Magnetic Vortices, Nielsen-Olesen - Nambu Strings and Abrikosov Lattices

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Discussions with Jürg Fröhlich, Gian Michele Graf, Peter Sarnak, Tom Spencer

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Ginzburg-Landau Equations

Equilibrium states of superconductors (macroscopically) and of the $U(1)$ Higgs model of particle physics are described by the Ginzburg-Landau equations:

$$-\Delta_A \Psi = \kappa^2 (1 - |\Psi|^2)\Psi$$
$$\operatorname{curl}^2 A = \operatorname{Im}(\bar{\Psi} \nabla_A \Psi)$$

where $(\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d,\ d = 2, 3,\ \nabla_A = \nabla - iA,\ \Delta_A = \nabla_A^2,$ the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau material constant.

Origin of Ginzburg-Landau Equations

Superconductivity. $\Psi : \mathbb{R}^d \to \mathbb{C}$ is called the *order parameter*; $|\Psi|^2$ gives the density of (Cooper pairs of) superconducting electrons. $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic potential. $\mathrm{Im}(\bar{\Psi}\nabla_A\Psi)$ is the superconducting current.

Particle physics. Ψ and A are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively. (Part of Weinberg - Salam model of electro-weak interactions/ a standard model.)

Geometrically, A is a connection on the principal $U(1)$- bundle $\mathbb{R}^2 \times U(1)$, and Ψ, a section of the associated bundle.

Similar equations appear in superfluidity, Bose-Einstein condensation and fractional quantum Hall effect.

Ginzburg-Landau Energy

Ginzburg-Landau equations are the Euler-Lagrange equations for the *Ginzburg-Landau energy functional*

$$\mathcal{E}_\Omega(\Psi, A) := \frac{1}{2}\int_\Omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A)^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}.$$

Superconductors: $\mathcal{E}(\Psi, A)$ is the difference in (Helmhotz) free energy between the superconducting and normal states.

In the $U(1)$ Higgs model case, $\mathcal{E}_\Omega(\Psi, A)$ is the energy of a static configuration in the $U(1)$ Yang-Mills-Higgs classical gauge theory.

Symmetries

The Ginzburg-Landau equations admit several symmetries, that is, transformations which map solutions to solutions.

Gauge symmetry: for any sufficiently regular function $\gamma : \mathbb{R}^2 \to \mathbb{R}$,

$$T_\gamma^{\mathrm{gauge}} : (\Psi(x), A(x)) \mapsto (e^{i\gamma(x)}\Psi(x), A(x) + \nabla\gamma(x));$$

Translation symmetry: for any $h \in \mathbb{R}^2$,

$$T_h^{\mathrm{trans}} : (\Psi(x), A(x)) \mapsto (\Psi(x+h), A(x+h));$$

Rotation symmetry: for any $\rho \in SO(2)$,

$$T_\rho^{\mathrm{rot}} : (\Psi(x), A(x)) \mapsto (\Psi(\rho^{-1}x), \rho^{-1}A((\rho^{-1})^T x)).$$

One of the analytically interesting aspects of the Ginzburg-Landau theory is the fact that, because of the gauge transformations, the symmetry group is infinite-dimensional.

Type I and II Superconductors

Two types of superconductors:

$\kappa < 1/\sqrt{2}$: Type I superconductors, exhibit first-order phase transitions from the non-superconducting state to the superconducting state (essentially, all pure metals);

$\kappa > 1/\sqrt{2}$: Type II superconductors, exhibit second-order phase transitions and the formation of vortex lattices (dirty metals and alloys).

For $\kappa = 1/\sqrt{2}$, Bogomolnyi has shown that the Ginzburg-Landau equations are equivalent to a pair of first-order equations.

Quantization of Flux

From now on we let $d = 2$. Finite energy states (Ψ, A) are classified by the topological degree

$$\deg(\Psi) := \deg\left(\frac{\Psi}{|\Psi|}\bigg|_{|x|=R} \right),$$

where $R \gg 1$. For each such state we have the quantization of magnetic flux:

$$\int_{\mathbb{R}^2} B = 2\pi \deg(\Psi) \in 2\pi\mathbb{Z},$$

where $B := \operatorname{curl} A$ is the magnetic field associated with the vector potential A.

Homogenous and Self-dual Solutions

Homogenous solutions:

(a) $(\Psi \equiv 1,\ A \equiv 0)$, the perfect superconductor,

(b) ($\Psi \equiv 0$, A with $\operatorname{curl} A$ constant), the normal metal.)

Self-dual case $\kappa = 1/\sqrt{2}$:

For $\kappa = 1/\sqrt{2}$ in GLE, Bogomolnyi found the lower bound for energy

$$\mathcal{E}(\Psi, A)\big|_{\kappa=1/\sqrt{2}} \geq \pi\, |\deg(\Psi)|$$

and showed that this bound is saturated when certain *first-order* equations are satisfied.
Using this Taubes described completely solutions of a given degree.

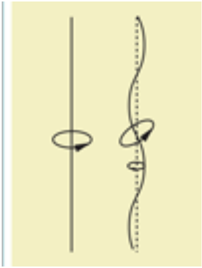
Vortices

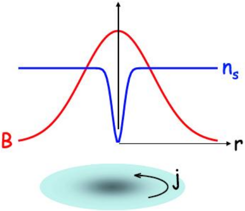
"Radially symmetric" (more precisely, *equivariant*) solutions:

$$\Psi^{(n)}(x) = f^{(n)}(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a^{(n)}(r)\nabla(n\theta),$$

where $n =$ integer and $(r,\theta) =$ polar coordinates of $x \in \mathbb{R}^2$.

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{curl} A^{(n)} = \deg(\Psi^{(n)}) = n \in \mathbb{Z}. \qquad \text{(Berger-Chen)}$$





$(\Psi^{(n)}, A^{(n)}) =$ the *magnetic n-vortex* (superconductors) or *Nielsen-Olesen* or *Nambu string* (the particle physics).

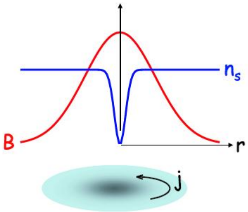
Vortex Profile

The profiles are exponentially localized:

$$|1 - f^{(n)}(r)| \leq ce^{-r/\xi}, \quad |1 - a^{(n)}(r)| \leq ce^{-r/\lambda},$$

Here ξ = *coherence length* and λ = *penetration depth*.

$$\sqrt{2}\kappa = \lambda/\xi.$$



The exponential decay is due to the Higgs mechanism of mass generation: massless A $\Rightarrow$ massive A, with $m_A = \lambda^{-1}$.

In the leading approximation the evolution of a superconductor is described by the gradient-flow-type equations

$$\gamma(\partial_t + i\Phi)\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$\sigma(\partial_t A - \nabla\Phi) = -\operatorname{curl}^2 A + Im(\bar{\Psi}\nabla_A\Psi),$$

$Re\gamma \geq 0$, the *time-dependent Ginzburg-Landau equations* or the *Gorkov-Eliashberg-Schmidt equations*. (Earlier versions: Bardeen and Stephen and Anderson, Luttinger and Werthamer.)

The last equation comes from two Maxwell equations, with $-\partial_t E$ neglected, (Ampère's and Faraday's laws) and the relations $J = J_s + J_n$, where $J_s = \mathrm{Im}(\overline{\Psi}\nabla_A\Psi)$, and $J_n = \sigma E$.

Time-Dependent Eqns. $U(1)$ Higgs Model

The time-dependent $U(1)$ Higgs model is described by $U(1)-$Higgs (or Maxwell-Higgs) equations ($\Phi = 0$)

$$(\partial_t + i\Phi)^2\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$-\partial_t(\partial_t A + \nabla\Phi)A = \operatorname{curl}^2 A - \operatorname{Im}(\bar{\Psi}\nabla_A\Psi),$$

coupled (covariant) wave equations describing the $U(1)$-gauge Higgs model of elementary particle physics.

In what follows we use the *temporal gauge* $\Phi = 0$.

Stability: Definition

We say the n-vortex is *asymptotic stable*, if for any initial data sufficiently close to the n-vortex, $u^{(n)} := (\Psi^{(n)}, A^{(n)})$, the solution converges, in the sense of the H^1-distance, as $t \to \infty$, to an element of the the manifold,

$$\mathcal{M}^{(n)} = \{ T_h^{\mathrm{trans}} T_\gamma^{\mathrm{gauge}} u^{(n)} : h \in \mathbb{R}^2,\ \gamma \in H^2(\mathbb{R}^2, \mathbb{R})\}.$$

This means that $\exists \quad g(t) := (h(t), \gamma(t)) \in \mathbb{R}^2 \times H^2(\mathbb{R}^2, \mathbb{R})$, s.t. the solution $u(t)$ of the time-dependent equation satisfies

$$\|u(t) - T_{h(t)}^{\mathrm{trans}} T_{g(t)}^{\mathrm{gauge}} u^{(n)}\|_{H^1} \to 0, \quad \text{as} \quad t \to \infty.$$

The weaker notion: orbital (cf. Lyapunov) stability.

From now on, $d = 2$.

Theorem

1. *For Type I superconductors all vortices are stable.*

2. *For Type II superconductors, the* ± 1*-vortices are stable, while the n-vortices with* $|n| \geq 2$*, are not.*

The statement of Theorem I was conjectured by Jaffe and Taubes on the basis of numerical observations (Jacobs and Rebbi, ...).

Vortex stability $\Longleftrightarrow$ Change in the surface tension.

In the *self-dual* case $\kappa = 1/\sqrt{2}$ in GLE, the surface tension is zero.

Ginzburg-Landau Equations. II

Recall that we are interested in solutions of the Ginzburg-Landau equations:

$$-\Delta_A \Psi = \kappa^2(1 - |\Psi|^2)\Psi$$
$$\text{curl}^2 A = \text{Im}(\bar{\Psi}\nabla_A \Psi)$$

where $(\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d$, $d = 2, 3$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau material constant.

These equations describe equilibrium states of superconductors and of the $U(1)$ Higgs model of particle physics.

We are also interested in dynamical versions of these equations.

Time-Dependent Eqns. Superconductivity

In the leading approximation the evolution of a superconductor is described by the gradient-flow-type equations

$$\gamma(\partial_t + i\Phi)\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$\sigma(\partial_t A - \nabla\Phi) = -\operatorname{curl}^2 A + Im(\bar{\Psi}\nabla_A\Psi),$$

$Re\gamma \geq 0$, the *time-dependent Ginzburg-Landau equations* or the *Gorkov-Eliashberg-Schmidt equations*. (Earlier versions: Bardeen and Stephen and Anderson, Luttinger and Werthamer.)

The last equation comes from two Maxwell equations, with $-\partial_t E$ neglected, (Ampère's and Faraday's laws) and the relations $J = J_s + J_n$, where $J_s = \mathrm{Im}(\overline{\Psi}\nabla_A\Psi)$, and $J_n = \sigma E$.

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The time-dependent $U(1)$ Higgs model is described by $U(1)-$Higgs (or Maxwell-Higgs) equations ($\Phi = 0$)

$$(\partial_t + i\Phi)^2\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
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coupled (covariant) wave equations describing the $U(1)$-gauge Higgs model of elementary particle physics.

In what follows we use the *temporal gauge* $\Phi = 0$.

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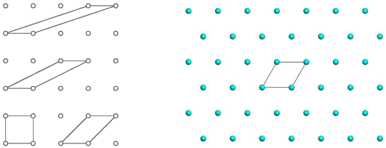
Both sets of time-dependent equations have the same static solution. We are interested in existence of these solutions and their stability w.r.to the both dynamics.

Abrikosov Vortex Lattice States

A pair (Ψ, A) for which all the physical characteristics

$$|\Psi|^2, \qquad B(x) := \operatorname{curl} A(x), \qquad J(x) := \operatorname{Im}(\bar{\Psi}\nabla_A \Psi)$$

are doubly periodic with respect to a lattice λ is called the *Abrikosov (vortex) lattice state*.



Quantization of magnetic flux: $\int_\omega \operatorname{curl} A = 2\pi \deg(\Psi) \in 2\pi\mathbb{Z}$,
where ω be an elementary cell of the lattice λ.

Vortices and vortex lattices are equivariant solutions for different subgroups of the group of rigid motions (subgroups of rotations and lattice translations, respectively).

Existence of Abrikosov Lattices

Let H_{c1} and $H_{c2} = \kappa^2$ be the 1st and 2nd critical magnetic fields and let ω be an elementary cell of the lattice λ.

Theorem (Existence for high magnetic fields)

For for every λ satisfying $\left|\frac{2\pi}{|\omega|} - \kappa^2\right| \ll 1$ and $(*)$, $\exists$ an Abrikosov lattice sol., with this λ and

$$b := \frac{1}{|\omega|}\int_\omega \operatorname{curl} A = \frac{2\pi}{|\omega|} \text{ (magnetic flux quantization).}$$

Theorem (Energy for high magnetic fields)

If $\kappa > 1/\sqrt{2}$ (Type II superconductors), then the minimum of the average energy per cell is achieved for the hexagonal lattice.

Theorem (Existence for low magnetic fields)

For every λ, with $|\omega|$ sufficiently large, $\exists$ an Abrikosov lattice solution, with this λ and $b := \frac{1}{|\omega|}\int_\omega \operatorname{curl} A = \frac{2\pi}{|\omega|}$.

References

- Aver. magn. field $\approx H_{c2} = \kappa^2$.

Existence for $(*)$ $b < \kappa^2$ and $\kappa > \frac{1}{\sqrt{2}}$: Odeh, Barany - Golubitsky - Tursky, Dutour, Tzaneteas - IMS

Existence for
$(*)$ $b < \kappa^2$ and $\kappa > \kappa_c(\lambda)$ or $b > \kappa^2$ and $\kappa < \kappa_c(\lambda)$,
where $\kappa_c(\lambda) := \sqrt{\frac{1}{2}\left(1 - \frac{1}{\beta(\lambda)}\right)}$ $(< \frac{1}{\sqrt{2}})$ (a new threshold in κ):
Tzaneteas - IMS

Energy minim. by triangular lattices: Dutour, Tzaneteas - IMS, using results of Aftalion - Blanc - Nier, Nonnenmacher - Voros.

- Aver. magn. field $\approx H_{c1}$ $(|\omega| \to \infty)$.
Existence: Aydi - Sandier and others $(\kappa \to \infty)$ and Tzaneteas - IMS (all κ's).

Stability of Abrikosov Lattices. I

Gauge -periodic perturbations: perturbations of the same periodicity as the Abrikosov lattice.

Recall $\kappa_c(\lambda) := \frac{1}{\sqrt{2}}\sqrt{1 - \frac{1}{\beta(\lambda)}}$. Note that $\kappa_c(\lambda) < \frac{1}{\sqrt{2}}$.

Theorem (Tzaneteas - IMS)

The Abrikosov vortex lattice solutions for high magnetic fields are

(i) *asymptotically stable for $\kappa > \kappa_c(\lambda)$;*

(ii) *unstable for $\kappa < \kappa_c(\lambda)$.*

Stability of Abrikosov Lattices. II

Let $(\Psi_\lambda, A_\lambda)$ = Abrikosov lattice solution specified by a lattice λ and $\mathcal{E}_\omega(\Psi, A)$ = *Ginzburg-Landau energy functional*

$$\mathcal{E}_\omega(\Psi, A) := \frac{1}{2}\int_\omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A)^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}.$$

Finite-energy perturbations: perturbations satisfying,

$$\lim_{Q \to \mathbb{R}^2} \left(\mathcal{E}_Q(\Psi, A) - \mathcal{E}_Q(\Psi_\lambda, A_\lambda)\right) < \infty, \text{ for some } \lambda.$$

Theorem (Tzaneteas - IMS)

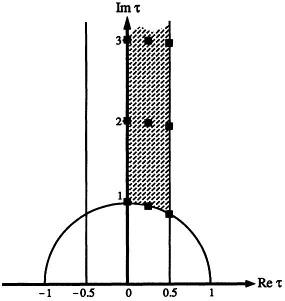
Let $b := \frac{1}{|\omega|}\int_\omega \operatorname{curl} A = \frac{2\pi}{|\omega|} \approx H_{c2}$ *(high magnetic fields).*
There is $\gamma(\lambda)$ *s.t. the Abrikosov vortex lattice solutions are*

(i) *asymptotically stable if* $\kappa > \frac{1}{\sqrt{2}}$ *and* $\gamma(\lambda) > 0$*;*

(ii) *unstable otherwise.*

Gamma Function

Let $\lambda = r(\mathbb{Z} + \tau\mathbb{Z})$, $r > 0$, $\tau \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$, and $\gamma(\tau) = \gamma(\lambda)$.
Then the function $\gamma(\tau)$ is invariant under modular group $SL(2, \mathbb{Z})$ and therefore can be reduced to the Poincaré strip, $\Pi^+/SL(2, \mathbb{Z})$,



Symmetries: $\gamma(-\bar{\tau}) = \gamma(\tau)$ and $\gamma(1 - \bar{\tau}) = \gamma(\tau)$
$\Rightarrow$ critical points at $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$

Work in progress: Estimating $\gamma(\tau)$ and checking the critical points. So far we have $\gamma(e^{i\pi/3}) > 0$ (numerics).

Stability Definition

The stability is defined w.r.to distance to the infinite-dimensional manifold of $\lambda-$lattice solutions

$$\mathcal{M} = \{T_g^{sym} u_\lambda : g \in G\},$$

where $T_g^{sym} = T_\gamma^{gauge} T_\rho^{rot}, \; g = (\gamma, \rho)$, is the action of the symmetry group

$$G = H^2(\mathbb{R}^2; \mathbb{R}) \times SO(2)$$

(semi-direct product) on Abrikosov vortex lattices $u_\lambda = (\Psi_\lambda, A_\lambda)$. Here T_γ^{gauge} and T_ρ^{rot} are the *gauge transformations and rotations*, i.e.

$$T_\gamma^{\text{gauge}}: \; (\Psi(x), \; A(x)) \mapsto (e^{i\gamma(x)}\Psi(x), \; A(x) + \nabla\gamma(x)).$$

Signature of Stability

Consider the hessian, $\mathcal{E}''(u_\lambda)$, of Ginzburg-Landau energy functional $\mathcal{E}(\Psi, A)$ at a Abrikosov lattice solution $u_\lambda = (\Psi_\lambda, A_\lambda)$.

(Recall that the Ginzburg-Landau equations are the Euler-Lagrange equations for $\mathcal{E}$.)

$\mathcal{E}''(u_\omega)$ has zero eigenvalues along the tangent space to $\mathcal{M} := G^{\mathrm{sym}} u_\lambda$ and u_λ

Signature of stability/instability is the sign of the lowest eigenvalue of $\mathcal{E}''(u_\lambda)$ in transversal direction to $\mathcal{M}$

$\Longrightarrow$ estimate the lowest eigenvalue of $[\mathcal{E}''(u_\lambda)]^{\mathrm{transv}}$

Abrikosov Lattices and Equivariance

Recall: the *Abrikosov (vortex) lattice* is a pair (Ψ, A) for which all the physical characteristics

$$|\Psi|^2, \quad B(x) := \operatorname{curl} A(x), \quad J(x) := \operatorname{Im}(\bar{\Psi} \nabla_A \Psi)$$

are doubly periodic with respect to a lattice λ.

Theorem. (Ψ, A) is an Abrikosov lattice state if and only if it is an equivariant pair for the group of lattice translations for a lattice λ:

$$T_s^{\mathrm{transl}}(\Psi, A) = T_{\gamma_s}^{\mathrm{gauge}}(\Psi, A), \ \forall s \in \lambda, \tag{1}$$

where T_h^{transl} and $T_\gamma^{\mathrm{gauge}}$ are the translations and *gauge transformations*,

$$T_\gamma^{\mathrm{gauge}} : \ (\Psi(x), \ A(x)) \mapsto (e^{i\gamma(x)}\Psi(x), \ A(x) + \nabla\gamma(x)).$$

$$(1) \Rightarrow \gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}.$$

Magnetic Translations

The key point: $u_\lambda = (\Psi_\lambda, A_\lambda)$ is equivariant $\Longrightarrow$ the Hessian $\mathcal{E}''(u_\lambda)$ commutes with magnetic translations,

$$T_s = T^{\rm gauge}_{\gamma_s} T^{\rm transl}_s,$$

where, recall, $T^{\rm transl}_s f(x) = f(x+s)$, and

$$T^{\rm gauge}_\gamma : \ (\psi(x), \ a(x)) \mapsto (e^{i\gamma(x)}\psi(x), \ a(x) + \nabla\gamma(x));$$

and $\gamma_s : \mathbb{R}^2 \to \mathbb{R}$ is a multi-valued differentiable function, satisfying

$$\gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}. \qquad (2)$$

$$(2) \quad \Rightarrow \quad T_{s+t} = T_s T_t.$$

($s \to T_s$ is a unitary repres. of $\mathcal{L}$ on $L^2(\mathbb{R}^2; \mathbb{C}) \times L^2(\mathbb{R}^2; \mathbb{R}^2)$.)

Direct Fibre Integral (Bloch Decomposition)

Since the Hessian operator $\mathcal{E}''(u_\lambda)$ commutes with T_s, it can be decomposed into the fiber direct integral

$$U\mathcal{E}''(u_\lambda)U^{-1} = \int_{\omega^*}^{\oplus} L_k d\mu_k$$

where ω^* is the fundamental cell of the reciprocal (dual) lattice, $U : L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \to \mathscr{H} = \int_{\omega^*}^{\oplus} \mathscr{H}_k d\mu_k$ is a unitary operator,

$$(Uv)_k(x) = \sum_{s \in \lambda} e^{-ik\cdot s} T_s v(x)$$

(decomposition into the Bloch waves, $v_k(x) = e^{ik\cdot x}\phi_k(x)$),
$\mathscr{H}_k := \{v \in L^2(\omega, \mathbb{C} \times \mathbb{R}^2) : T_s v(x) = e^{ik\cdot s} v(x),\ s \in \text{basis}\}$,
L_k is the restriction of the operator $\mathcal{E}''(u_\lambda)$ to $\mathscr{H}_k$.

ϑ-function

In the leading order in $\epsilon := \sqrt{\kappa^2 - b}$, the two lowest eigenvalues of the fiber operators, L_k are given by

$$\mu_{k0} = (1 + O(\epsilon^2))|k|^2, \qquad \mu_{k1} = \gamma_k(\tau)\epsilon^2 + O(\epsilon^3),$$

where

$$\gamma_k(\tau) := 2\frac{\langle|\vartheta_k(\tau)|^2|\vartheta_0(\tau)|^2\rangle}{\langle|\vartheta_k(\tau)|^2\rangle\langle|\vartheta_0(\tau)|^2\rangle} + \dots - \frac{\langle|\vartheta_0(\tau)|^4\rangle}{\langle|\vartheta_0(\tau)|^2\rangle^2}.$$

Here $\vartheta_k(z,\tau)$, $k \in \Omega^*$, are the *modified theta functions*, i.e. entire functions satisfying ($\sqrt{\frac{2\pi}{\operatorname{Im}\tau}}i(a\tau + b) = k_1 + ik_2$)

$$\begin{cases} \vartheta_k(z+1,\tau) = e^{2\pi i a}\vartheta_k(z,\tau), \\ \vartheta_k(z+\tau,\tau) = e^{-2\pi i b}e^{-\pi i \tau z - 2\pi i z}\vartheta_k(z,\tau). \end{cases}$$

Conclusion of Sketch

The relations

- $U\mathcal{E}''(u_\lambda)U^{-1} = \int_{\omega^*}^{\oplus} L_k d\mu_k$ and
- $\inf L_k = \min(\gamma_k(\tau)\epsilon^2 + O(\epsilon^3), (1 + O(\epsilon^2))|k|^2)$, where $\gamma_k(\tau) := 2\frac{\langle|\vartheta_k(\tau)|^2|\vartheta_0(\tau)|^2\rangle}{\langle|\vartheta_k(\tau)|^2\rangle\langle|\vartheta_0(\tau)|^2\rangle} + \dots,$

imply

$$\inf \mathcal{E}''(u_\lambda) = \min(\underbrace{\inf_{k\in\omega^*} \gamma_k(\tau)}_{\gamma(\tau)}\epsilon^2 + O(\epsilon^3), 0).$$

Hence the Abrikosov lattice is

- linearly stable if $\gamma(\tau) > 0$
- linearly unstable if $\gamma(\tau) < 0$.

Conclusions

In the context of superconductivity and particle physics, we described

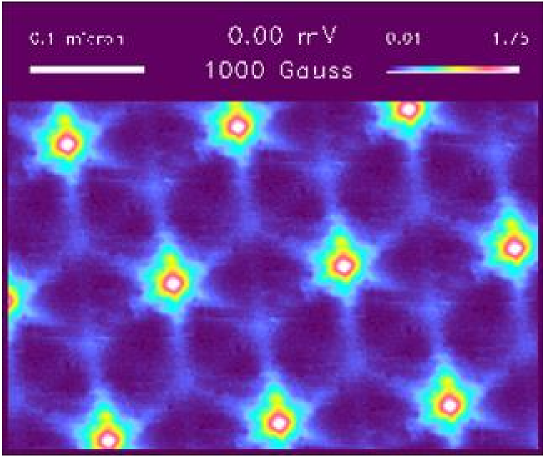
- existence and stability of magnetic vortices and vortex lattices
- a new threshold $\kappa_c(\tau)$ in the Ginzburg-Landau parameter appears in the problem of existence of vortex lattices
- while Abrikosov lattice energetics is governed by Abrikosov function $\beta(\tau)$, a new automorphic function $\gamma(\tau)$ emerges controlling stability of Abrikosov lattices.

We gave some indications how to prove the latter results. While the proof of existence leads to standard theta functions, the proof of stability leads to theta functions with characteristics.

Interesting extensions:

- unconventional/high T_c supercond.,
- Weinberg - Salam model of electro-weak interactions,
- microscopic/quantum theory.

Abrikosov Lattice. Experiment



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