - 1)} - \frac{z^2 - 2v^kz + 1}{2(v^k - 1)} = \frac{1}{2v^k} \left( -z^2 + 2(v^k + 1)z - 1 \right).
\]

Observe that the coefficients of \(p_k^2(z)\) are the entries in the second row of \((A_k^3)^{-1}\).

Example 2: Let
\[
a_{5,1}^k(z) = \frac{(z + 1)^4(z^2 + 2v^kz + 1)}{16(v^k + 1)}
\]
be the \(k\)-level symbol of the first \(C^4\) approximating scheme proposed in [11] and
\[
a_{5,1}^k = \frac{1}{16(v^k + 1)} \begin{bmatrix} 1 & 2v^k + 2 & 8v^k + 7 & 4(3v^k + 2) & 8v^k + 7 & 2(v^k + 2) & 1 \end{bmatrix}
\]
the associated mask. Since, again, \(a_{5,1}^k(z)\) and \(a_{5,1}^k(-z)\) are relatively prime, running the Appint Algorithm, at each step we construct the matrix
\[
A_{5,1}^k = \frac{1}{16(v^k + 1)} \begin{bmatrix} 2(v^k + 2) & 4(3v^k + 2) & 2(v^k + 2) & 0 & 0 \\
1 & 8v^k + 7 & 8v^k + 7 & 1 & 0 \\
0 & 2(v^k + 2) & 4(3v^k + 2) & 2(v^k + 2) & 0 \\
0 & 1 & 8v^k + 7 & 8v^k + 7 & 1 \\
0 & 0 & 2(v^k + 2) & 4(3v^k + 2) & 2(v^k + 2) 
\end{bmatrix}
\]
and by taking the rows of its inverse we get the following five interpolatory masks
\[ m_{5,1,1}^k = \begin{bmatrix} 32(v^k)^2+2v^k+2 & 1 & 0 & -70(v^k)^3-44(v^k)^2+7v^k+2 \\ 0 & 42(v^k)^3+20(v^k)^2-v^k+2 & 0 & 5v^k+2 \\ 0 & 0 & 20(v^k)^3+40(v^k)^2+39v^k+6 & 32v^k(v^k+1)^2 \\ 0 & 0 & 0 & 64v^k(v^k+1)^2 \end{bmatrix} \]

\[ m_{5,1,2}^k = \begin{bmatrix} -5v^k+2 & -20(v^k)^3+3v^k+2 & 0 & 30(v^k)^3+60(v^k)^2+17v^k-2 \\ 0 & 20(v^k)^3+3(v^k)^2+3v^k+2 & 0 & 32v^k(v^k+1)^2 \\ 0 & 0 & 20(v^k)^3+40(v^k)^2+39v^k+6 & 64v^k(v^k+1)^2 \\ 0 & 0 & 0 & 64v^k(v^k+1)^2 \end{bmatrix} \]

\[ m_{5,1,3}^k = \begin{bmatrix} v^k+2 & 0 & 0 & v^k+2 \\ 0 & (3v^k+2)(6v^k)^2+3v^k+2 & 0 & 64v^k(v^k+1)^2 \\ 0 & 0 & 20(v^k)^3+39v^k+6 & 64v^k(v^k+1)^2 \\ 0 & 0 & 0 & 64v^k(v^k+1)^2 \end{bmatrix} \]

\[ m_{5,1,4}^k = \begin{bmatrix} v^k+2 & 0 & 0 & v^k+2 \\ 0 & 4(v^k)^3+2(v^k)^2+3v^k+2 & 0 & 32v^k(v^k+1)^2 \\ 0 & 0 & 20(v^k)^3+40(v^k)^2+39v^k+6 & 64v^k(v^k+1)^2 \\ 0 & 0 & 0 & 64v^k(v^k+1)^2 \end{bmatrix} \]

\[ m_{5,1,5}^k = \begin{bmatrix} 5v^k+2 & 0 & 0 & 5v^k+2 \\ 0 & 64v^k(v^k+1)^2 & 0 & 42(v^k)^3+20(v^k)^2-v^k+2 \\ 0 & 0 & 70(v^k)^3+60(v^k)^2+17v^k-2 & 64v^k(v^k+1)^2 \\ 0 & 0 & 0 & 64v^k(v^k+1)^2 \end{bmatrix} \]

Note that \( m_{5,1,3}^k \) defines the mask of the interpolatory six-point scheme proposed in [11, Section 4.1], whose symbol is

\[ m_{5,1,3}(z) = \frac{v^k+2}{4(v^k+1)} a_{5,1}^k(z) - \frac{(v^k+2)^2}{20(v^k+1)} 2^2 a_{5,1}^k(z) + \frac{4(v^k)^2+9v^k+6}{20(v^k+1)} z^2 a_{5,1}^k(z) - \frac{(v^k+2)^2}{20(v^k+1)} z a_{5,1}^k(z) + \frac{v^k+2}{4(v^k+1)} a_{5,1}^k(z). \]

Fig. 2 Comparison between limit curves obtained through the interpolatory schemes \( \{ m_{5,2}^k, k \geq 0 \} \), (left) and \( \{ m_{5,1,3}^k, k \geq 0 \} \), (right) starting from the same initial value \( v^{-1} = 1.5 \).
Exactly like the non-stationary approximating subdivision scheme correspondent to \( \{ a^{k+1}_k(z), k \geq 0 \} \), such a scheme can reproduce functions in the space \( \langle 1, x, x^2, e^{\theta x}, e^{-\theta x} \rangle \) for any non-negative real or imaginary constant \( \theta \). Hence, it allows an exact representation of cubics, circles and conic sections (see Figure 1) when the vertices of the starting polyline are equally spaced samples and the initial parameter \( v^{-1} \) is chosen as in (7.34). Moreover the limit curves generated by such a scheme are featured by \( C^2 \) smoothness for any choice of the starting parameter \( v^{-1} \in (-1, +\infty) \).

An application example of this subdivision scheme to an arbitrary starting polyline is given in Figure 2.

**Example 3:** We continue by considering another \( C^4 \) approximating scheme proposed in [11]. Its \( k \)-level symbol is of the form

\[
a^k_{5,2}(z) = \frac{(z + 1)^2(z^2 + 2v^k z + 1)(z^2 + 2(2v^k)^2 - 1)z + 1}{16(v^k)^2(v^k + 1)}
\]

and the associated mask is

\[
a^k_{5,2} = \frac{1}{16(v^k)^2(v^k + 1)} \begin{bmatrix}
4(v^k)^2 + 2v^k & 16(v^k)^3 + 8(v^k)^2 - 4v^k & 4(v^k)^2 + 2v^k & 0 & 0 \\
1 & 8(v^k)^3 + 8(v^k)^2 - 1 & 8(v^k)^3 + 8(v^k)^2 - 1 & 1 & 0 \\
0 & 4(v^k)^2 + 2v^k & 16(v^k)^3 + 8(v^k)^2 - 4v^k & 4(v^k)^2 + 2v^k & 0 \\
0 & 0 & 8(v^k)^3 + 8(v^k)^2 - 1 & 8(v^k)^3 + 8(v^k)^2 - 1 & 1 \\
0 & 0 & 4(v^k)^2 + 2v^k & 16(v^k)^3 + 8(v^k)^2 - 4v^k & 4(v^k)^2 + 2v^k
\end{bmatrix}.
\]

Now being \( a^k_{5,2}(z) \) and \( a^k_{5,2}(-z) \) relatively prime, we can apply the Appint Algorithm where the matrix to be inverted, besides a factor \( \frac{1}{16(v^k)^2(v^k + 1)} \), reads as

\[
\mathcal{A}^k_{5,2} = \begin{bmatrix}
4(v^k)^2 + 2v^k & 16(v^k)^3 + 8(v^k)^2 - 4v^k & 0 & 0 \\
1 & 8(v^k)^3 + 8(v^k)^2 - 1 & 1 & 0 \\
0 & 4(v^k)^2 + 2v^k & 16(v^k)^3 + 8(v^k)^2 - 4v^k & 0 \\
0 & 0 & 8(v^k)^3 + 8(v^k)^2 - 1 & 1 \\
0 & 0 & 4(v^k)^2 + 2v^k & 16(v^k)^3 + 8(v^k)^2 - 4v^k
\end{bmatrix}.
\]

The 5 interpolatory masks built upon the rows of its inverse are
Example 4: We conclude by showing that the \( C^4 \) approximating scheme with \( k \)-level symbol

\[
a_{5,3}^k(z) = \frac{(z + 1)^2(z^2 + 2v^kz + 1)^2}{8(v^k + 1)^2}
\]

and mask

\[
a_{5,3}^k = \frac{1}{8(v^k + 1)^2} \left[ 1 \quad 2(2v^k - 1) \quad 4(v^k)^2 + 8v^k + 3 \quad 4(2(v^k)^2 + 2v^k + 1) \right]
\]

Note that \( \{a_{5,2,3}^k, k \geq 0\} \), defines the \( C^2 \) interpolatory six-point scheme presented in [11, Section 4.2]. Such a scheme is featured by the capability of reproducing functions in the space \( \{1, x, e^{\theta x}, e^{-\theta x}, e^{2\theta x}, e^{-2\theta x}\} \) for any non-negative real or imaginary constant \( \theta \), exactly as it happens for the corresponding non-stationary approximating scheme \( \{a_{5,2}^k, k \geq 0\} \). Hence, it allows an exact representation of algebraic curves of the second order (i.e. described by means of parametric equations involving order-2 trigonometric functions) such as the cardioid, Pascal’s limacon, ... (see Figure 3) whenever the starting parameter \( v^{-1} \) is defined according to (7.34).
Fig. 3 Exact reconstruction of the cardioid (a), the deltoid (b), the lemniscate of Gerono (c), Pascal’s limacon (d), the piriform (e) and the eight curve (f) through the interpolating scheme \( \{ m_{k,2,3}^5, \ k \geq 0 \} \), with starting parameter \( v^{-1} = \frac{1}{2} \). Dashed lines denote the chosen starting polygons made of 6 points sampled at a uniform parameter spacing of \( \frac{\pi}{3} \).

recently proposed in [11], satisfies the property that \( a_k^{5,3}(z) \) and \( a_k^{5,3}(-z) \) are relatively prime and hence can be transformed into a family of interpolating schemes with the same reproduction properties. To this purpose we compute the inverse of the associated matrix which, besides a factor \( \frac{1}{8(v^k+1)^2} \), reads as

\[
A_{5,3}^k = \begin{pmatrix}
2(2v^k + 1) & 4(2(v^k)^2 + 2v^k + 1) & 2(2v^k + 1) & 0 & 0 \\
1 & 4(v^k)^2 + 8v^k + 3 & 4(v^k)^2 + 8v^k + 3 & 1 & 0 \\
0 & 2(2v^k + 1) & 4(2(v^k)^2 + 2v^k + 1) & 2(2v^k + 1) & 0 \\
0 & 1 & 4(v^k)^2 + 8v^k + 3 & 4(v^k)^2 + 8v^k + 3 & 1 \\
0 & 0 & 2(2v^k + 1) & 4(2(v^k)^2 + 2v^k + 1) & 2(2v^k + 1)
\end{pmatrix}
\]

and apply the Appint Algorithm. The 5 interpolatory masks that we obtain are
m_{5,1}^k = \begin{bmatrix}
\frac{(2^k+1)(8(\nu)^4+12(\nu)^2+1)}{64(\nu)\theta(\nu+1)^{5}} & 1 & \frac{(2^k+1)(32(\nu)^4+48(\nu)^2+32(\nu)^2-8(\nu)^2+1)}{64(\nu)\theta(\nu+1)^{3}} & 0 & -\frac{(2^k+1)(16(\nu)^4+4(\nu)^2+2(\nu)^2-1)}{32(\nu)\theta^2(\nu+1)^{2}}
\frac{64(\nu)^2+48(\nu)^2+8(\nu)^2-8(\nu)^2+6(\nu)^2+2(\nu)^2+1}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{(2^k+1)(16(\nu)^4+4(\nu)^2+2(\nu)^2-1)}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{4(\nu)^2+2(\nu)^2+1}{64(\nu)\theta^2(\nu+1)^{2}}
0 & \frac{(2^k+1)(24(\nu)^4+12(\nu)^2+1)}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{16(\nu)^4+8(\nu)^2+6(\nu)^2-2(\nu)^2-1}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{2^k+1}{64(\nu)\theta^2(\nu+1)^{2}}
\frac{2^k+1}{64(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{(2^k+1)(12(\nu)^2+2(\nu)^2-1)}{32(\nu)\theta^2(\nu+1)^{2}} & 1 & \frac{(2^k+1)(24(\nu)^4+12(\nu)^2+1)}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{2^k+1}{64(\nu)\theta^2(\nu+1)^{2}}
n\frac{2^k+1}{64(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{(2^k+1)(8(\nu)^2+2(\nu)^2-1)}{64(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{(2^k+1)(8(\nu)^2+2(\nu)^2-1)}{64(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{2^k+1}{64(\nu)\theta^2(\nu+1)^{2}}
\frac{2^k+1}{64(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{(2^k+1)(24(\nu)^4+12(\nu)^2+1)}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{4(\nu)^2+2(\nu)^2+1}{64(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{2^k+1}{64(\nu)\theta^2(\nu+1)^{2}}
0 & \frac{(2^k+1)(16(\nu)^4+4(\nu)^2+2(\nu)^2-1)}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{64(\nu)^2+48(\nu)^2+8(\nu)^2-8(\nu)^2+6(\nu)^2+2(\nu)^2+1}{32(\nu)\theta^2(\nu+1)^{2}} & 0 & \frac{(2^k+1)(8(\nu)^2+12(\nu)^2+1)}{64(\nu)\theta^2(\nu+1)^{2}} & 1 & \frac{(2^k+1)(8(\nu)^4+12(\nu)^2+1)}{64(\nu)\theta^2(\nu+1)^{2}}
\end{bmatrix}

Note that \{m_{5,3}^k, k \geq 0\} defines a C^2 interpolatory six-point scheme which corresponds to the one presented in [11, Section 4.3]. Like its approximating correspondent \{a_{5,3}^k, k \geq 0\}, it is featured by the capability of reproducing functions in the space \(l, x, e^{\theta x}, e^{-\theta x}, x e^{\theta x}, x e^{-\theta x}\) for any non-negative real or imaginary constant \(\theta\). It means that an exact representation of spirals, of great interest in engineering such as the circle involute which represents the curve employed in most gear-tooth profiles, (see Figure 4) can be obtained. Like in the previous cases, the reproduction of these curves is allowed whenever the subdivision process is started from equally spaced samples and the initial parameter \(\nu^{-1}\) has been opportunely set as described in (7.34).

8 Conclusions and Future Work

A novel approach based on polynomial and structured matrix technology has been presented for the computation of a family of interpolatory non-stationary subdivision
schemes from a symmetric non-stationary, non-interpolatory one. The approach reduces the functional problem either to the inversion of certain structured matrices of Hurwitz or Sylvester resultant form or to the solution of certain Bezout-like polynomial equations. Efficient computational procedures for both problems have been proposed. It was shown that the interpolatory schemes are capable of generating the same functional space as the approximating one. In principle our approach could be extended to more general approximating schemes of arity $p > 2$. The extension requires to investigate the structural properties of Sylvester-type resultant matrices for polynomial sets consisting of more than 2 polynomials. This will be the subject of a forthcoming paper. A fundamental issue is concerned with the analysis of the convergence properties of the subdivision schemes generated by our techniques. The semi–explicit representations of the symbols obtained by solving the polynomial formulation would be useful to provide some insights on this difficult topic. Finally, the use of parametric symbols leads to symbolic inversion problems for structured matrices with Hurwitz or Sylvester resultant form. The adaptation of customary methods to work efficiently in this case is an ongoing research.

References