Elements of Partial Differential Equations

Hamilton - Jacobi equations

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Chapter 1

Equazioni del prim'ordine e metodo delle caratteristiche

1.1 Trasporto lungo un campo vettoriale.

Sia $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Siamo interessati per ora alla seguente tipologia molto particolare di E.D.P. del primo ordine:

$$\sum_{j=1}^{n} a_j(x)\,\partial_{x_j} u(x) = 0 \tag{1.1.1}$$

detta anche E.D.P. del trasporto lungo il campo vettoriale $(a_1(x), \ldots, a_n(x))$.

Osserviamo che se l'operatore

$$u(x) \to L(u)(x) = \sum_{j=1}^{n} a_j(x)\,\partial_{x_j} u(x)$$

é lineare rispetto u. Cio" implica che (1.1.1) é lineare come E.D.P., ossia

$$\mathscr{L}(u_1 + u_2)(x) = \mathscr{L}(u_1)(x) + \mathscr{L}(u_2)(x)\mathscr{L}(cu)(x) = c\mathscr{L}(u)(x)$$

dove u, u_1, u_2 sono funzioni della variabile indipendente $x \in \mathbb{R}^n$, $c \in \mathbb{R}\mathscr{L}(u)(x) = \sum_{j=1}^{n} a_j(x)\partial_{x_j} u(x)$.

Possiamo effetuare cambiamento di variabili

$$x \to y = F(x)$$

con matrice Jacobiana

$$J(f) = \left(\partial_{x_j} F_k(x)\right)_{j,k=1}^{n}$$

invertibile e quindi localmente possiamo definire la funzione

$$y \to x = G(y) = F^{-1}(y).$$

Usando le relazioni

$$\partial_{x_j} = \sum_{k=1}^{n} \partial_{x_j} F_k \partial_{y_k},$$

si puovedere che se $u(x)$ e C^1 soluzione del problema (1.1.1),

$$v(y) = u(G(y))$$

e soluzione di

$$\sum_{j=1}^{n} b_k\left(y\right) \partial_{y_k} v(y) = 0, \tag{1.1.2}$$

dove

$$b_k(y) = \sum_{j=1}^{n} a_j(G(y)) \left(\partial_{x_j} F_k\right)(G(y)).$$

Definition 1.1.1. *Curva caratteristica e ogni soluzione* $y(s) = \big(y_1(s), \ldots, y_n(s)\big)$ *del sistema E.D.O.*

$$\begin{cases} \frac{d}{ds} y_1(s) = a_1\left(y_1(s), \ldots, y_n(s)\right) \\ \cdots \\ \frac{d}{ds} y_n(s) = a_n\left(y_1(s), \ldots, y_n(s)\right) \\ y_1(0) = \alpha_1, \\ \cdots \\ y_n(0) = \alpha_n \end{cases} \tag{1.1.3}$$

Ponendo $a(x) = (a_1(x), \cdots, a_n(x))$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ possiamo rescrivere (1.1.5) come

$$\begin{cases} \frac{d}{ds} y(s) = a(y(s)) \\ y(0) = \alpha, \end{cases} \tag{1.1.4}$$

Il problema e nonlineare e quindi in generale abbiamo solo soluzioni locali in s. Useremo la notazione $y(s; \alpha)$ per la soluzione (locale) del problema (1.1.4).

Abbiamo la seguente proprietá

Lemma 1.1.1. *Sia* $y(s) = \big(y_1(s),\dots,y_n(s)\big) \in \mathbb{R}^n$ *una soluzione del seguente sistema di E.D.O.*

$$\frac{d}{ds}y(s) = a(y(s)) \tag{1.1.5}$$

ed $u\,(x) \in \mathbb{R}$ *soluzione di* (1.1.1). *Allora*

$$\frac{d}{ds}u(y(s)) = 0$$

Quindi la soluzione u e'costante lungo la traiettoria descritta dall'immagine di $y(s)$ (le curve caratteristiche). In generale si e' interessati non solo a cercare soluzioni (osserviamo che ad esempio nel caso dell" E.D.P. (1.1.1) le funzioni costanti sono certamente soluzioni), ma si studia il seguente problema di Cauchy con dati su una superficie S in $\mathscr{R}^n$

$$\begin{cases} \sum_{j=1}^n a_j\,(x)\,\partial_{x_j}u(x) = 0, \\ u(x) = \varphi(x), x \in S. \end{cases} \tag{1.1.6}$$

Possiamo rescrivere il problema come segue.

$$\begin{cases} \langle a\,(x)\,, \nabla u(x)\rangle = 0, \\ u|_S = \varphi \end{cases} \tag{1.1.7}$$

Per avere unica soluzione del problema (1.1.7) si puo prendere $\alpha_0 \in S$ e un piccolo intorno $U \subset \mathbb{R}^n$ di α_0. Per trovare una soluzione $u \in C^1(U)$ del

$$\begin{cases} \langle a\,(x)\,, \nabla u(x)\rangle = 0, x \in U \\ u|_{S\cap U} = \varphi \end{cases} \tag{1.1.8}$$

possiamo usare le curve caratteristiche ponendo ipotesi

(H1) per $\forall\ \alpha \in S \cap U$ la superficie S e trasversale alla curva $y(s;\alpha)$

L'ipotesi (H1) é cruciale e significa che il vettore $a(\alpha)$ non é tangente alla S.

Visto che cerchiamo soluzioni locali (solo in U) possiamo supporre che S e definita in U con l'equazione

$$f(x) = 0.$$

Lemma 1.1.1 suggerisce a cercare la soluzione di (1.1.8) nella forma

$$u(y(s,\alpha)) = \varphi(\alpha), \alpha \in S.$$

Usando il teorema della funzione inversa si puo vedere che

Lemma 1.1.2. *L'ipotesi* (H1) *implica che la mappa*

$$s \in (-\varepsilon, \varepsilon), \alpha \in S \cap U \;\to\; x = y(s, \alpha)$$

sia localmente invertibile.

Proof. Il fatto che l'equazione di trasporto rimane equazione di trasporto dopo cambiamento di variabili (1.1.2) ci premette a supporre S definita localmente con $x_n = 0$. L'ipotesi di trasversalita implica $a_n(\alpha) \neq 0$ e

$$\alpha \in S \iff \alpha_n = 0.$$

Quindi dobbiamo vedere che la mappa

$$s \in (-\varepsilon, \varepsilon), \alpha' = (\alpha_1, \cdots, \alpha_{n-1}) \in \mathbb{R}^{n-1} \to x = y(s; \alpha', 0)$$

e invertibile. La sua matrice Jacobiana e

$$\begin{pmatrix} I_{n-1} & 0 \\ a'(\alpha) & a_n(\alpha) \end{pmatrix}$$

dove

$$a' = (a_1, \cdots, a_{n-1}) \in \mathbb{R}^{n-1}$$

Ovviamente $a_n(\alpha) \neq 0$ implica invertibilita. □

1.2 Hamilton - Jacobi equation

L'equazione di Hamilton – Jacobi é un'equazione differenziale alle derivate parziali non lineare del primo ordine che ha la forma

$$H(x, \nabla_x S) = 0, x \in \mathbb{R}^n. \tag{1.2.9}$$

We are trying to solve this equation imposing

$$S(y) = \alpha(y), \quad y \in \Sigma, \tag{1.2.10}$$

where Σ is a surface in$\mathbb{R}^n$.

The Hamiltonian $H : \mathbb{R}^n_{\mathrm{p}} \times \mathbb{R}^n_{\mathrm{x}} \to \mathbb{R}$, enables us to define the bicharacteristic curves as solutions to the Hamiltonian equations

$$\begin{cases} \dot{\xi}(s) = -\nabla_{\mathbf{x}} H(\xi(s), \mathbf{x}(s)) \\ \dot{\mathbf{x}}(s) = \nabla_{\xi} H(\xi(s), \mathbf{x}(s)) \end{cases} \tag{1.2.11}$$

since H is given, this is an system of $2n$ ordinary differential equations, with $\xi(s)$ and $\mathbf{x}(s)$ as the unknowns. If initial data is given (for example, if $\xi(0) = \xi_0$ and $\mathbf{x}(0) = \mathbf{x}_0$), then the ODE can be solved and we can describe the motion of the system. We shall denote by $\rho = (x,\xi)$ the points in $T^*(\mathrm{R}^n) = R^{2n}$. Then

$$\rho(t) = \rho(s,\rho_0),\ \rho_0 = (x_0,\xi_0)$$

is the solution to the system (1.2.11) of the bicharacteristics.

Example 1.2.1. *If*

$$H(x,\xi) = \frac{1}{2}|\xi|^2,\ \xi = (\xi_1,\cdots,\xi_n) \in \mathrm{R}^n, n \geq 2,$$

then

$$\xi(s) = \xi_0, x(s) = x_0 + \xi_0 s.$$

Example 1.2.2. *If*

$$H(t,x,\tau,\xi) = \tau - |\xi|^2,\ \xi = (\xi_1,\cdots,\xi_n) \in \mathrm{R}^n, n \geq 2,$$

then

$$\tau(s) = \tau_0, \xi(s) = \xi_0, t(s) = t_0 + s,\ x(s) = x_0 - 2\xi_0 s,\ \ \tau_0 = |\xi_0|^2.$$

These bi characteristic shall be used to solve the propblem

$$\begin{aligned} &\partial_t S - (\partial_x S)^2 = 0 \\ &S(0,y) = \alpha(y). \end{aligned} \tag{1.2.12}$$

Example 1.2.3. *If we use rectangular coordinates, the Hamiltonian for a particle of mass m in a force field is*

$$H(\xi,\mathbf{x}) = \frac{1}{2m}|\xi|^2 + V(\mathbf{x})$$

where $V : \mathbb{R}^n_x \to \mathbb{R}$ *is the potential energy. Then equations* (1.2.11) *reduce to* $\dot{\xi} = -\nabla_x V(\mathbf{x})$ *and* $\dot{\mathrm{x}} = \frac{1}{m}\xi$. *The first equation is a statement of Newton's second law* $\mathbf{F} = ma$. *The second equation relates the classical position and momentum vectors.*

Remark 1.2.1. *Observe that in Example 1.2.3, the Hamiltonian is equal to the total energy of the system. This is no coincidence. To see why, fix a Hamiltonian* $H : \mathbb{R}^n_{\mathrm{p}} \times \mathbb{R}^n_x \to \mathbb{R}$, *and let* $(\mathrm{p}(t),\mathrm{x}(t))$ *be a solution to Hamilton's equations. Then set* $E(t) = H(\xi(t),\mathrm{x}(t))$. *We have*

$$\dot{E}(t) = \nabla_\xi H(\xi(t),\mathbf{x}(t))\cdot\dot{\xi}(t) + \nabla_{\mathbf{x}} H(\xi(t),\mathbf{x}(t))\cdot\dot{\mathbf{x}}(t) = 0 \tag{1.2.13}$$

where the equality on the right is due to (1.2.11). *Thus, the quantity* $E(t)$ *is indeed conserved, and the Hamiltonian admits an interpretation as energy.*

To be specific, if S is a solution of (1.2.9) then the graph of the gradient (or differential) of the function S

$$\Gamma = \{(x, S_x(x))\}$$

(where $S_x = \partial S/\partial x$) is invariant with respect to the flow defined by the system (1.2.11), since if $(x(t), \xi(t))$ is a solution of (1.2.11) and $\xi(0) = S_x(x(0))$ then

$$\begin{aligned}\frac{d}{dt}(\xi(t) - S_x(x(t))) &= \dot{\xi}(t) - S_{xx}(x(t))\dot{x}(t)\\ &= -H_x(x(t)) - S_{xx}(x(t))H_\xi(x(t), S_x(x(t)))\\ &= -\frac{d}{dx}H(x, S_x(x))\Big|_{x=x(t)} = 0.\end{aligned}$$

Lemma 1.2.1. *If $S(x)$ is a solution to the Hamilton - Jacobi equation* (1.2.9) *and $\rho(t) = (x(t), \xi(t))$, $t \in I$ is a bi characteristic in the interval I, then*

$$\xi(t) - S_x(x(t)),\ \forall t \in I.$$

We note that in the case of most importance for us, when the function $H = H(x, \xi)$ is positive-homogeneous in ξ (of any degree m), the solution $S = S(x)$ of the Hamilton-Jacobi equation (3.33) is constant along the projections $x(t)$ of the trajectories $(x(t), \xi(t))$ of the Hamiltonian system (1.2.11) lying on the graph of the gradient of the function S, since by Euler's identity

$$\frac{d}{dt}S(x(t)) = S_x(x(t)) \cdot \dot{x}(t) = \xi(t) \cdot H_\xi(x(t)) = mH(x(t), \xi(t)) = 0$$

Lemma 1.2.2. *If $S(x)$ is a solution to the Hamilton - Jacobi equation* (1.2.9), *$H(x, \xi)$ is homogeneous of order m in ξ and $\rho(t) = (x(t), \xi(t))$, $t \in I$ is a bi characteristic in the interval I, then*

$$\frac{d}{dt}S(x(t)) = 0,\ \forall t \in I.$$

Now we turn to the general solution of the Hamilton Jacobi equation, knowing the Hamltonian flow $\rho = \rho(t, \rho_0)$.

First we consider the surface $\Sigma \subset \mathbb{R}^n$ that is transversal to $H(x, \xi)$ at point $(x, \xi) \in \Sigma \times \mathbb{R}^n$, such that

$$H(x, \xi) = 0. \tag{1.2.14}$$

Definition 1.2.1. *We shall say that $\Sigma \subset \mathbb{R}^n$ that is transversal to H at the point $(x, \xi) \in \Sigma \times \mathbb{R}^n$ satisfying* (1.2.16) *if*

$$N(x).\nabla_\xi H(x, \xi) \neq 0 \tag{1.2.15}$$

with $N(x)$ being the unit normal vector at $x \in \Sigma$.

One can easily see that if $\Sigma \subset \mathbb{R}^n$ that is transversal to H at the point $(\widetilde{x}, \widetilde{\xi}) \in \Sigma \times \mathbb{R}^n$ satisfying

$$H(\widetilde{x}, \widetilde{\xi}) = 0, \tag{1.2.16}$$

then the map defined through Hamiltonian flow

$$t \in (-\varepsilon, \varepsilon) \times y \in \Sigma \cap \{|y - \widetilde{x}| \leq \varepsilon\} \to x = x(t, y, \eta) \in \mathbb{R}^n \tag{1.2.17}$$

is invertible for $\eta \in R^n$ close to $\widetilde{\xi}$.

Indeed, the proof is similar to the proof of Lemma 1.1.2 and we omit it.

Our next step is to choose appropriate initial data

$$x(0) = y, \ \ \xi(0) = \eta$$

for the Hamiltonian flow (1.2.11) that is compatible with the data (1.2.10) on the surface Σ.

First we choose

$$x(0) = y \in \Sigma. \tag{1.2.18}$$

Further, we note that for any $y \in \Sigma$ and for any vector $\eta \in R^n$ we may use decomposition of η into tangential and normal parts respectively

$$\eta = \eta^{(\tau)} + \eta^{(\nu)} N(y), \ \ \eta^{(\nu)} = \eta . N(y), \tag{1.2.19}$$

where $N(y)$ being the unit normal vector at $y \in \Sigma$. From the data (1.2.10) on the surface we have

$$S(y) = \alpha(y), \ \ y \in \Sigma, \tag{1.2.20}$$

so taking into account Lemma 1.2.1 we can make the tangential projections of

$$\xi(t) - \nabla_x S(x(t))$$

at the point $x(0) = y \in \Sigma$ and find

$$\eta^{(\tau)} - \nabla_y \alpha(y) = 0. \tag{1.2.21}$$

To define the normal part $\eta^{(\nu)}$ of η we impose the condition

$$H(y, \eta) = 0. \tag{1.2.22}$$

One can solve this equation locally near a point $y_0 \in \Sigma$, $\eta_0 \in \mathbb{R}^n$, such that

$$\eta_0^{(\tau)} - \nabla_y \alpha(y_0) = 0 \tag{1.2.23}$$

and

$$H(y_0,\eta_0)=0. \tag{1.2.24}$$

imposing the transversality of Σ at (y_0,η_0) by using the implicit function theorem.

Indeed, if Σ is defined locally by $x_n=0$, then we use $y=(y_1,\cdots,y_{n-1})$ as parametrization of Σ. For any $\eta\in\mathbb{R}^n$ we see that its tangential part is

$$\eta^{(\tau)}=(\eta_1,\cdots,\eta_{n-1},0)$$

while the normal part is η_n. Let

$$(y_0,\eta_0),\, y_0=0,\;\; \eta_0=(\eta_1^{(0)},\cdots,\eta_{n-1}^{(0)},\eta_n^{(0)})$$

are chosen so that (1.2.23) and (1.2.22) hold, i.e.

$$\eta_j^{(0)}-\partial_{y_j}\alpha(0)=0,\;\; j=1,\cdots,n-1 \tag{1.2.25}$$

and

$$H(0,\eta_0)=0. \tag{1.2.26}$$

Then to solve (1.2.23) and (1.2.22) with $(y,\eta)\in\Sigma\times\mathbb{R}^n$ close to $(0,\eta_0)$ we have to find only η_n that solves

$$H(y,\nabla_y\alpha,\eta_n)=0.$$

The transversality condition at $(0,\eta_0)$ reads as

$$\partial_{\xi_n}H(0,\eta_0)\neq 0,$$

so implicit function theorem guarantees local solvability of

$$H(y,\nabla_y\alpha,\eta_n)=0$$

with respect to η_n.

Next, we define $z(t,y,\eta)$ to be the solution to

$$\begin{cases} z'=\xi(t).H_\xi(x(t),\xi(t)) \\ z(0)=\alpha(y). \end{cases} \tag{1.2.27}$$

Lemma 1.2.3. *If*

$$x=x(t,y,\eta),\xi=\xi(t,y,\eta)$$

is the Hamiltonian flow determined by (1.2.11) *with initial data* $(x(0),\xi(0))=(y,\eta)$, *satisfying* (1.2.21) *and* (1.2.22), *then if* $S(x)=z(t(x),y(x))$, *where* $(y(x,\eta),t(x,\eta))$ *is the inverse of* (1.2.17) *and* z *is a solution of* (1.2.27), *then we have the properties*

a)

$$\xi(t,y,\eta) = \nabla_x S(x(t,y,\eta)). \tag{1.2.28}$$

b) *the function $S(x)$ is a solution to the Hamilton - Jacobi equation*

$$H(x,\nabla_x S(x)) = 0.$$

Proof. We have the relations

$$\frac{d}{dt}S(x(t,y,\eta)) = \frac{d}{dt}z(t,y) = \nabla_\xi H(x(t,y,\eta),\xi(t,y,\eta)).\xi(t,y,\eta) = x'(t,y,\eta)).\xi(t,y,\eta)$$

and since

$$\frac{d}{dt}S(x(t,y,\eta)) = \nabla_x S(x(t,y,\eta)).x'(t,y,\eta)),$$

we conclude that

$$\left(\nabla_x S(x(t,y,\eta)) - \xi(t,y,\eta)\right).x'(t,y,\eta)) = 0. \tag{1.2.29}$$

Without loss of generality we can assume that $\Sigma : x_n = 0$ and then we shall show

$$\left(\nabla_x S(x(t,y,\eta)) - \xi(t,y,\eta)\right).\partial_{y_j} x(t,y,\eta)) = 0, \; j1,\cdots,(n-1)). \tag{1.2.30}$$

Indeed, we have the relations

$$\partial_{y_j} S(x(t,y,\eta)) = \partial_{y_j} z(t,y)$$

and

$$\partial_{y_j} S(x(t,y,\eta)) = \nabla_x S(x(t,y,\eta)).\partial_{y_j} x(t,y,\eta)),$$

therefore we need to show that

$$\partial_{y_j} z(t,y) = \xi(t,y,\eta).\partial_{y_j} x(t,y,\eta). \tag{1.2.31}$$

For the purpose we use the relations

$$\partial_{y_j} z'(t,y) = \nabla^2_{\xi,x} H(x(t,y,\eta),\xi(t,y,\eta))(\partial_{y_j} x(t,y,\eta)).\xi(t,y,\eta) +$$

$$+\nabla^2_{\xi,\xi} H(x(t,y,\eta),\xi(t,y,\eta))(\partial_{y_j}\xi(t,y,\eta)).\xi(t,y,\eta) +$$

$$+\nabla_\xi H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}\xi'(t,y,\eta)).$$

Moreover, we have

$$\frac{d}{dt}\left(\xi(t,y,\eta).\partial_{y_j} x(t,y,\eta)\right) = -\nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).\partial_{y_j} x(t,y,\eta) +$$

$$+\xi(t,y,\eta).\partial_{y_j}x'(t,y,\eta) = -\nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).\partial_{y_j}x(t,y,\eta)+$$
$$+\nabla^2_{\xi,x}H(x(t,y,\eta),\xi(t,y,\eta))(\partial_{y_j}x(t,y,\eta)).\xi(t,y,\eta)+$$
$$+\nabla^2_{\xi,\xi}H(x(t,y,\eta),\xi(t,y,\eta))(\partial_{y_j}\xi(t,y,\eta)).\xi(t,y,\eta)$$

and we arrive at

$$\frac{d}{dt}\Big(\partial_{y_j}z(t,y) - \Big(\xi(t,y,\eta).\partial_{y_j}x(t,y,\eta)\Big)\Big) =$$

$$= \nabla_\xi H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}\xi(t,y,\eta)) + \nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}x(t,y,\eta)).$$

Since we are working on null bi characteristics $H(x(t,y,\eta),\xi(t,y,\eta)) = 0$ so we have also

$$\nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}x(t,y,\eta)) + \nabla_\xi H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}\xi(t,y,\eta)) = 0$$

so we arrive at

$$\frac{d}{dt}\Big(\partial_{y_j}z(t,y) - \Big(\xi(t,y,\eta).\partial_{y_j}x(t,y,\eta)\Big)\Big) = 0.$$

For $t = 0$ we have

$$\partial_{y_j}z(0,y) - \xi(0,y,\eta).\partial_{y_j}x(0,y,\eta) = 0$$

so we arrive at (1.2.31). This relation implies (1.2.30) and combining with (1.2.29), we find (1.2.28). Therefore we have the property a). The property b) follows from a) and (1.2.13). □

Example 1.2.4. *We continue to study the Example 1.2.2. Recall that the null bi -characteristics are defined by*

$$\begin{aligned} &\tau(s) = \tau_0, \xi(s) = \xi_0, \ \ \tau_0 = |\xi_0|^2, \\ &t(s) = t_0 + s, \ \ x(s) = x_0 - 2\xi_0 s, \end{aligned} \tag{1.2.32}$$

We want to solve the problem (1.2.12) *with data on* $t = 0$

$$S(0,x) = \alpha(x)$$

we take $x_0 = y$, $t_0 = 0$ *with* $y \in \mathbb{R}^n$.

Note that the relation (1.2.28) *with* $t = 0$ *implies*

$$\xi_0 = \nabla_y \alpha(y). \tag{1.2.33}$$

As in (1.2.27) *we define the function*

(1.2.34)
$$\begin{cases} z'(s) = \xi(s,y,\xi_0).H_\xi(t(s),x(s),\tau(s),\xi(s)) + \tau(s)H_\tau(t(s),x(s),\tau(s),\xi(s)) \\ z(0) = \alpha(y) \end{cases}$$

From (1.2.32) *we find*

$$\xi(s,y,\xi_0).H_\xi(t(s),x(s),\tau(s),\xi(s))+\tau(s)H_\tau(t(s),x(s),\tau(s),\xi(s)) = -2|\xi_0|^2+\tau_0 = -|\xi_0|^2.$$

Hence

$$z(s,y,\xi_0) = \alpha(y) - s|\xi_0|^2.$$

Then

$$S(t(s,y,\xi_0),x(s,y,\xi_0)) = z(s,y,\xi_0)$$

shall be the solution to our problem so

$$S(s,y-2\xi_0 s) = \alpha(y) - s|\xi_0|^2.$$

We have to make change of variables

$$t = s,\ x = y - 2\nabla_y\alpha(y)s$$

and we get

$$S(t,x) = \alpha(y(t,x)) - t|\nabla_y\alpha(y(t,x))|^2.$$

Example 1.2.5. *For given initial data $g(x)$, we consider a classical solution to the* PDE

(1.2.35)
$$\partial_t u + \frac{1}{2}(\partial_x u)^2 = 0 \quad on\ \mathbb{R}_x \times (0,\infty)_t$$
$$u = g \quad on\ \mathbb{R}_x \times \{0\}_t$$

(By classical, we simply mean that u is differentiable, so that the PDE can be satisfied everywhere on $\mathbb{R}_x \times (0,\infty)_t$ (1.2.35)*. Characteristic lines. Suppose u is a classical solution to* (1.2.35)*. Furthermore, suppose u is twice differentiable in the variable x. since u is differentiable, we can let $v(x,t) = \partial_x u(x,t)$. Then v satisfies the equation*

(1.2.36)
$$\partial_t v + v\cdot\partial_x v = 0 \quad on\ \mathbb{R}_x \times (0,\infty)_t$$

with initial data

$$v = h \quad on\ \mathbb{R}_x \times \{0\}_t$$

where $h(x) := g'(x)$ The PDE (1.2.36) *is known as the inviscid Burgers ' equation and plays an important role in fluid mechanics. We have the following property for solutions of Burgers' equation.*

Lemma 1.2.4. *If v is solution to* (1.2.36)*, then v is a constant along the lines $x = x_0 + th(x_0)$ for each $x_0 \in \mathbb{R}$.*

Example 1.2.6. Non-existence of a classical solution.

Suppose we have initial data $g(x) = -|x|$ to the Hamilton-Jacobi equation (1.2.35)*. Then v has initial data*

$$h(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0 \end{cases}$$

However, we immediately see a problem with our given initial data. Take, for example, the characteristic lines $x = 1 - t$ and $x = -1 + t$ must be constant along these two lines. This is fine for all $t < 1$, but the two characteristics intersect at $(x, t) = (0, 1)$. (This intersection is called a shock. since the values of v along the two lines are different, there cannot be a differentiable solution).

1.3 Symplectic manifolds

Symplectic manifold is a smooth manifold, M , equipped with a closed nondegenerate differential 2-form ω called the symplectic form.

Example 1.3.1. *Let $\{v_1, \dots, v_{2n}\}$ be a basis for R^{2n} We define our symplectic form ω on this basis as follows:*

$$\omega(v_i, v_j) = \begin{cases} 1 & j - i = n \text{ with } 1 \leqslant i \leqslant n \\ -1 & i - j = n \text{ with } 1 \leqslant j \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

In this case the symplectic form reduces to a simple quadratic form. If I_n denotes the $n \times n$ identity matrix then the matrix, Ω of this quadratic form is given by the $2n \times 2n$ block matrix:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Example 1.3.2. *If (x, ξ) are local coordinates, then one such form is*

$$\omega = dx \wedge d\xi.$$

So if we have vector field

$$v_1 = \sum_{j=1}^{n} a_j^{(1)} \partial_{x_j} + b_j^{(1)} \partial_{\xi_j}, \; v_2 = \sum_{j=1}^{n} a_j^{(2)} \partial_{x_j} + b^{(2)})_j \partial_{\xi_j},$$

then

$$\omega(v_1, v_2) = a^{(1)}.b^{(2)} - a^{(2)}.b^{(1)},$$

where

$$a^{(1)} = (a_1^{(1)}, a_2^{(1)}, \cdots, a_n^{(1)}), a^{(2)} = (a_1^{(2)}, a_2^{(2)}, \cdots, a_n^{(2)}),$$
$$b^{(1)} = (b_1^{(1)}, b_2^{(1)}, \cdots, b_n^{(1)}), b^{(2)} = (b_1^{(2)}, b_2^{(2)}, \cdots, b_n^{(2)}).$$

and $a.b$ *denotes the scalar product of the vectors* a, b.

The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds. For example, in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field, the set of all possible configurations of a system is modeled as a manifold, and this manifold's cotangent bundle describes the phase space of the system.

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