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# Manipulating Polynomials in Generalized Form

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We present new algorithms using structured matrix methods for manipulating polynomials expressed in generalized form, that is, relative to a polynomial basis $\{p_i(x)\}$ satisfying a three-term recurrence relation. This includes computing the Euclidean remainders as well as the greatest common divisor of two polynomials $u(x)$ and $v(x)$ of degrees less than $n$. The computational schemes involve solving linear systems with nested Hankel matrices represented in the given generalized basis and they reach the optimal sequential complexity of $O(n^2)$ arithmetical operations.

## 1. Introduction

Developing algorithms for the Extended Euclidean scheme computation applied to two polynomials $u(x)$ and $v(x)$ over a field $\mathbf{F}$ is a highly important research area of Computer Algebra ( see Collins (1967), Brown (1971), Brown and Traub (1971), and Knuth (1981)) and of its applications such as rational function arithmetic (Loos (1982)) and squarefree and shiftfree factorizations (Loos (1983) and Yun (1976)). Despite the wide literature dealing with the standard power form case, however a little attention has been addressed to the problem of manipulating polynomials in generalized form, that is, relative to a given basis satisfying a three–term recurrence relation, without at any stage converting polynomials into power form.

Specifically, the author is aware of only one method dealing with generalized polynomials in this way. Such a method was developed by Barnett (Barnett (1970), (1984), (1988)) which applied a classical elimination procedure to the matrix $u(A)$, where $A$ is a suitable generalized analogue of the Frobenius matrix associated with $u(x)$, in order to arrive at a quasitriangular form which reveals the coefficients of the Euclidean remainders in generalized form. If $u(x)$ has degree $n$ greater than the degree of $v(x)$, then this approach requires $O(n^3)$ arithmetical operations and, therefore, it does not seen to lead to any good upper bound on the sequential complexity of computing both the Euclidean remainders and the greatest common divisor ($gcd$).

Recently, many authors (Bini and Gemignani (1990), (1995); Sendra and Llovet (1992)) have proposed algorithms for the Extended Euclidean scheme computation, applied to two polynomials expressed in power form, which require at most $O(n^2)$ arithmetical

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operations which is the optimum among classical algorithms. These methods reduces the evaluation of the polynomial sequence to determining the block triangular factorization of a $n \times n$ Hankel matrix $H$ generated by the coefficients of the formal power series of the rational function $k(x)/u(x)$ where $k(x)$ satisfies the Bezout's equality

$$k(x)v(x) = \ gcd(u(x), v(x)) \ (mod \ u(x)).$$

The evaluation of this decomposition relies on a general approach to the solution of linear systems with structured coefficient matrices, namely the displacement rank theory (Kailath and Sayed (1995), Heinig and Rost (1984)), which leads to $O(n^2)$ fast algorithms.

In this article we propose a generalization of these efficient computational schemes in order to deal with generalized polynomials

$$u(x) = p_n(x) + \tilde{u}_{n-1}p_{n-1}(x) + \ldots + \tilde{u}_0 p_0(x),$$
$$v(x) = p_m(x) + \tilde{v}_{m-1}p_{m-1}(x) + \ldots + \tilde{v}_0 p_0(x)$$

where $\{p_i(x)\}_{i=0}^n$ is a system of polynomials which satisfies a three-term recurrence relation

$$p_i(x) = (x - \alpha_i)p_{i-1}(x) - \beta_{i-1}p_{i-2}(x)$$
$$p_{-1}(x) = 0, \quad p_0(x) = 1.$$

We have been motivated by some new results concerning preconditioning techniques of the usually ill-conditioned Hankel matrices (Tyrtyshnikov (1994)) by means of lower triangular matrices defined by the coefficients of the classical orthogonal polynomials (Fasino (1995)). It is in fact a standard property (Akhiezer (1965)) that the Hankel matrix $H$ can be factorized as $VSV^T$ where $V$ is the Vandermonde matrix relative to the zeros of $u(x)$. Hence the advantage of considering $LHL^T$ in the place of $H$ consists mainly of replacing $V$ with the Vandermonde-like matrix $LV$ which can be much better conditioned (Gautschi (1983), Reichel and Opfer (1991)).

In terms of polynomials, we first observe that in the generalized case the matrix $LHL^T$, where $L$ is the $n \times n$ lower triangular matrix whose $(i+1)-$th row is defined by the coefficients of the monic polynomial $p_i(x)$, plays a similar role to that of the Hankel matrix $H$ in the standard case. In fact, the block triangular factorization of the matrix $LHL^T$ gives the coefficients of the Euclidean remainders represented in generalized form with respect to the basis $\{p_i(x)\}_{i=0}^n$ . In this way, since the computation of a block triangular factorization of an Hankel matrix amounts to solve a nested sequence of Hankel linear systems, the above mentioned results suggest that computing the Euclidean scheme in generalized form can be better conditioned than computing the Euclidean scheme with respect to the standard power form.

Then, by introducing a new suitable displacement operator, we are able to construct recursively a sequence of vectors which defines the wanted triangular decomposition of $LHL^T$. This recursion generalizes well known recurrences for the standard power form case (Gragg and Lindquist (1983)). Moreover, by a computational point of view, it involves essentially matrix by vector products where the matrix is the real tridiagonal one associated with the sequence $\{p_i(x)\}_{i=0}^n$ in the sense that the characteristic polynomial of its $i \times i$ leading principal submatrix coincides with $p_i(x)$. This allows us to show that this recursion takes $O(n^2)$ arithmetical operations only.

Furthermore, we prove that it is possible to start this recursion without explicitly computing all the entries of the modified Hankel matrix $LHL^T$. What we need are the first and the last two columns of the matrix and they can be determined at the cost of $O(n^2)$ arithmetical operations. In this way, we finally obtain the block triangular factorization of the matrix $LHL^T$ and, therefore, the coefficients of the Euclidean remainders in generalized form at the overall computational cost of $O(n^2)$ arithmetical operations. Surely, all the $O(n^2)$ coefficients cannot be computed with less than $O(n^2)$ arithmetical operations and, therefore, our method is optimum with respect to the sequential complexity.

## 2. Generalized Hankel Matrices and Polynomial Remainders

Let $\{p_i(x)\}_{i=0}^n$ be a system of polynomials which forms a basis of the linear space $\mathbf{F}_n[x]$ of all polynomials over $\mathbf{F}$ in $x$ whose degree not exceed $n$. Let $u(x) = \sum_{i=0}^n \tilde{u}_i p_i(x)$ and $v(x) = \sum_{i=0}^m \tilde{v}_i p_i(x)$, $m < n$, be two polynomials belonging to $\mathbf{F}_n[x]$. Let

$$f(z,w) = \sum_{i,j=0}^{n} a_{i,j} p_i(z) p_j(w)$$

be a polynomial over $\mathbf{F}$ in the variables $z$ and $w$. We associate with $f(z,w)$ the matrix $A_{n+1} = (a_{i,j})$, $i,j = 0,1,\ldots,n$, and we refer to $f(z,w)$ as to the *generating function* of $A_{n+1}$ with respect to the given basis $\{p_i(x)\}_{i=0}^n$.

The expression

$$\frac{u(z)v(w) - v(z)u(w)}{z - w} = \sum_{i,j=0}^{n-1} b_{i,j} p_i(z) p_j(w) \tag{2.1}$$

which is easily verified to be a polynomial in $z$ and $w$, is called *Bezoutian* of $u(x)$ and $v(x)$ with respect to the basis $\{p_i(x)\}_{i=0}^n$. The Bezoutian is a rather special bilinear form which appears in the context of the theory of equations, in stability theory and in the classical elimination theory (Helmke and Fuhrmann (1989)). The $n \times n$ matrix $B(u,v,\{p_i(x)\}_{i=0}^n) = (b_{i,j})$ whose generating function is 2.1 is called the *Bezout matrix* or, more simply, the *Bezoutian* of $u(x)$ and $v(x)$ with respect to the basis $\{p_i(x)\}_{i=0}^n$ (Gohberg and Olshevsky (1994), Rost (1993)).

Since $\{p_i(x)\}_{i=0}^n$ defines a basis of $\mathbf{F}_n[x]$, then there exists a $n \times n$ nonsingular matrix $L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)$ such that

$$\mathbf{e}_i^T L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)(1, x, \ldots, x^{n-1}) = p_i(x)$$

where $\mathbf{e}_i$ denotes the $i-$th column of the $n \times n$ identity matrix. In this way we find that the matrix $B(u,v,\{p_i(x)\}_{i=0}^n)$ can be represented as

$$L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)^{-T} B(u,v,\{x^i\}_{i=0}^n) L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)^{-1}. \tag{2.2}$$

On the other hand, it is well known (Heinig and Rost (1984)) that the matrix

$$B(u,1,\{x^i\}_{i=0}^n)^{-1} B(u,v,\{x^i\}_{i=0}^n) B(u,1,\{x^i\}_{i=0}^n)^{-1} = H(u,v,\{x^i\}_{i=0}^n) \tag{2.3}$$

is an Hankel matrix,

$$H(u,v,\{x^i\}_{i=0}^n) = \begin{pmatrix} h_0 & h_1 & \ldots & h_{n-1} \\ h_1 & \iddots & \iddots & \vdots \\ \iddots & \iddots & \iddots & \vdots \\ h_{n-1} & h_{n+1} & \ldots & h_{2n-2} \end{pmatrix}, \tag{2.4}$$

generally referred to as the Hankel matrix of $u(x)$ and $v(x)$ with respect to the canonical basis $\{x^i\}_{i=0}^n$. In view of 2.2 and 2.3 the matrix

$$H(u,v,\{p_i(x)\}_{i=0}^n) = L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n) H(u,v,\{x^i\}_{i=0}^n) L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)^T$$

is a generalization of the ordinary Hankel matrices, where $p_i(x) = x^i$, and will be referred to as the Hankel matrix of $u(x)$ and $v(x)$ with respect to the basis $\{p_i(x)\}_{i=0}^n$.

Now let us assume that $\{p_i(x)\}_{i=0}^n$ represents a system of monic polynomials such that deg $(p_i(x)) = i$, $i = 0, \ldots, n$. Then, it follows that $L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)$ is a $n \times n$ lower triangular matrix with unit diagonal entries.

Let $0 = m_0 < m_1 < \ldots < m_l = k$ be the integers such that $\det(H(u,v,\{p_i(x)\}_{i=0}^n))_{m_i} \neq 0$, $1 \leq i \leq l$ and $\det(H(u,v,\{p_i(x)\}_{i=0}^n))_j = 0$ if $j \neq m_i$, $j \leq n$. Here and hereafter $(H(u,v,\{p_i(x)\}_{i=0}^n))_k$ denotes the $k \times k$ leading principal submatrix of the matrix $H(u,v,\{p_i(x)\}_{i=0}^n)$.

A slight generalization of the block triangular factorization of $H(u,v,\{x^i\}_{i=0}^n)$ (Gragg and Lindquist (1983), Heinig and Rost (1984)) produces an upper triangular matrix

$$R(u,v,\{p_i(x)\}_{i=0}^n) = (\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}),$$

$$\mathbf{r}_i = (\bar{\mathbf{r}}_i^T, 1, \mathbf{0}^T)^T, \quad 1 \leq i \leq n-1,$$

having unit diagonal entries such that

$$R(u,v,\{p_i(x)\}_{i=0}^n)^T H(u,v,\{p_i(x)\}_{i=0}^n) R(u,v,\{p_i(x)\}_{i=0}^n) = D. \tag{2.5}$$

The block diagonal matrix $D$ satisfies

$$D = diag(D_1, \ldots, D_l, 0), \quad D_i \in \mathbf{F}^{\delta_i \times \delta_i}, \quad \delta_i = m_i - m_{i-1},$$

with

$$D_i = \begin{pmatrix} & & & d_1^{(i)} \\ & \bigcirc & d_1^{(i)} & d_2^{(i)} \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ d_1^{(i)} & d_2^{(i)} & \ldots & d_{\delta_i}^{(i)} \end{pmatrix}, \quad d_1^{(i)} \neq 0.$$

It is interesting to point out that this generalized block triangular factorization is not unique. However, since

$$(H(u,v,\{p_i(x)\}_{i=0}^n))_{m_i} (\bar{\mathbf{r}}_{m_{i-1}}^T, 1, \mathbf{0}^T)^T = d_1^{(i)} (\mathbf{0}^T, 1)^T, \quad 2 \leq i \leq l \tag{2.6}$$

and

$$(H(u,v,\{p_i(x)\}_{i=0}^n))_{m_i} (\bar{\mathbf{r}}_{m_i}) = -(h_{m_i}, \ldots, h_{2m_i-1})^T, \quad 1 \leq i \leq l, \tag{2.7}$$

and $(H(u,v,\{p_i(x)\}_{i=0}^n))_{m_i}$ is nonsingular, then any upper triangular matrix

$$\tilde{R}(u,v,\{p_i(x)\}_{i=0}^n) = (\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_{n-1})$$

satisfies $\tilde{\mathbf{r}}_{m_i} = \mathbf{r}_{m_i}$.

Let us introduce the polynomials

$$q_i(x) = (p_0(x), , \ldots, p_{n-1}(x)) \mathbf{r}_{m_i}, \quad 0 \leq i \leq l. \tag{2.8}$$

Starting from the well known results for the case where $p_i(x) = x^i$ (Gutknecht (1992)), then we find that the generalized polynomials $q_i(x)$ are orthogonal with respect to the symmetric bilinear form on the space $\mathbf{F}_{n-1}[x]$ of the polynomials over $\mathbf{F}$ of degree less than or equal to $n-1$

$$\langle \phi, \psi \rangle = \mathbf{z}^T (H(u, v, \{p_i(x)\}_{i=0}^n))\mathbf{w}, \tag{2.9}$$

where

$$\phi = (p_0(x), \ldots, p_{n-1}(x))\mathbf{z}, \quad \psi = (p_0(x), \ldots, p_{n-1}(x))\mathbf{w}. \tag{2.10}$$

Moreover, the polynomials $q_i(x)$, $0 \le i \le l$, satisfy the three-term recurrence relation,

$$q_{i+1}(x) = c_{i+1}(x)q_i(x) - b_i q_{i-1}(x), \quad q_{-1}(x) = 0, \; q_0(x) = 1 \tag{2.11}$$

where $c_i(x)$ are monic polynomials of degree $\delta_i$,

$$c_i(x) = p_{\delta_i}(x) + \sum_{j=0}^{\delta_i - 1} c_{i,j} p_j(x). \tag{2.12}$$

It can be easily proved that the monic polynomial $q_l(x)$ satisfies the Bezout's equality

$$v(x)q_l(x) = \; gcd(u(x), v(x)) \; ( \; mod \; u(x))$$

(Bini and Gemignani (1995)). This means that we may obtain the generalized Euclidean remainders generated by the Euclidean algorithm applied to $u(x)$ and to $v(x)$ at the cost of computing the block triangular factorization of two Hankel matrices with respect to the given basis. Further, by using the same arguments, we are able to reduce the computation of the greatest common divisor of $u(x)$ and $v(x)$ expressed in a generalized form to the recursive evaluation of the polynomials $q_i(x)$, $0 \le i \le l$. In particular, by adapting a result of Bini and Gemignani (1995) for the standard case, we characterize the greatest common divisor ($gcd$) of the two polynomials $u(x)$ and $v(x)$ in terms of suitable vectors in the kernel of the matrix $H(u, v, \{p_i(x)\}_{i=0}^n)$.

Let us assume that $r(x) = \gcd(u(x), v(x))$ be a monic polynomial of degree $n-k$. Then we have that

$$\mathrm{rank}(H(u, v, \{p_i(x)\}_{i=0}^n)) = k,$$

and, moreover,

$$\det(H(u, v, \{p_i(x)\}_{i=0}^n))_k \ne 0, \; \det(H(u, v, \{p_i(x)\}_{i=0}^n))_i = 0 \text{ for } i > k.$$

Furthermore, if

$$(H(u, v, \{p_i(x)\}_{i=0}^n))_{k+1}\mathbf{w} = \mathbf{0}, \; \mathbf{w} = (w_i)_{i=0,\ldots,k}, \; w_k = 1,$$

or, equivalently,

$$(H(u, v, \{p_i(x)\}_{i=0}^n))_k \begin{pmatrix} w_0 \\ \vdots \\ w_{k-1} \end{pmatrix} = - \begin{pmatrix} h_k \\ \vdots \\ h_{2k-1} \end{pmatrix}$$

holds, then we find that

$$u(x) = r(x) \sum_{i=0}^{k} w_i p_i(x).$$

In the next section we will face the problem of computing the block triangular decomposition of an Hankel matrix with respect to a given basis satisfying a three-term recurrence relation without any assumption on the singularity of its leading principal submatrices. Moreover, we will show that we have not to compute all the entries of the modified Hankel matrix $H(u, v, \{p_i(x)\}_{i=0}^n)$ since the algorithm needs of three rows or columns at the most. In this way, we reach the optimum upper bound of $O(n^2)$ arithmetical operations for the sequential complexity of computing the Euclidean remainders in generalized form.

## 3. Derivation of the Algorithm

Let us consider the upper triangular matrix

$$R(u, v, \{x^i\}_{i=0}^n) = (\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}).$$

If the polynomials $q_i(x)$ of 2.11 are represented with respect to the standard power basis, then the polynomial relation 2.11 can be viewed in matrix form as

$$\mathbf{r}_{m_{j+1}} = (P^{\delta_{j+1}} + \sum_{i=0}^{\delta_{j+1}-1} c_{j+1,i} P^i)\mathbf{r}_{m_j} - b_j \mathbf{r}_{m_{j-1}} \tag{3.1}$$

where $P = (p_{i,j})$ is the shift operator, $p_{i,j} = \delta_{i,j+1}$ and $\delta_{h,k} = 1$ if $h = k$, $\delta_{h,k} = 0$ otherwise.

Now, let $\{p_i(x)\}_{i=0}^n$ be a polynomial sequence such that $p_i(x)$ is a monic polynomial of degree $i$ which satisfies a three-term recurrence

$$p_i(x) = (x - \alpha_i)p_{i-1}(x) - \beta_{i-1}p_{i-2}(x) \tag{3.2}$$

$$p_{-1}(x) = 0, \quad p_0(x) = 1. \tag{3.3}$$

In order to evaluate the coefficients of the Euclidean remainders $q_i(x)$ represented in the given basis $\{p_i(x)\}_{i=0}^n$, that is, in view of the above section, the entries of suitable columns of the matrix

$$R(u, v, \{p_i(x)\}_{i=0}^n) = (\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_{n-1}),$$

we have then to compute the vectors

$$L^{-T}\mathbf{r}_{m_j} = L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)^{-T}\mathbf{r}_{m_j} = \tilde{\mathbf{r}}_{m_j}$$

which satisfy

$$\tilde{\mathbf{r}}_{m_{j+1}} = L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)^{-T}(P^{\delta_{j+1}} + \sum_{i=0}^{\delta_{j+1}-1} c_{j+1,i}P^i)\mathbf{r}_{m_j} - b_j\tilde{\mathbf{r}}_{m_{j-1}}. \tag{3.4}$$

It is well known that the sequence $\{p_i(x)\}_{i=0}^n$ coincides with the sequence of the characteristic polynomials of the leading principal submatrices of a tridiagonal matrix $T_n$, that is,

$$p_i(x) = \det(xI - T_i), \quad i \geq 1,$$

where

$$T_i = \begin{pmatrix} \alpha_1 & 1 & & \\ \beta_1 & \alpha_2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & \beta_{i-1} & \alpha_i \end{pmatrix}.$$

Moreover, it can be seen (Bini and Gemignani (1990)) that

$$L^{-1}T_nL = L(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n)^{-1}T_nL(\{p_i(x)\}_{i=0}^n, \{x^i\}_{i=0}^n) = F(p_n) \tag{3.5}$$

where

$$F(u) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -u_0 & \dots & \dots & -u_{n-1} \end{pmatrix}$$

denotes the Frobenius matrix associated with the monic polynomial $u(x)$. Furthermore, it has been proved (Pták (1984)) that

$$F(u)H(u, v, \{x^i\}_{i=0}^n) = H(u, v, \{x^i\}_{i=0}^n)F(u)^T.$$

By setting

$$F(p_n) = F(u) + \mathbf{e}_n\gamma^T, \tag{3.6}$$

we find that

$$T_nH(u, v, \{p_i(x)\}_{i=0}^n) - H(u, v, \{p_i(x)\}_{i=0}^n)T_n^T =$$

$$L(F(p_n)H(u, v, \{x^i\}_{i=0}^n) - H(u, v, \{x^i\}_{i=0}^n)F(p_n)^T)L^T$$

which implies

$$T_nH(u, v, \{p_i(x)\}_{i=0}^n) - H(u, v, \{p_i(x)\}_{i=0}^n)T_n^T =$$

$$L(\mathbf{e}_n\gamma^T H(u, v, \{x^i\}_{i=0}^n) - H(u, v, \{x^i\}_{i=0}^n)\gamma\mathbf{e}_n^T)L^T.$$

In this manner we obtain that

$$T_nH(u, v, \{p_i(x)\}_{i=0}^n) - H(u, v, \{p_i(x)\}_{i=0}^n)T_n^T = \mathbf{e}_n\delta^T - \delta\mathbf{e}_n^T. \tag{3.7}$$

Now, let us recall relations 3.1 and 3.4. Since we have

$$F(p_n)^T = P - \mathbf{p}_n\mathbf{e}_n^T,$$

relation 3.5 implies that

$$L^{-T}P^k = (T_n^T + L^{-T}\mathbf{p}_n\mathbf{e}_n^T)^kL^{-T}.$$

In this way, we find that

$$\tilde{\mathbf{r}}_{m_{j+1}} = ((T_n^T)^{\delta_{j+1}} + \sum_{i=0}^{\delta_{j+1}-1} c_{j+1,i}(T_n^T)^i + \sum_{i=0}^{\delta_{j+1}-1} \tilde{c}_{j+1,i}\chi_i^{(j)}\mathbf{e}_{n-i}^T)\tilde{\mathbf{r}}_{m_j} - b_j\tilde{\mathbf{r}}_{m_{j-1}}. \tag{3.8}$$

Since we have

$$\mathbf{e}_{n-i}^T\tilde{\mathbf{r}}_{m_j} = 0, \quad \text{if} \quad 0 \le i \le \delta_{j+1} - 1,$$

then it follows that

$$\tilde{\mathbf{r}}_{m_{j+1}} = ((T_n^T)^{\delta_{j+1}} + \sum_{i=0}^{\delta_{j+1}-1} c_{j+1,i}(T_n^T)^i)\tilde{\mathbf{r}}_{m_j} - b_j\tilde{\mathbf{r}}_{m_{j-1}}. \tag{3.9}$$

The coefficients $c_{j+1,i}$ and $b_j$ can be determined by means of the orthogonality conditions which follows from relations 2.5, 2.8 and 2.9. In fact, by choosing a particular upper triangular matrix $R(u, v, \{x^i\}_{i=0}^n)$ which exploits the Hankel structure, we obtain the following orthogonality properties (Gemignani (1993), Gemignani (1995))

$$(P^k \mathbf{r}_{m_j})^T H(u, v, \{x^i\}_{i=0}^n) P^h \mathbf{r}_{m_i} = \begin{cases} 0, & \text{if } i \neq j; \\ 0, & \text{if } i = j,\ k + h < \delta_{i+1} - 1; \\ d^{(i+1)}_{h+k+2-\delta_{i+1}}, & \text{elsewhere} \end{cases}$$

where $0 \leq k \leq \delta_{j+1} - 1$ and $0 \leq h \leq \delta_{i+1} - 1$. This means that in the generalized case we have

$$((T_n^T)^k \tilde{\mathbf{r}}_{m_j})^T H(u, v, \{p_i(x)\}_{i=0}^n)(T_n^T)^h \tilde{\mathbf{r}}_{m_i} = \begin{cases} 0, & \text{if } i \neq j; \\ 0, & \text{if } i = j,\ k + h < \delta_{i+1} - 1; \\ d^{(i+1)}_{h+k+2-\delta_{i+1}}, & \text{elsewhere} \end{cases}$$

where $0 \leq k \leq \delta_{j+1} - 1$ and $0 \leq h \leq \delta_{i+1} - 1$.

By imposing these orthogonality conditions for determining both the coefficients of the monic polynomials $c_i(x)$ and $b_i$, we arrive at the following relations:

$$b_i = \frac{d_1^{(i+1)}}{d_1^{(i)}} \tag{3.10}$$

and

$$c_{i,k} = -\frac{1}{d_1^{(i)}} \sum_{j=k+1}^{\delta_i} c_{i,j} ((T_n^T)^j \tilde{\mathbf{r}}_{m_{i-1}})^T H(u, v, \{p_i(x)\}_{i=0}^n)(T_n^T)^{\delta_i - k - 1} \tilde{\mathbf{r}}_{m_{i-1}}. \tag{3.11}$$

Relations 3.9, 3.10 and 3.11 are the means of computing a block triangular factorization of the matrix $H(u, v, \{p_i(x)\}_{i=0}^n)$. The sequential complexity of the resulting algorithm is dominated by the computation of the quotients $c_i(x)$. In order to compute efficiently the scalar products involved in the equality 3.11, we look for a recursive construction of the vectors $H(u, v, \{p_i(x)\}_{i=0}^n)(T_n^T)^h \tilde{\mathbf{r}}_{m_j}$, $0 \leq h \leq \delta_{j+1}$.

Relation 3.9 implies that the vector $H(u, v, \{p_i(x)\}_{i=0}^n)\tilde{\mathbf{r}}_{m_{j+1}}$ can be represented as

$$H(u, v, \{p_i(x)\}_{i=0}^n)((T_n^T)^{\delta_{j+1}} + \sum_{i=0}^{\delta_{j+1}-1} c_{j,i}(T_n^T)^i)\tilde{\mathbf{r}}_{m_j} - b_j H(u, v, \{p_i(x)\}_{i=0}^n)\tilde{\mathbf{r}}_{m_{j-1}}. \tag{3.12}$$

By means of $\delta_{j+1}$ applications of 3.7, we may therefore construct recursively the vectors $H(u, v, \{p_i(x)\}_{i=0}^n)$ $\tilde{\mathbf{r}}_{m_{j+1}}$ at the cost of $O(n\delta_{j+1})$ arithmetical operations. Then, one further application of 3.7 gives the vector $H(u, v, \{p_i(x)\}_{i=0}^n)(T_n^T)\tilde{\mathbf{r}}_{m_{j+1}}$, at the computational cost of $O(n)$ arithmetical operations. In this way, we perform the computation of the quotients $c_i(x)$ at the cost of $O(n\delta_i)$ arithmetical operations.

Now, we observe that the above recursions require only the last row or column $\delta$ of the matrix

$$T_n H(u, v, \{p_i(x)\}_{i=0}^n) - H(u, v, \{p_i(x)\}_{i=0}^n) T_n^T.$$

The computation of $\delta$ can be easily reduced to evaluate the last two columns of the modified Hankel matrix $H(u, v, \{p_i(x)\}_{i=0}^n)$. The classical Barnett's decomposition (Barnett (1983)), suitably extended to generalized Bezoutian and Hankel matrices, states that

$$B(u, 1, \{p_i(x)\}_{i=0}^n)^{-1} B(u, v, \{p_i(x)\}_{i=0}^n) = v(LF(u)L^{-1}) \tag{3.13}$$

which, in view of 2.3, gives

$$H(u, v, \{p_i(x)\}_{i=0}^n) B(u, 1, \{p_i(x)\}_{i=0}^n) = v(LF(u)L^{-1}). \tag{3.14}$$

In this way, by combining 3.5 and 3.6, we obtain that

$$H(u, v, \{p_i(x)\}_{i=0}^n) B(u, 1, \{p_i(x)\}_{i=0}^n) = v(T_n - \mathbf{e}_n \gamma^T L^{-1}). \tag{3.15}$$

Now, from 3.6 there follows that

$$\gamma^T L^{-1} = (\tilde{u}_0, \ldots, \tilde{u}_{n-1})^T$$

where $\tilde{u}_i$ are the coefficients of $u(x)$ expressed in the given basis $\{p_i(x)\}_{i=0}^n$. It is then a standard property of the matrix $v(T_n - \mathbf{e}_n \gamma^T L^{-1})$ (Barnett (1988)) that its rows $\rho_i$ satisfy the recurrence 3.3, namely

$$\rho_i = \rho_{i-1}(T_n - \mathbf{e}_n \gamma^T L^{-1} - \alpha_i I) - \beta_{i-1} \rho_{i-2} \tag{3.16}$$

where

$$\rho_1 = (\tilde{v}_0, \ldots, \tilde{v}_{m-1}, 1, 0, \ldots, 0)$$

and $\tilde{v}_i$ are the coefficients of $v(x)$ expressed in the given basis $\{p_i(x)\}_{i=0}^n$. The matrix $v(T_n - \mathbf{e}_n \gamma^T L^{-1})$ can be therefore contructed at the cost of $O(n^2)$ arithmetical operations given the generalized representations of $u(x)$ and $v(x)$.

Now, in order to conveniently represent the matrix $B(u, 1, \{p_i(x)\}_{i=0}^n)^{-1}$, we recall that the equality

$$H(u, 1, \{x^i\}_{i=0}^n) = B(u, 1, \{x^i\}_{i=0}^n)^{-1}$$

holds (Heinig and Rost (1984)). This implies the following decomposition

$$B(u, 1, \{p_i(x)\}_{i=0}^n)^{-1} = LH(u, 1, \{x^i\}_{i=0}^n) L^T. \tag{3.17}$$

Since we have (Pták (1984))

$$H(u, 1, \{x^i\}_{i=0}^n) = (\mathbf{e}_n, F(u)\mathbf{e}_n, \ldots, F(u)^{n-1}\mathbf{e}_n),$$

then we find that

$$LH(u, 1, \{x^i\}_{i=0}^n) = (\mathbf{e}_n, (T_n - \mathbf{e}_n \gamma^T L^{-1})\mathbf{e}_n, \ldots, (T_n - \mathbf{e}_n \gamma^T L^{-1})^{n-1}\mathbf{e}_n) \tag{3.18}$$

which says that the matrix $LH(u, 1, \{x^i\}_{i=0}^n)$ can be constructed by using $O(n^2)$ arithmetical operations.

Finally, by combining relations 3.15, 3.16, and 3.18, one readily obtains an algorithm for computing the needed columns of the modified Hankel matrix $H(u, v, \{p_i(x)\}_{i=0}^n)$ at the cost of $O(n^2)$ arithmetical operations. In this way, we are able to compute the generalized Euclidean remainders generated by the Euclidean Algorithm applied to $u(x)$ and $v(x)$ at the overall computational cost of $O(n^2)$ arithmetical operations. Surely, all the $O(n^2)$ coefficients of the polynomials $q_i(x)$ cannot be computed with less than $O(n^2)$ arithmetical operations and, therefore, our method is optimum with respect to the sequential complexity.

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