Solving Bezout-like polynomial equations for the design of interpolatory subdivision schemes

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1 Motivations: The applicative problem.

2 Results: Numeric-symbolic methods using structured matrix technology for solving the problem

3 Open problems and current research issues
A subdivision scheme of *arity* \( p \geq 2 \) recursively defines a sequence of points

\[
P^{(k+1)}: = \{ p_{i}^{(k+1)} \in \mathbb{R}^3, i \in \mathbb{Z} \}
\]

\[
p_{i}^{(k+1)} = \sum_{j \in \mathbb{Z}} a_{i-pj}^{(k)} p_{j}^{(k)}, \ k = 0, 1, \ldots.
\]

By looking at each component we can reduce to consider a sequence of scalar points \( \{ f_{i}^{(k)}, i \in \mathbb{Z} \} \)

The sequence \( a^{(k)} = \{ a_{i}^{(k)}, i \in \mathbb{Z} \} \) is the associated mask at the \( k \)-th step with support \( \sigma(a^{(k)}) = \{ j: a_{j}^{(k)} \neq 0 \} \).

The Laurent polynomial \( a_k(z) = \sum_{j \in \mathbb{Z}} a_{j}^{(k)} z^j \) is the associated symbol.
The subdivision scheme is called **interpolating** if

\[ P^{(k)} \subset P^{(k+1)}, \quad k = 0, 1, \ldots, \]

The general form of an interpolating scheme is

\[
\begin{align*}
  f^{(k+1)}_{p_i} &= f^{(k)}_i; \\
  f^{(k+1)}_{p_i + \ell} &= \sum_{j \in \mathbb{Z}} a^{(k)}_{\ell + pj} f^{(k)}_{i-j}
\end{align*}
\]

The associated mask satisfies

\[ a^{(k)}_{p_i} = \delta_{i,0}, \quad i \in \mathbb{Z}. \]

Equivalently, the associated symbol satisfies

\[
\sum_{j=0}^{p-1} a_k(r_j z) = p, \quad r_j = e^{2\pi ij/p}.
\]
The DD-Scheme

\[
\begin{align*}
  f_{2i}^{(k+1)} &= f_i^{(k)}, \\
  f_{2i+1}^{(k+1)} &= \frac{9}{16} (f_i^{(k)} + f_{i+1}^{(k)}) - \frac{1}{16} (f_{i-1}^{(k)} + f_{i+2}^{(k)})
\end{align*}
\]
A Quaternary Scheme

\[
\begin{align*}
\left\{
\begin{array}{l}
\frac{f_{4i}^{(k+1)}}{f_{4i+1}^{(k+1)}} = a_1 f_{i-1}^{(k)} + a_2 f_i^{(k)} + a_3 f_{i+1}^{(k)} + a_4 f_{i+2}^{(k)}; \\
\frac{f_{4i+1}^{(k+1)}}{f_{4i+2}^{(k+1)}} = a_5 f_{i-1}^{(k)} + a_6 f_i^{(k)} + a_7 f_{i+1}^{(k)} + a_8 f_{i+2}^{(k)}; \\
\frac{f_{4i+2}^{(k+1)}}{f_{4i+3}^{(k+1)}} = a_8 f_{i-1}^{(k)} + a_7 f_i^{(k)} + a_6 f_{i+1}^{(k)} + a_5 f_{i+2}^{(k)}; \\
\frac{f_{4i+3}^{(k+1)}}{f_{4i+4}^{(k+1)}} = a_4 f_{i-1}^{(k)} + a_3 f_i^{(k)} + a_2 f_{i+1}^{(k)} + a_1 f_{i+2}^{(k)}.
\end{array}
\right.
\end{align*}
\]

\[
\begin{align*}
a_1 &= (7/32) - (7/64)w, & a_2 &= (29/64) + (13/64)w, \\
a_3 &= (5/16) - (5/64)w, & a_4 &= (1/64) - (1/64)w, \\
a_5 &= (15/128) - (5/64)w, & a_6 &= (57/128) + (7/64)w, \\
a_7 &= (49/128) + (1/64)w, & a_8 &= (7/128) - (3/64)w.
\end{align*}
\]

Notice that the coefficients of the mask depend on a parameter $w$.
The Subdivision Operator

Let $S_{a(k)}(k) = (a_{i-pj}^{(k)})$, $i, j \in \mathbb{Z}$, be the banded bi-infinite matrix associated with the subdivision scheme. For the the DD-scheme

$$S_a = \begin{bmatrix}
\ldots & \ldots & 0 & 1 & 0 & \ldots & \ldots & \ldots \\
\ldots & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & \ldots \\
\ldots & \ldots & \ldots & 0 & 1 & 0 & \ldots & \ldots \\
\ldots & \ldots & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{bmatrix}$$

1. Toeplitz-like structure (Hurwitz matrix with step $p$)
   $$a_{i,j}^{(k)} = a_{i-pj}^{(k)} = a_{i+p,j+1}^{(k)}$$

2. The interpolation condition $a_{pi}^{(k)} = \delta_{i,0}$, $i \in \mathbb{Z}$ gives the unit vector rows
Goal(s)

- Given the mask \( a = \{ a_i, i \in \mathbb{Z} \} \) of a stationary \textit{approximating} subdivision scheme
- Find an easy and computationally feasible mechanism for generating a mask \( \hat{a} = \{ \hat{a}_i, i \in \mathbb{Z} \} \) that is related with \( a \) and defines a stationary \textit{interpolating} subdivision scheme
- The relationship between \( a \) and \( \hat{a} \) should guarantee that the (reproduction) properties of \( a \) are maintained.
- In the non-stationary case the reduction would be applied step-by-step
Let \( t(z) = \sum_{j=-\hat{h}}^{\hat{h}} t_j z^j \) be a Laurent polynomial and \( \mathcal{T} \) the bi-infinite Toeplitz matrix associated with \( t(z) \), namely, \( \mathcal{T} = (t_{i-j}) \).

For the product operator \( \mathcal{S} : = \mathcal{T} \cdot \mathcal{S}_a = (s_{i,j}) \) we have

\[
s_{i,j} = \sum_{r=i-\hat{h}}^{i+h} t_{i-r} a_{r-pj} = \sum_{\ell=-\hat{h}}^{\hat{h}} t_{\ell} a_{i-pj-\ell} = s_{i+p,j+1}, \quad i, j \in \mathbb{Z}.
\]

Therefore \( \mathcal{S} = \mathcal{S}_q \) is the subdivision operator of a scheme of arity \( p \) with symbol \( q(z) = a(z) \cdot t(z) \).
The sought polynomial \( t(z) \) should be chosen to satisfy

\[
\sum_{\ell=0}^{p-1} q(r_{\ell}z) = \sum_{\ell=0}^{p-1} a(r_{\ell}z) \cdot t(r_{\ell}z) = p, \quad r_{j} = e^{2\pi ij/p}
\]

- From an algebraic point of view the problem reduces to solve a very special Bezout-like polynomial equation possibly involving more than two polynomials \( a_{j}(z) = a(r_{j}z) \), \( 0 \leq j \leq p - 1 \), depending on the value of \( p \).
- From a computational point of view we are interested in solutions with a small support almost centered around zero.
Let \( a(z) = \sum_{\ell=0}^{2N} a_\ell z^\ell \)

Assume that \((p - 1) | (2N)\) and set
\[
m + 1 = \frac{2N}{p-1} : = \text{degree}(t(z))
\]

Then the polynomial equation reduces to a linear system of size \((m + 1)p\)

\[
\mathcal{R} \cdot \begin{bmatrix} t_0^T & t_1^T & \cdots & t_{p-1}^T \end{bmatrix}^T = p \cdot e_{jp+1},
\]

If \( p = 2 \) (binary case) \( \mathcal{R} \) is the Sylvester resultant matrix associated with \( a(z) \) and \( a(-z) \)

If \( p > 2 \) \( \mathcal{R} \) IS NOT a resultant matrix
The basic idea is to decompose $R$ as direct sum of smaller matrices.

**Th:** Let $R \in \mathbb{R}^{(4N) \times (4N)}$ be the Sylvester resultant matrix associated with $a(z)$ and $a(-z)$. There exist two invertible matrices $G_1$ and $G_2$ such that

$$G_1 \cdot R \cdot G_2 = H_0 \oplus H_1.$$

1. The invertibility of $H_0$ and $H_1$ is deduced from the invertibility of $R$ that is equivalent to require that $b(z)$ and $b(-z)$ are relatively prime polynomials.

2. Similar decompositions hold also for $p > 2$ but the corresponding $R$ is not a resultant matrix.
How to Find the Solutions: The General Case

Theorem: A set of solutions of the Bezout equation for $a(z) = \sum_{\ell=0}^{2N} a_\ell z^\ell$ is determined by the coefficients of the rows of the inverse of $H_k^T$, $0 \leq k \leq p - 1$, $H_k \in \mathbb{R}^{(m+1)\times(m+1)}$, defined by

$$h_{i,j}^{(k)} = a_{k-i+1+(j-1)p}, \quad 1 \leq i, j \leq m + 1$$

whenever the matrix is nonsingular.

1. The correction is computed via structured matrix inversion of certain finite sections of the subdivision operator.

2. If the scheme is defined in terms of a free parameter $\rightarrow$ we can employ numeric-symbolic algorithms.
A non-stationary scheme depending on a parameter

\[ a_k(z) = \frac{1}{2} (z + 1)^2 \frac{z^2 + 2v_kz + 1}{2(v_k + 1)}, \]

\[ v_k \in (0, +\infty), \quad k \geq 0, \quad v_k = \sqrt{\frac{v_k-1 + 1}{2}} \]

\( a_k(z) \) is Hurwitz and, hence, \( a_k(z) \) and \( a_k(-z) \) are relatively prime

\[ \hat{a}_k(z) = -\frac{1}{8v_k(v_k + 1)} + \frac{(2v_k + 1)^2}{8v_k(v_k + 1)}z^2 + z^3 \]

\[ + \frac{(2v_k + 1)^2}{8v_k(v_k + 1)}z^4 - \frac{1}{8v_k(v_k + 1)}z^6 \]
The Reproduction Properties

- Exact reconstruction of conic sections through the interpolating scheme with symbols $\hat{a}_k(z)$. Dashed lines depict the assumed starting polygons. The corresponding initial parameter is respectively chosen as (a) $\nu_{-1} = 1$, (b) $\nu_{-1} = \frac{1}{2}$, (c) $\nu_{-1} = \cosh\left(\frac{3}{5}\right)$.
Future Work

1. Solvability conditions for the Bezout equation in the case $p > 2$.
   1. Relating the invertibility of $\mathcal{R}$ to suitable conditions on the roots of $a(z)$
   2. Invertibility of all finite sections implies the invertibility of $\mathcal{R}$

2. Extension to bivariate and multivariate schemes
   1. Multi-index or multi-level Toeplitz-like matrices
   2. Bezout-like equation for bivariate and multivariate polynomials