Solving Bezout-like polynomial equations for the design of interpolatory subdivision schemes

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ABSTRACT
Subdivision schemes are nowadays customary in curve and surface modeling. In this paper the problem of designing interpolatory subdivision schemes is considered. The idea is to modify a given approximating subdivision scheme just enough to satisfy the interpolation requirement. This leads to the solution of a generalized Bezout polynomial equation possibly involving more than two polynomials. By exploiting the matrix counterpart of this equation it is shown that small-degree solutions can be generally found by inverting an associated structured matrix of Toeplitz-like form.

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Subdivision scheme, Bezout equation, structured matrix

1. INTRODUCTION
Subdivision schemes are widely used in several contexts like in Computer Aided Geometric Design to represent a smooth curve or surface as the limit of successive refinements on denser and denser grids of points. At each refinement step new points are generated by means of an affine combination of the existing ones. The number of inserted points characterizes the concept of arity of the considered scheme. For instance, if the points are doubled at each iteration the scheme is called binary its arity being $p = 2$. For processes defined in the real two- or three-dimensional Euclidean space the analysis can be immediately reduced to the scalar univariate case by restriction to each coordinate. A stationary univariate subdivision scheme with arity $p \geq 2$ is an iterative process that starting with some initial points attached to the integer grid, i.e., with $\mathbf{q} = \mathbf{q}^{(0)} = \{q^{(0)}_i : i \in \mathbb{Z}\}$, iteratively computes a sequence $\mathbf{q}^{(n)} := S_n \mathbf{q}^{(n-1)} = S_n^p \mathbf{q}^{(0)}$ for $n \geq 1$, by repeated application of the stationary rule

$$ (\mathbf{q}^{(n)})_i = \left( S_n \mathbf{q}^{(n-1)} \right)_i = \sum_{j \in \mathbb{Z}} a_{i-pj} q^{(n-1)}_j, \quad i \in \mathbb{Z}, \quad (1) $$

which relies upon the coefficients $a_i, i \in \mathbb{Z}$. These coefficients identify the subdivision operator $S_n$ and the so called refinement mask $\mathbf{a} = (a_i : i \in \mathbb{Z})$. It is assumed that $\sigma(\mathbf{a}) := \{j \in \mathbb{Z} : a_j \neq 0\} \subseteq [-N,N]$ for a certain $N \in \mathbb{N}$ so that $\mathbf{a} \in \ell_p(\mathbb{Z})$, i.e., the space of compactly supported sequences of real values. For the purpose of analysis it is also useful to introduce the symbol

$$ a(z) = \sum_{i=-N}^{N} a_i z^i, \quad z \in \mathbb{C} \setminus \{0\}, $$

which is a Laurent polynomial associated with the mask $\mathbf{a}$, and, moreover, the associated sub-symbols

$$ a_j(z) = \sum_{i=-N}^{N} a_{i+j} z^i, \quad z \in \mathbb{C} \setminus \{0\}, \quad 0 \leq j \leq p-1. $$

These are related to the symbol by the equation

$$ a(z) = \sum_{i=0}^{p-1} z^i a_i(z^p). $$

By assigning the values of $S_n^p \mathbf{q}$, $n \in \mathbb{N}_0$, to the denser and denser grids $p^{-n}\mathbb{Z}$, one can then establish a notion of $L_{\infty}$-convergence to a continuous limit function by requiring the existence of a uniformly continuous and bounded function $f_{\mathbf{q}}$ (depending on the starting sequence $\mathbf{q}$) satisfying

$$ \lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \left| (S_n^p \mathbf{q})_j - f_{\mathbf{q}}(p^{-n}j) \right| = 0. \quad (2) $$

We say that the subdivision scheme (1) is convergent if it converges for any initial vector $\mathbf{q} = \mathbf{q}^{(0)} \in \ell_{\infty}(\mathbb{Z})$, i.e., $\|\mathbf{q}\|_{\infty} := \sup_{i \in \mathbb{Z}} |q_i| < \infty$, and, moreover, $f_{\mathbf{q}} \neq 0$ for at least some initial data $\mathbf{q} \in \ell_{\infty}(\mathbb{Z})$. Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee.

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An important class of subdivision schemes are those that refine the sequence \( q \) while keeping the “original data” in the sense that \( (S_n q)_n = q, \ i \in \mathbb{Z} \). For obvious reasons, such schemes are called interpolatory and their refinement mask is of special type since it satisfies

\[
a_p = \delta_{i,0}, \ i \in \mathbb{Z}. \tag{3}
\]

The condition (3) can also be easily reformulated in terms of the associated symbol: The Laurent polynomial \( a(z) \) is the symbol of an interpolatory subdivision scheme of arity \( p \) if and only if it satisfies

\[
\sum_{j=0}^{p-1} a_j(r_j z) = p, \quad z \in \mathbb{C} \setminus \{0\},
\tag{4}
\]

where \( r_j = e^{2\pi i j/p}, \ j = 0, \ldots, p-1 \) are the \( p \)-th roots of the unity. Despite of their importance and of the recent burgeoning literature in the field of subdivision schemes (see for instance [5] and [11] and the references therein), very few interpolatory examples are known so far even in the univariate setting. The most celebrated example is the class of Dubuc-Deslauriers (DD) symmetric schemes, first presented in [3].

In this paper we present a general approach for deriving a family of interpolatory subdivision schemes associated with the subdivision operator defined in (1). The approach generalizes previous results in [2] [4], [7] and [8]. Since in a matrix setting this (linear) operator can be represented by a bi-infinite banded Toeplitz-like matrix \( S_a \) we argue that a novel subdivision operator \( S_b \) of the same form as \( S_a \) can be obtained by multiplication by a bi-infinite banded Toeplitz matrix \( T \), i.e., \( S_b = T \cdot S_a \). Furthermore, there follows that the symbol \( b(z) \) of \( S_b \) satisfies \( b(z) = a(z) \cdot t(z) \), where \( t(z) \) is the symbol of \( T \). In this way we may determine the Laurent polynomial \( t(z) \) so that \( b(z) \) fulfills the interpolation requirement (4), i.e.,

\[
a(z) \cdot t(z) + a(r_1 z) \cdot t(r_1 z) + \ldots + a(r_{p-1} z) \cdot t(r_{p-1} z) = p. \tag{5}
\]

The solution of such a polynomial equation is at the core of our method for the construction of interpolatory subdivision schemes. By setting \( t_i(z) := t(r_i z) \) it is found that (5) is a specific instance of the Bezout-like equation

\[
a(z) \cdot t_0(z) + a(r_1 z) \cdot t_1(z) + \ldots + a(r_{p-1} z) \cdot t_{p-1}(z) = p.
\]

In a matrix setting its solution reduces to solving a complex Sylvester-like linear system. Under mild assumptions it is shown that the coefficient matrix can be decomposed in terms of smaller real structured matrices of Hurwitz type. The decomposition affords an effective means for solving (5) as well as for investigating the existence of solutions under given assumptions.

The paper is organized as follows. In Section 2 we present the derivation of the polynomial equation (5) and develop solution methods using structured matrix computations. Some computational examples, a brief discussion of the results and possible further developments are finally drawn in Section 3.

2. THE POLYNOMIAL APPROACH AND ITS MATRIX COUNTERPARTS FOR GENERATING INTERPOLATORY SUBDIVISION SCHEMES

In the matrix environment the linear operator \( S_a \) defined in (1) is represented by a bi-infinite Toeplitz-like matrix \( S_a = (a_{i-j}), \ i, j \in \mathbb{Z} \). Since \( a(z) = \sum_{j=-\infty}^{\infty} a_j z^j \) is a Laurent polynomial it follows that \( S_a \) is banded with bandwidth \( [N/p] \) at most.

Let \( t(z) = \sum_{j=-N}^{h} t_j z^j, \ t_{-h} \neq 0 \), be another Laurent polynomial and denote by \( T \) the bi-infinite Toeplitz matrix associated with \( t(z) \), namely, \( T = (t_{i-j}) \). Observe that \( T \) is again banded with bandwidth \( h \) at most. For the product operator

\[
S_b := T \cdot S_a = (s_{i-j}), \quad i, j \in \mathbb{Z},
\]

we have

\[
s_{i,j} = \sum_{r=-h}^{h} a_{r-pj} \ = \sum_{\ell=-h}^{h} t_{\ell a_{i-pj-\ell}} = s_{i+p,j+1}, \quad i, j \in \mathbb{Z}.
\]

This means that the product operator \( S_b \) is a bi-infinite Toeplitz-like matrix of the same form as the subdivision operator \( S_a \) with entries \( s_{i,j} = s_{i-j} \), \( i, j \in \mathbb{Z} \). By setting

\[
b(z) = a(z) \cdot t(z) = \sum_{j=-h-N}^{h} b_j z^j, \quad (b_j = 0 \text{ if } |j| > h + k),
\]

we find that

\[
b_j = \sum_{i=-h}^{h} t_i a_{i-j}, \quad -h - N \leq j \leq h + N,
\]

and, therefore,

\[
b_{i-pj} = s_{i,j} = s_{i-j}, \quad i, j \in \mathbb{Z}.
\]

The product operator \( S_b \) can therefore be seen as the subdivision operator associated with the Laurent polynomial \( b(z) \). The unknown coefficients of \( t(z) \) might be determined in such a way that the resulting polynomial \( b(z) \) verifies the analogue of condition (3), i.e.,

\[
b_j = \delta_{i,0}, \quad i \in \mathbb{Z}. \tag{6}
\]

Let

\[
b(z) = \sum_{\ell=0}^{p-1} z^\ell b_\ell(z^p), \quad t(z) = \sum_{\ell=0}^{p-1} z^\ell t_\ell(z^p) \quad z \in \mathbb{C} \setminus \{0\},
\]

be the representations of \( b(z) \) and \( t(z) \), respectively, in terms of their sub-symbols. The condition (6) is equivalent to \( b_0(z^p) = 1 \). In addition, from

\[
\sum_{\ell=0}^{p-1} b_j r_j z^j = \sum_{\ell=0}^{p-1} t_j z^\ell b_\ell(z^p) = \sum_{\ell=0}^{p-1} z^\ell b_\ell(z^p) \sum_{\ell=0}^{p-1} r_j \]

where \( r_j = e^{2\pi i j/p}, \ j = 0, \ldots, p-1 \), we conclude that \( b(z) \) is the symbol of an interpolatory subdivision scheme of arity \( p \) if and only if
The Laurent polynomials $t(z)$ that are solutions of (7) are suited to convert the approximating subdivision scheme associated with $a(z)$ into a corresponding interpolating subdivision scheme associated with $b(z) = a(z) \cdot t(z)$.

In the sequel of this section we investigate conditions under which the (generalized) Bezout equation (7) is solvable by developing effective computational methods for its solution in these cases. First of all, suppose that the symbol $a(z)$ is possibly padded with zeros coefficients in such a way that $2N$ is an integer multiple of $p - 1$, namely, $2N = (m + 1) \cdot (p - 1)$ with $m \geq p - 1$. Set $N + h = j \cdot p + k$ with $0 \leq k \leq p - 1$. We look for a Laurent polynomial $t(z)$ solving (7) and determined by $m + 1$ nonzero coefficients at most. This polynomial can be represented in the form

$$t(z) = t_\cdot h \cdot z^{-h} + t_{-h+1} \cdot z^{-h+1} + \ldots + t_{-h+m} \cdot z^{-h+m}.$$

For such a given polynomial $t(z)$ the product Laurent polynomial $b(z) = a(z) \cdot t(z)$ can be expressed as

$$b(z) = \sum_{j=-m}^{2N-jp-m-k} b_j z^j = \sum_{j=-m}^{-(j-p-1)} b_j z^j.$$

Therefore we can determine the $m+1$ unknown coefficients of $t(z)$ to satisfy the following equation

$$a(z) \cdot t(z) + a(r_1z) \cdot t(r_1z) + \ldots + a(r_{p-1}z) \cdot t(r_{p-1}z) = p z^{p\cdot s},$$

for $-j \leq s \leq m - j$. Indeed, it is clear that a solution of (8) provides a solution of (7) simply obtained by simply adjusting the value of $h$.

The solution of (8) reduces to solving a structured linear system of order $2N + m + 1 = (m + 1) p$ whose coefficient matrix is Sylvester-like. Let $a_0 = (a_{-N}, \ldots, a_0, \ldots, a_N)^T \in \mathbb{R}^{2N+1}$ denote the coefficient vector of the Laurent polynomial $a(z)$. The associated extended coefficient vector $\tilde{a}_0 \in \mathbb{R}^{(m+1)p}$ is defined by $\tilde{a}_0 = (a_0^T, 0, \ldots, 0)$. Similarly let us introduce the extended coefficient vectors $\tilde{a}_j \in \mathbb{C}^{(m+1)p}$, $1 \leq j \leq p - 1$, associated with the polynomials $a(r_jz)$, $1 \leq j \leq p - 1$, respectively. Moreover let $Z = (\delta_{i,j}) \in \mathbb{R}^{(m+1)p \times (m+1)p}$ be the down-shift matrix given by $z_{i,j} = \delta_{i-1,j}$, where $\delta_{i,j}$ is the Kronecker delta symbol. The vector $\tilde{R}_j \in \mathbb{C}^{(m+1)p \times (m+1)p}$ is the related Toeplitz matrix

$$\tilde{R}_j = [\tilde{a}_j | Z \tilde{a}_j | \ldots | Z^{m-1} \tilde{a}_j], \quad 0 \leq j \leq p - 1.$$

The coefficient matrix of the linear system (8) is

$$\tilde{R} = [\tilde{R}_0 | \tilde{R}_1 | \ldots | \tilde{R}_{p - 1}] \in \mathbb{C}^{(m+1)p \times (m+1)p}.$$

The linear system can be written as

$$\tilde{R} \cdot [t_0^T | t_1^T | \ldots | t_{p-1}^T]^T = p \cdot e_{p+(j+j)+1+k},$$

where $j \leq s \leq m - j, 0 \leq k \leq p - 1$, $t_k \in \mathbb{R}^{m+1}$ is the coefficient vector of $t(r_kz)$, $0 \leq k \leq p - 1$, and $e_{p+(j+j)+1}$ is the column of given index of the identity matrix of order $(m + 1)p$. This means that, whenever $\tilde{R}$ is nonsingular any complete set of solutions of (8) can be determined by inverting the matrix $\tilde{R}$.

Remark 2.1. It is worth noting that in the case $p = 2$ the matrix $\tilde{R}$ reduces to the classical Sylvester resultant matrix which is invertible if and only if $a(z)$ and $a_1(z) = a(-z)$ are relatively prime. This property is exploited in [2] to derive a complete characterization of the solutions of (8). Differently, it is shown below that for $p > 2$ the matrix $\tilde{R}$ is not a resultant matrix as it can be singular even if the polynomials $a(r_jz)$, $0 \leq j \leq p - 1$ have no common factors. Extensions of the classical resultant theorem for more than two polynomials $(p > 2)$ are established in [1, 13], but they employ triangularized resultant matrices.

Example 2.2. Let

$$a(z) = z^3 - 2z^{-1} + 1 + 2z + z^3,$$

with $N = 3$, $p = 3$ and $m = 2$. Although the polynomials $a(z)$, $a(e^{\frac{2\pi i}{3}} z)$ and $a(e^{\frac{4\pi i}{3}} z)$ are pairwise relatively prime, it is easily found that the corresponding $9 \times 9$ matrix $\tilde{R}$ is singular.

A column permutation applied to $\tilde{R}$ enables the linear system (9) to be rearranged in a more convenient way. Let $\tilde{P} \in \mathbb{R}^{(m+1)p \times (m+1)p}$ be the permutation matrix such that

$$\tilde{R} = \tilde{R} \cdot \tilde{P} = [\tilde{R}_0 | Z_{\tilde{R}_0} | \ldots | Z_{m-1} \tilde{R}_0], \quad \tilde{R}_0 = [\bar{a}_0 | \bar{a}_1 | \ldots | \bar{a}_{p-1}]^T.$$

We find that

$$\tilde{P}^T \cdot [t_0^T | t_1^T | \ldots | t_{p-1}^T]^T = \tilde{T} \cdot \begin{bmatrix} \omega_0^T & \ldots & \omega^{p-1}_0 \end{bmatrix}^T,$$

where

$$\omega_0 = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}, \quad 0 \leq \ell \leq m.$$

Now let $\Omega \in \mathbb{C}^{p \times p}$ the matrix with columns $\omega_{\ell}, 0 \leq \ell \leq p - 1$, that is,

$$\Omega = \begin{bmatrix} r^\omega_0^T & \ldots & r^\omega_{p-1}_0 \end{bmatrix} = \mathcal{F}^T, \quad \mathcal{F} = \begin{bmatrix} \omega_0^T & \ldots & \omega_{p-1}^T \end{bmatrix},$$

where $\mathcal{F}$ is the Fourier matrix constructed from the $p$-th roots of unity. Further, let $\tilde{D}_B$ the block diagonal matrix given by

$$\tilde{D}_B = \begin{bmatrix} \Omega^{-1} & C \cdot \Omega^{-1} & \ldots & \Omega^{-1} \cdot C^{m-1} \cdot \Omega^{-1} \end{bmatrix},$$

where $C \in \mathbb{R}^{p \times p}$ denotes the generator of the circulant matrix algebra, i.e.,

$$C = \begin{bmatrix} 0 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \end{bmatrix}.$$

Then $\tilde{D}_B$ is invertible and, moreover,

$$\tilde{D}_B \cdot [t_{-h} \omega_0^T | \ldots | t_{-h} \omega_m^T | \ldots | t_{-h} \omega_{m-1}^T | \ldots | t_{-h} \omega_{m-1}^T] = \begin{bmatrix} \ldots & \ldots & \ldots \end{bmatrix}. $$
In the light of (10) the structure of $\tilde{R} \cdot D_B^{-1}$ follows from that one of $R_0 \cdot \Omega$. Let us denote $\tilde{a}(z)$ the Laurent polynomial associated with the extended coefficient vector $\tilde{a}_0$, i.e.,

$$\tilde{a}(z) = a_{-N} z^{-N} + \ldots + a_N z^N + a_{N+1} z^{N+1} + \ldots + a_{N+m} z^{N+m},$$

with $a_{N+i} = \ldots = a_{N+m} = 0$. Observe that

$$\tilde{a}(z) = z^{-N} (\tilde{a}_0(z^p) + z\tilde{a}_1(z^p) + \ldots + z^{p-1}\tilde{a}_{p-1}(z^p))$$

with

$$\tilde{a}_0(z) = a_{-N+i} z^{-N+i} + a_{N-1+i} z^{N-1+i} + \ldots + a_{N+m+i} z^{N+m+i}, \quad 0 \leq i \leq p - 1.$$ It holds

$$\sum_{s=0}^{p-1} \sum_{h=0}^{p-1} \sum_{z \in k} \left[ \sum_{r=0}^{N-h-k} \tilde{a}_h(z^p) = \sum_{s=0}^{p-1} \sum_{h=0}^{p-1} \sum_{z \in k} \tilde{a}_h(z^p)$$

$$\sum_{h=0}^{p-1} \sum_{z \in k} \tilde{a}_h(z^p) = 0, \quad \text{for } j = j(\ell) \text{ satisfying } j + \ell - k \equiv 0 \pmod{p}. \text{ There follows that}$$

$$\tilde{R}_0 \cdot \tilde{\omega}_0 = p \tilde{R}_0 = p \left[ Z^k \tilde{a}_k | Z^{(1)}(1) \tilde{a}_0 | \ldots | Z^{(p-1)}(p-1) \tilde{a}_0 \right],$$

where $\tilde{a}_0, 0 \leq \ell \leq p - 1$, is the extended coefficient vector associated with $\tilde{a}_0(z^p)$. Summing up the linear system (9) can be rewritten in the following form

$$[\tilde{R}_0 \ldots Z^m \tilde{R}_0 C^m \tilde{e}_1^T \ldots \tilde{e}_m] = e_p(s+j)+1+k.$$

It is straightforward to show that the linear system can be reduced to an equivalent system of smaller order. We find that

$$[Z^k \tilde{a}_k | \ldots | Z^p \tilde{a}_0 | \tilde{a}_{p-1} \ldots] \left[ \begin{array}{c} t_{-h} \\ t_{1-h} \\ \vdots \\ t_{m-h} \end{array} \right] = e_{p(s+j)+1+k}.$$ By performing a suitable rearrangement of rows the system can be finally rewritten in the following form

$$\mathcal{H}_k \left[ \begin{array}{c} t_{-h} \\ t_{1-h} \\ \vdots \\ t_{m-h} \end{array} \right] = e_{s+j+1}, \quad -j \leq s \leq m,$$

where $\mathcal{H}_k \in \mathbb{R}^{(m+1) \times (m+1)}$, $0 \leq k \leq p - 1$, is a finite matrix of Hurwitz type

$$\mathcal{H}_k = (h_{i,j}^{(k)}), \quad h_{i,j}^{(k)} = a_{N+i-1+j+1}(i-1)p, \quad 1 \leq i, j \leq m.$$ In this way we arrive at the result

**Theorem 2.3.** Let us assume that the coefficient matrix $\mathcal{R}$ of (9) is nonsingular. Then, for any given $k$, $0 \leq k \leq p - 1$, $\mathcal{H}_k$ is also nonsingular and, moreover, the complete set of the solutions of (9) is determined by the coefficients of the rows of the inverse of $\mathcal{H}_k^T$.

Observe that the matrices $\mathcal{H}_k$, $0 \leq k \leq p - 1$, are finite sections of order $m + 1$ of the infinite operator $\mathcal{S}_k$. The previous result also describes a suitable decomposition of the Sylvester-like matrix $\mathcal{R}$ in terms of the Hurwitz-like matrices $\mathcal{H}_k$, $0 \leq k \leq p - 1$. This decomposition establishes the intimate connection among the construction of the modified operator $\mathcal{S}_k$, the inversion of the matrices $\mathcal{H}_k$ and their polynomial analogues.

**3. CONCLUSION**

In this section we discuss some computational issues related to the application of Theorem 2.3. Let us start with a computational example to illustrate the result.

**Example 3.1.** Following [12], the generalization of binary B-splines of order $m+1$ to arity $p > 2$ is given by the basic limit function associated with the symbol

$$a(z) = z^{-N} \left( \frac{z^p - 1}{z - 1} \right)^{m+1},$$

with $2N = (p - 1) \cdot (m + 1)$ and

$$a(z) = \frac{1}{27} (\cdots, 0, 1, 4, 10, 16, 19, 16, 10, 4, 1, 0, \cdots),$$

and we apply the strategy discussed above to get a complete family of corresponding interpolatory ternary subdivision schemes. We have

$$a(z) = \frac{1}{27} (z^4 - 4z^3 + 10z^2 - 16z + 19 - 16z - 19 + 16z + 10z^2 + 4z^3 + z^4),$$

and, hence, $N = 4$, $p = 3$ and $m = 3$. For $k = 2$ we find that

$$\mathcal{H}_2^T = \frac{1}{27} \begin{bmatrix} 10 & 16 & 1 & 0 \\ 4 & 19 & 4 & 0 \\ 1 & 16 & 10 & 0 \\ 0 & 10 & 16 & 1 \end{bmatrix}$$

whose inverse is

$$\mathcal{H}_2^{-T} = \frac{1}{3} \begin{bmatrix} 14 & -16 & 5 & 0 \\ -4 & 11 & -4 & 0 \\ 5 & -16 & 14 & 0 \\ -40 & 146 & -184 & 27 \end{bmatrix}.$$ By using the coefficients in the second row we define the Laurent correction

$$t(z) = \frac{1}{3} \left( -4z^{-1} + 11 - 4z \right)$$

and the “corrected” interpolatory symbol $b(z) = a(z) \cdot t(z)$ with associated mask

$$b = \frac{1}{8} \left( \cdots, 0, -4, -5, 0, 30, 60, 81, 60, 30, 0, -5, -4, 0, \cdots \right).$$

Thus we observe that we can derive the Dubuc-Deslauriers ternary 4-point scheme (b) exactly from the ternary cubic B-spline (a).

We conclude by noticing that for $m+1 = 3, 4, 5, 6, \cdots$ many other $m+1$–point schemes of general arity $p$ can be generated in a similar way. In fact, by using the decomposition

$$\left( 1 - z \right)^{-(m+1)} = \sum_{\ell=0}^{\infty} \left( m + \ell \right) \frac{x^\ell}{m!}.$$
after multiplication of the latter one by \((1 - z^p)^{m+1}\) we can compute the coefficients \(a_i = a_i(p)\) of the corresponding mask and then generate the Laurent correction by symbolic inversion of the Hurwitz-type matrix of order \(m+1\). These schemes include the ones derived in \([9, 10]\) for \(m+1 = 3, 4, 5, 6\) by using some specific techniques.

This example shows that Theorem 2.3 yields an effective way to construct a large family of interpolatory subdivision schemes starting from an initial approximating scheme. The construction essentially reduces to invert some small structured matrices of Hurwitz type defined as finite sections of appropriate order of the linear operator \(S_m\) associated with the approximating scheme. The invertibility of these matrices can be characterized in terms of the invertibility of a corresponding cumulative Sylvester-like matrix. This latter formulation enables the inversion problems for the Hurwitz matrices to be recast into a polynomial setting thus leading to a generalized Bezout-like equation involving \(p \geq 2\) polynomials.

In the case \(p = 2\) addressed in \([2]\), the Sylvester-like matrix is a resultant matrix and, therefore, the polynomial and the matrix formulations are completely equivalent making possible to translate both the invertibility analysis and the inversion methods into a polynomial framework. In particular, the assumption made in \([2]\) that \(a(z)\) is a symmetric Hurwitz polynomial implies that \(R\) is nonsingular and, moreover, the corresponding generalized Bezout equation can be solved by some adaptations of the extended Euclidean algorithm. The resulting polynomial solution methods seem to be computationally interesting since they are able to better exploit specific structures and/or sparsity properties occurring in the symbolic representation.

In the case \(p > 2\) the Sylvester-like matrix is not generally a resultant matrix. The analysis of the relationships among the properties of the initial symbol and the properties of the associated Sylvester-like matrix is an interesting research field. In particular, it would be very interesting to relate the invertibility of \(R\) with some property about the distribution of the roots of the symbol in the complex plane. Furthermore, the possibility of devising effective polynomial solution methods for the case \(p > 2\) is also an open issue. Finally, we recall that the Hurwitz type matrices \(H_m\) inherit a displacement structure. The interplay between this structure and the polynomial formulation resulting from the Bezout-like equation is under investigation.

4. REFERENCES


