On some classes of structured matrices with algebraic trigonometric eigenvalues

L. Gemignani

Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

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Abstract

Diagonal plus semiseparable matrices are constructed, the eigenvalues of which are algebraic numbers expressed by simple closed trigonometric formulas.

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1. Introduction

Finding classes of structured matrices with known eigenvalues is important for testing purposes as well as for proving specific properties of the associated sequences of numbers and polynomials. Three well-known collections of test matrices are given in [10, 17, 11], which also include examples of structured matrices with band and/or displacement structure. The paper [14] describes some elementary applications of matrix eigenvalue theory in computational algebra for deciding whether some numbers are algebraic or not.

A quite recent advance in numerical linear algebra has been the design of fast numerical methods for eigenvalue computation of *rank structured* matrices. The theory of such matrices originated in the work of F. P. Gantmakher

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Email addresses: `gemignan@dm.unipi.it` (L. Gemignani)

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and M. G. Krein [6] concerning the structure of the inverses of certain tridiagonal matrices. The seminal paper [3] laid the bases of fast eigenvalue algorithms for rank-structured matrices by introducing suitable generalizations of the inverses of banded matrices. An up-to-date survey of these algorithms with several applications can be found in [15].

The structure of *diagonal-plus-semiseparable* matrices [2, 8, 13] (dpss for short) is perhaps the simplest generalization of that one of the inverses of tridiagonal matrices. A dpss matrix $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ is such that

$$a_{i,j} = \begin{cases} u_i v_j, & \text{if } i > j; \\ p_i q_j, & \text{if } i < j; \end{cases}$$

with $u_i, v_i, p_i, q_i \in \mathbb{C}$. The diagonal entries of A, $a_{i,i}$, $1 \leq i \leq n$, are arbitrary. The rank structure of A follows from

$$\max_{1\leq k\leq n-1} \operatorname{rank} A[k+1\colon n; 1\colon k] \leq 1, \qquad \max_{1\leq k\leq n-1} \operatorname{rank} A[1\colon k; k+1\colon n] \leq 1.$$

In this note we present two classes of dpss matrices whose eigenvalues can be explicitly determined by means of closed trigonometric formulas. Specifically, by following the approach given in [5] it is shown that the sequences of characteristic polynomials of these matrices are closely related to Chebyshev orthogonal polynomials. The connection yields simple formulas for the roots of the characteristic polynomials that are the values of the cotangent function evaluated at certain rational multiples of π. The result also provides an answer in the spirit of [14] to the issue raised in [1, 9] of giving a direct elementary proof of the well-known fact that the values of cotangent and tangent functions evaluated at rational multiples of π are algebraic numbers.

2. Preliminaries

In this section we review the main result in [5], which under some mild additional assumptions exhibits a three-term recurrence relation for the sequence of the characteristic polynomials of a dpss matrix. The result has been generalized in [4] whereas the same approach has been used in [7] and [12] to obtain sparse and structured representations of banded-plus semiseparable matrices.

The method proposed in [5] makes use of *Neville elimination* to transform the initial dpss matrix into banded form by means of a congruence transformation. Neville elimination is a classical elimination technique which, differently from the customary Gaussian method, at each step combines two consecutive equations (rows) to progressively make zeros in the lower left corner of the linear system. In the matrix language this means that Neville method employs bidiagonal matrices in the annihilation scheme and, therefore, it may virtually lead to a representation of the input matrix as product of bidiagonal matrices.

To be specific, let $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ be a dpss matrix with the entries defined as follows:

$$a_{i,j} = \begin{cases} u_i v_j, \text{ if } i > j; \\ p_i q_j, \text{ if } i < j; \\ d_i, \text{ if } i = j; \end{cases}$$

where $u_j \neq 0$ and $q_j \neq 0$ for $2 \leq j \leq n-1$ and $d_i \in \mathbb{C}$, $1 \leq i \leq n$, are arbitrary. Denote by L_u the lower bidiagonal matrix of order n with unit diagonal entries and subdiagonal entries $(L_u)_{i+1,i} = \alpha_i$ with α_1 arbitrary and $\alpha_i \colon = -u_{i+1}/u_i$, $2 \leq i \leq n-1$. It is found that $H = (h_{i,j}) = L_u \cdot A$ is upper Hessenberg with possibly nonzero entries given by

$$h_{i,j} = \begin{cases} \alpha_{i-1} d_{i-1} + u_i v_{i-1} \text{ if } j = i-1; \\ \alpha_{i-1} p_{i-1} q_i + d_i \text{ if } j = i; \\ (\alpha_{i-1} p_{i-1} + p_i) q_j \text{ if } j > i; \end{cases}$$

where $\alpha_0 \colon = 0$. It is worth noting that the rank structure in the upper triangular part of A is maintained. Thus let L_q be the lower bidiagonal matrix of order n with unit diagonal entries and subdiagonal entries $(L_u)_{i+1,i} = \beta_i$ with β_1 arbitrary and $\beta_i \colon = -q_{i+1}/q_i$, $2 \leq i \leq n-1$. The matrix $T = (t_{i,j}) = H \cdot L_q^T = L_u \cdot A \cdot L_q^T$ is tridiagonal with entries

$$t_{i,j} = \begin{cases} h_{i,i-1} \text{ if } j = i-1; \\ h_{i,i-1}\beta_{i-1} + h_{i,i} \text{ if } j = i; \\ h_{i,i}\beta_i + h_{i,i+1} \text{ if } j = i+1; \end{cases}$$

where $\beta_0 \colon = 0$. Hence, there follows that

$$t_{i,j} = \begin{cases} \alpha_{i-1} d_{i-1} + u_i v_{i-1} \text{ if } j = i-1; \\ \alpha_{i-1}\beta_{i-1} d_{i-1} + d_i + \beta_{i-1} u_i v_{i-1} + \alpha_{i-1} p_{i-1} q_i \text{ if } j = i; \\ \beta_i d_i + p_i q_{i+1} \text{ if } j = i+1. \end{cases} \tag{1}$$

Observe that $\Delta = (\delta_{i,j}) = L_u \cdot L_q^T$ is also tridiagonal with entries

$$\delta_{i,j} = \begin{cases} \alpha_{i-1} \text{ if } j = i-1; \\ \alpha_{i-1}\beta_{i-1} + 1 \text{ if } j = i; \\ \beta_i \text{ if } j = i+1. \end{cases} \tag{2}$$

Let us consider the sequence of the characteristic polynomials $\psi_k(\lambda)$, $1 \le k \le n$, of the leading principal submatrices $A_k\colon = A[1\colon k, 1\colon k]$, $1 \le k \le n$, of A , that is,

$$\psi_k(\lambda)\colon = \det(\lambda I_k - A_k), \quad 1 \le k \le n.$$

From

$$\det(\lambda I - A) = \det(\lambda \Delta - T),$$

by using (1) and (2) we obtain that these polynomials satisfies a three-term recurrence relation, namely,

$$\psi_{k+1}(\lambda) = \gamma_k(\lambda)\psi_k(\lambda) - \rho_{k-1}(\lambda)\psi_{k-1}(\lambda), \quad 0 \le k \le n-1, \tag{3}$$

where

$$\gamma_k(\lambda) = \alpha_k\beta_k(\lambda - d_k) + (\lambda - d_{k+1}) - \beta_k u_{k+1} v_k - \alpha_k p_k q_{k+1},\ 0 \le k \le n-1,$$

$$\rho_{k-1}(\lambda) = [\alpha_k(\lambda - d_k) - u_{k+1}v_k]\,[\beta_k(\lambda - d_k) - p_k q_{k+1}]\,,\ 1 \le k \le n-1,$$

with $\psi_0(\lambda)\colon = 1$ and $\psi_{-1}(\lambda)\colon = 0$.

In [5] it is shown that the three-term relation (3) can be decoupled in two different recurrent relations which do not require any assumption about the invertibility of the elements u_i and q_i, $2 \le i \le n-1$, and, therefore, can be evaluated more stably. The evaluation is at the core of the divide-and-conquer eigenvalue method for real symmetric dpss matrices proposed in [5]. The properties of (3) are also exploited in [4] for devising a Sturm-like bisection method for the numerical computation of the eigenvalues of a real symmetric dpss matrix. Hereafter in this paper we shall consider some specializations of the recurrence (3) for different choices of the parameters u_i, v_i, p_i, q_i, d_i. Under these specializations the polynomials $\psi_k(\lambda)$ become closely related to Chebyshev orthogonal polynomials and this connection makes possible to find closed formulas for their zeros.

3. A dpss matrix related to first-kind Chebyshev polynomials

The parameters $u_j = q_j = 1$, $2 \le j \le n$, $v_j = \bar{p}_j = \mathtt{i}$, where $\mathtt{i}$ stands for the imaginary unit, and $d_j = 0$ generate the Hermitian dpss matrix $A = \mathcal{T}_n \in \mathbb{C}^{n\times n}$. For $n = 5$ the matrix is displayed below

$$\mathcal{T}_5 = \begin{bmatrix} 0 & -\mathtt{i} & -\mathtt{i} & -\mathtt{i} & -\mathtt{i} \\ \mathtt{i} & 0 & -\mathtt{i} & -\mathtt{i} & -\mathtt{i} \\ \mathtt{i} & \mathtt{i} & 0 & -\mathtt{i} & -\mathtt{i} \\ \mathtt{i} & \mathtt{i} & \mathtt{i} & 0 & -\mathtt{i} \\ \mathtt{i} & \mathtt{i} & \mathtt{i} & \mathtt{i} & 0 \end{bmatrix}.$$

For $\alpha_1 = \beta_1 = -1$ it is found that the corresponding sequence of polynomials $\{\psi_k(\lambda)\}$ satisfies

$$\psi_{k+1}(\lambda) = 2\lambda\psi_k(\lambda) - (\lambda^2+1)\psi_{k-1}(\lambda), \quad 1 \le k \le n-1,$$

with the initializations $\psi_0(\lambda)\colon = 1$ and $\psi_1(\lambda)\colon = \lambda$.

These polynomials are related to the Chebyshev polynomials of the first kind $t_k(\lambda)$, $k \ge 0$, defined by

$$t_0(\lambda)\colon = 1;\ t_1(\lambda)\colon = \lambda;\ t_{k+1}(\lambda)\colon = 2\lambda t_k(\lambda) - t_{k-1}(\lambda),\ k \ge 1.$$

By setting $\lambda = \cos\theta$, $0 \le \theta \le \pi$, we obtain the closed form of the Chebyshev polynomials

$$t_k(\cos\theta) = \cos k\theta, \quad k = 0, 1, \ldots,$$

which gives the well-known expression for the roots $\xi_j^{(k)}$, $1 \le j \le k$, of $t_k(\lambda)$,

$$\xi_j^{(k)} = \cos\frac{(2j-1)\pi}{2k}, \quad 1 \le j \le k,\ k \ge 1. \tag{4}$$

For $k \ge 0$ let us denote by $f_k(\lambda)$ the function

$$f_k(\lambda)\colon = \left(\sqrt{1+\lambda^2}\right)^k t_k\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right), \quad k \ge 0.$$

It follows that $f_0(\lambda) = 1$, $f_1(\lambda) = \lambda$ and, moreover,

$$f_{k+1}(\lambda) = 2\lambda f_k(\lambda) - (1+\lambda^2) f_{k-1}(\lambda),\ k \ge 1.$$

Hence, it is found $f_k(\lambda)$ is a polynomial of degree k which coincides with $\psi_k(\lambda)$. In this way we arrive at the following result for the eigenvalues of $\mathcal{T}_n$, $n \ge 1$.

Theorem 1. *The eigenvalues* $\lambda_j^{(n)}$, $1 \le j \le n$ *of* $\mathcal{T}_n$, $n \ge 1$, *satisfy*

$$\lambda_j^{(n)} = \cot\frac{(2j-1)\pi}{2n} = \frac{\cos\frac{(2j-1)\pi}{2n}}{\sin\frac{(2j-1)\pi}{2n}}, \quad 1 \le j \le n.$$

For the purpose of testing numerically the performance of fast structured QR algorithms for dpss matrices it is worth noting that the eigenvalue problem for $\mathcal{T}_n$ is perfectly well conditioned and, therefore, the problem should be solved in an accurate way even for large values of n. In addition, Theorem 2 provides a simple matrix proof of the well-known fact that the numbers $\lambda_j^{(n)}$, $1 \le j \le n$, are algebraic. By reversing the coefficients of $\psi_n(\lambda)$ we obtain that the reciprocals $1/\lambda_j^{(n)}$ when defined are algebraic.

4. A dpss matrix related to second-kind Chebyshev polynomials

A dpss matrix with ill-conditioned eigenvalues can be obtained by means of the following set of parameters: $u_j = 1$, $q_j = \frac{2(n+1-j)\mathtt{i}}{n+1}$, $2 \leq j \leq n$; $p_j = 1$, $v_j = -\frac{2\mathtt{i}j}{n+1}$, $p_j = 1$, $1 \leq j \leq n-1$; $d_j = \mathtt{i}\frac{n-2j+1}{n+1}$, $1 \leq j \leq n$. The resulting dpss matrix $A = \mathcal{U}_n \in \mathbb{C}^{n \times n}$ is shown below for $n = 5$

$$\mathcal{U}_5 = \frac{\mathtt{i}}{6} \begin{bmatrix} 4 & 8 & 6 & 4 & 2 \\ -2 & 2 & 6 & 4 & 2 \\ -2 & -4 & 0 & 4 & 2 \\ -2 & -4 & -6 & -2 & 2 \\ -2 & -4 & -6 & -8 & -4 \end{bmatrix}.$$

This matrix looks very interesting for testing purposes. From one hand it is observed that $\mathcal{U}_n$ satisfies

$$\mathcal{U}_n = J_n \cdot \mathcal{U}_n^H \cdot J_n,$$

where J_n denotes the permutation (reversion) matrix with unit antidiagonal entries of order n. Thus the computation of the eigenvalues of $\mathcal{U}_n$ can be a good test problem for fast adaptations of QR-like eigenvalue routines using not orthogonal transformations like the HR and XHR methods [16]. From the other hand it is easily shown that

$$\operatorname{rank}(\mathcal{U}_n - \mathcal{U}_n^H) \leq 2,$$

and, therefore, the rank structure of $\mathcal{U}_n$ is also maintained under the customary QR eigenvalue algorithm. This makes the matrix a good test case for fast adaptations of the QR method, too.

For $\alpha_1 = -1$ and $\beta_1 = -\frac{n-1}{n}$ we obtain that

$$\frac{n-k}{n-k+1}(\lambda - \mathtt{i}\frac{n-2k+1}{n+1}) + (\lambda - \mathtt{i}\frac{n-2k-1}{n+1}) - \mathtt{i}\frac{n-k}{n-k+1}\frac{2k}{n+1} + \mathtt{i}\frac{2(n-k)}{n+1} = \\ \frac{n-k}{n-k+1}(\lambda - \mathtt{i}) + (\lambda + \mathtt{i}) = \gamma_k(\lambda), \; 1 \leq k \leq n-1,$$

and

$$\left[-(\lambda - \mathtt{i}\frac{n-2k+1}{n+1}) + \mathtt{i}\frac{2k}{n+1}\right]\left[-\frac{n-k}{n-k+1}(\lambda - \mathtt{i}\frac{n-2k+1}{n+1}) - \mathtt{i}\frac{2(n-k)}{n+1}\right] = \\ \frac{n-k}{n-k+1}(1 + \lambda^2) = \rho_{k-1}(\lambda), \quad 1 \leq k \leq n-1.$$

The sequence $\{\psi_k(\lambda)\}$ of the characteristic polynomials of the leading principal submatrices of $\mathcal{U}_n$ verifies the recurrence

$$\psi_{k+1}(\lambda) = \left[\frac{n-k}{n-k+1}(\lambda - \mathtt{i}) + \lambda + \mathtt{i}\right]\psi_k(\lambda) - \frac{n-k}{n-k+1}(1+\lambda^2)\psi_{k-1}(\lambda), \tag{5}$$

with $1 \leq k \leq n-1$ and $\psi_0(\lambda) \colon = 1$ and

$$\psi_1(\lambda) \colon = \lambda - \mathtt{i}\frac{n-1}{n+1}.$$

These polynomials are related with the second-kind Chebyshev polynomials $u_k(\lambda)$, $k \geq 0$, defined by

$$u_0(\lambda) \colon = 1;\ u_1(\lambda) \colon = 2\lambda;\ u_{k+1}(\lambda) \colon = 2\lambda u_k(\lambda) - u_{k-1}(\lambda),\ k \geq 1.$$

In order to enlighten this connection we consider the auxiliary sequences of polynomials

$$f_k(\lambda) \colon = \left(\sqrt{1+\lambda^2}\right)^k t_k\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right), \quad k \geq 0,$$

and

$$g_k(\lambda) \colon = \left(\sqrt{1+\lambda^2}\right)^k u_k\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right), \quad k \geq 0,$$

where $t_k(\lambda)$ and $u_k(\lambda)$ are the Chebyshev polynomials of degree k of the first and the second kind, respectively. Observe that

$$f_{k+1}(\lambda) = 2\lambda f_k(\lambda) - (1+\lambda^2) f_{k-1}(\lambda),\ k \geq 1, \tag{6}$$

with $f_0(\lambda) = 1$, $f_1(\lambda) = \lambda$ and, similarly,

$$g_{k+1}(\lambda) = 2\lambda g_k(\lambda) - (1+\lambda^2) g_{k-1}(\lambda),\ k \geq 1, \tag{7}$$

with $g_0(\lambda) = 1$, $q_1(\lambda) = 2\lambda$. From these two recurrences it also follows that for $k \geq 1$

$$g_{k+1}(\lambda) - f_{k+1}(\lambda) = 2\lambda(g_k(\lambda) - f_k(\lambda)) - (1+\lambda^2)(g_{k-1}(\lambda) - f_{k-1}(\lambda))$$

with $g_0(\lambda) - f_0(\lambda) = 0$ and $g_1(\lambda) - f_1(\lambda) = \lambda$. Hence, by induction we find that

$$g_{k+1}(\lambda) - f_{k+1}(\lambda) = \lambda g_k(\lambda), \qquad k \geq 0. \tag{8}$$

Let us now introduce the sequence of polynomials defined by

$$(n+1)\rho_k(\lambda)\colon = g_k(\lambda) + (n-k)f_k(\lambda) - \mathtt{i}(n-k)g_{k-1}(\lambda), \quad k \geq 0.$$

Observe that

$$\rho_0(\lambda) = 1 = \psi_0(\lambda), \quad \rho_1(\lambda) = \lambda - \mathtt{i}\frac{n-1}{n+1} = \psi_1(\lambda).$$

Indeed, from (6), (7) and (8) it is shown that $\{\psi_k(\lambda)\}$ satisfies the recurrence (5) and, therefore, we can conclude that

$$\psi_k(\lambda) = \rho_k(\lambda), \quad 0 \leq k \leq n.$$

For $\lambda = \dfrac{\cos\theta}{\sin\theta}$, $0 < \theta < \pi$, by using the well-known trigonometric forms of the Chebyshev polynomials we f obtain that for $0 \leq k \leq n$,

$$\rho_k(\lambda) = \rho_k(\cot\theta) = \frac{(\sin\theta)^{-k}}{n+1}\left[\frac{\sin(k+1)\theta}{\sin\theta} + (n-k)e^{-\mathtt{i}k\theta}\right].$$

This implies that

$$\rho_n(\lambda) = \rho_n(\cot\theta) = \frac{1}{n+1}\frac{\sin(n+1)\theta}{(\sin\theta)^{n+1}}, \quad n \geq 0.$$

To our knowledge these polynomials have been first studied in [9]. Their closed trigonometric form makes possible to explicitly determine the eigenvalues of $\mathcal{U}_n$.

Theorem 2. *The eigenvalues* $\lambda_j^{(n)}$, $1 \leq j \leq n$ *of* $\mathcal{U}_n$, $n \geq 1$, *satisfy*

$$\lambda_j^{(n)} = \cot\frac{j\pi}{n+1} = \frac{\cos\frac{j\pi}{n+1}}{\sin\frac{j\pi}{n+1}}, \quad 1 \leq j \leq n.$$

A closed form for the eigenvectors of $\mathcal{U}_n$ can also be given. Let

$$q_n(\lambda) = \frac{\lambda^{n+1}-1}{\lambda-1} = 1 + \lambda + \ldots + \lambda^n = \prod_{j=1}^{n}(\lambda - \omega_{n+1}^j),$$

where $\omega_{n+1} = \cos\frac{2\pi}{n+1} + \mathtt{i}\sin\frac{2\pi}{n+1}$ is a $(n+1)$–st primitive root of unity. If we set

$$q_n^{(k)}(\lambda) = \frac{q_n(z)}{\lambda - \omega_{n+1}^k} = \prod_{j=1, j\neq k}^{n}(\lambda - \omega_{n+1}^j), \quad 1 \leq k \leq n,$$

and $\boldsymbol{q}_n^{(k)}$ denotes the coefficient vector of order n associated with $q_n^{(k)}(\lambda)$, then it holds

$$\boldsymbol{q}_n^{(k)^T}\mathcal{U}_n = \lambda_k^{(n)}\boldsymbol{q}_n^{(k)^T}, \quad 1 \leq k \leq n.$$

Since it is well-known that the coefficients of the polynomials $q_n^{(k)}(\lambda)$ grow exponentially as n increases, we remark that $\mathcal{U}_n$ can have seriously ill-conditioned eigenvalues. For $n = 32$ and $n = 64$ the MatLab[1] routine *condeig* applied to $\mathcal{U}_n$ reports the maximum value $cmax = 9.27e+05$ and $cmax = 1.21e+16$, respectively.

5. Conclusion

Two classes of dpss matrices whose eigenvalues can be expressed in closed form by means of trigonometric formulas have been exhibited. The characteristic polynomials of these matrices are shown to be related to Chebyshev orthogonal polynomials. The matrices can be useful for testing fast adaptations of customary eigenvalue algorithms for rank structured matrices as well as for an elementary treatment of some issues in number theory.

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