Structured Matrix Methods for Polynomial Root-Finding

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Summary of the Talk

- **Polynomial Root-finding via matrix methods based on the reduction to a matrix eigenvalue problem**

- **Structured matrix technology for the efficient iterative solution of the eigenvalue problem**

- **Fixed or variable precision floating point arithmetic depending on the iterative method employed**
The companion matrix $C$ associated with a polynomial $p(z)$ expressed in the power basis:

$$p(z) = \sum_{i=0}^{n} p_i z^i, \quad p_n \neq 0, \quad C = \begin{bmatrix}
0 & \ldots & 0 & -p_0/p_n \\
1 & \ddots & \vdots & \vdots \\
\vdots & \ddots & 0 & \vdots \\
0 & \ldots & 1 & -p_{n-1}/p_n
\end{bmatrix}$$

$p(z) = 0 \iff \det(C - zI_n) = 0$

Properties of $C$:

1. sparse – NOT REALLY IMPORTANT !!!!!
2. small rank (rank-one) perturbation of a unitary matrix with submatrices of small rank (<=1) in the lower triangular part – REALLY IMPORTANT !!!!
Structures in Polynomial Root-Finding II

The companion matrix $A$ associated with a polynomial $p(z)$ expressed using the Lagrange basis with real nodes

\[ p(z) = \sum_{i=1}^{n} \frac{p(s_i)}{w'(s_i)} \prod_{j \neq i} (z - s_i) + \prod_{i=1}^{n} (z - s_i) \]

\[ A = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix} - \begin{bmatrix} \frac{p(s_1)}{w'(s_1)} \\ \frac{p(s_2)}{w'(s_2)} \\ \vdots \\ \frac{p(s_n)}{w'(s_n)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \]

\[ p(z) = 0 \iff \det(A - zI_n) = 0 \]

1. small rank (rank-one) perturbation of a real diagonal matrix – REALLY IMPORTANT !!!!

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The companion matrix $T$ of a polynomial $p(z)$ expressed using Chebyshev polynomials $\{t_k(z)\}$ of the first kind

$$p(z) = u_0 t_0(z) + \sqrt{2} \sum_{i=1}^{n-2} u_i t_i(z) + u_{n-1} t_{n-1}(z) - (t_n(z) - z t_{n-1}(z))$$

$$T = \begin{bmatrix}
0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \frac{1}{2} \\
& & & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\
& & & & \frac{1}{\sqrt{2}} & 0
\end{bmatrix} + \begin{bmatrix}
u_0 \\
u_1 \\
\vdots \\
u_{n-1}
\end{bmatrix} \begin{bmatrix}
0 & \cdots & 0 & 1
\end{bmatrix}$$

$$p(z) = 0 \iff \det(T - zI_n) = 0.$$
The companion pencil \((A, B)\) of a polynomial \(p(z)\) expressed in the Bernstein basis [Winkler, LAA, 2003]

\[
p(z) = \sum_{i=0}^{n} b_i \binom{n}{i} (1 - z)^{n-i} z^i, \quad b_n = 1
\]

\[
A = \begin{bmatrix}
0 & \ldots & 0 & -b_0 \\
1 & \ddots & \ddots & \ddots \\
\ddots & 0 & \ddots & \\
1 & \ldots & -b_{n-1}
\end{bmatrix}, \quad B = \begin{bmatrix}
\binom{n}{1} / \binom{n}{0} \\
\ddots \\
\binom{n}{n} / \binom{n}{n-1}
\end{bmatrix}
\]

\[p(z) = 0 \iff z = \frac{x}{1 + x}, \quad \det(A -xB) = 0.\]

1. combination of unitary and symmetric rank structures
2. \(B^{-1}A\) is NOT a rank-one perturbation of a unitary matrix
Structures in Eigenvalue Problems

The QR method is customary for solving eigenvalue problems numerically

\[ A_0 = A \]

\[ q_k(A^{(k)}) = Q^{(k)} R^{(k)}, \quad \text{(QR factorization)} \]

\[ A^{(k+1)} := Q^{(k)H} A^{(k)} Q^{(k)}, \]

In the symmetric case matrix shapes preserved under the QR method reduce to staircase – generalized banded matrices \([\text{Arbenz & Golub, NLAA, 1993}]\)

Question: Are there any other structures – NOT SHAPES– preserved under the QR iteration?
Rank Structures in Eigenvalue Problems

\[ A \xrightarrow{\text{QR steps}} B, \quad B = R \cdot A \cdot R^{-1}, \quad B = Q^H \cdot A \cdot Q \]

\( A \in \mathbb{C}^{n \times n} \) has a rank structure \( (A \in \mathcal{F}(p, q)) \) if:

\[ \max_{1 \leq k \leq n-1} \text{rank} \ A(k + 1 : n, 1 : k) \leq p, \quad (p << n) \]

\[ \max_{1 \leq k \leq n-1} \text{rank} \ A(1 : k, k + 1 : n) \leq q, \quad (q << n) \]

\( B = R \cdot A \cdot R^{-1} \Rightarrow B \in \mathcal{F}(p, *) \)

\( A = H + U \cdot V^T, \) \( H \) is Hermitian and/or unitary, \( U, V \in \mathbb{C}^{n \times r} \)

\( B = Q \cdot A \cdot Q^H \Rightarrow B \in \mathcal{F}(p, q + 2 \cdot r) \)

Numerical Issues

- The representation of rank structures is not unique and different representations have different numerical properties. Differently speaking, the choice of generators is critical.

\[
\begin{bmatrix}
1 & \epsilon \\
\epsilon & \epsilon^2
\end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon \end{bmatrix} = \begin{bmatrix} \epsilon \\ \epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}
\]

- How to preserve the additive structure of the eigenvalue problem (unitary/Hermitian plus low rank)

\[
A = H + U \cdot V^T, \quad A \xrightarrow{QR \text{ steps}} B, \quad B = Q^H A Q, \quad B - Q^H (U \cdot V^T) \cdot Q
\]

projection on the manifold of unitary matrices is NOT trivial [Bini & Eidelman & G. & Gohberg, Math. Comput., 2007]
Fast QR method for companion matrices [Bini & Eidelman & G.& Gohberg, SIMAX, 2007].
Computational cost: \(\simeq 180n\) flops per iteration. Faster than the customary implementation for \(n \geq 100\).

\[
p(z) = z^n - 1
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(rcond)</th>
<th>(err)</th>
<th>(it)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5.2e-15</td>
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<td>512</td>
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\(p(z) = z^n - 1\)
### Numerical Results II

Fast QR method for companion matrices in the Chebyshev basis

<table>
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<tr>
<th>$N$</th>
<th>DGEEV (sec)</th>
<th>FastQR (sec)</th>
<th>max_err($\Re$)</th>
<th>max_err($\Im$)</th>
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<td>0.58e-02</td>
<td>0.49e-14</td>
<td>0.46e-14</td>
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<td>0.17e-14</td>
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<td>0.34</td>
<td>0.14e-13</td>
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<tr>
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<td>0.22e-13</td>
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<td>148.62</td>
<td>4.78</td>
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<td>0.28e-13</td>
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<td>3200</td>
<td>1450.4</td>
<td>18.54</td>
<td>0.6e-13</td>
<td>0.17e-12</td>
</tr>
</tbody>
</table>
Numerical Results II

Fast QR method for companion matrices in the Chebyshev basis

(a) Time ratio vs. n

(b) Number of iterations per eigenvalue vs. n
Beyond the QR Method

- Investigate other iterative methods for solving the matrix eigenvalue problem
  1. How to enlarge the class of rank structured matrices which can be dealt with

\[
A \xrightarrow{LR \text{ steps}} B, \quad B = R \cdot A \cdot R^{-1}, \quad B = L^{-1} \cdot A \cdot L
\]

\[
B = R \cdot A \cdot R^{-1} \Rightarrow B \in \mathcal{F}(p, \ast)
\]

\[
B = L^{-1} \cdot A \cdot L \Rightarrow B \in \mathcal{F}(\ast, q)
\]

2. Not unitary transformations \(\Rightarrow\) from fixed to variable precision floating point arithmetic
QR-like Methods

The Bernstein companion matrix $B^{-1}A$ is a “scaled” companion matrix, i.e.,

$$B^{-1}A = D \cdot C \cdot D^{-1}, \quad D \text{ diagonal} \quad C \text{ companion}$$

$$B^{-1}A = U + \text{rank one}, \quad U \cdot D^2 \cdot U^H = D^2$$

- Suggestion: Consider the QR method where $Q$ is orthogonal w.r.t. the scalar product $\langle x, y \rangle = x^T D^2 y$

- Generalization: If $Q \in \mathbb{C}^{n \times n}$ is $D$–orthogonal, $D = \text{diag}[1, \pm 1]$, QR method $\Rightarrow$ DQR algorithm [Uhlig, Numer. Math., 19977]

1. The DQR algorithm applies to the Lagrange companion matrix – (block) diagonal plus rank-one matrix– in linear time per iteration
Conclusions/ Future Work

- Efficient numerical methods for a companion matrix
  1. small perturbation of an Hermitian matrix
  2. small perturbation of a unitary matrix
- Theoretically generalized QR algorithms enable this set to be enlarged
- Practically we need to consider the effects of using NOT unitary transformations
  1. If we limit the size of the entries of $D$—orthogonal transformations to below $10^p$ then we could guarantee about $m - 4p$ accurate eigenvalue digits of the $m$ being carried
- What’s about the behavior of the fast QR algorithms in multi-precision arithmetic?