The Local Limit Theorem and the Almost Sure Local Limit Theorem

Rita Giuliano

Department of Mathematics "L. Tonelli"
University of Pisa
ITALY

Outline

Local Limit Theorems (LLT)

- The DeMoivre–Laplace theorem
- Gnedenko's Theorem
- Necessary and sufficient conditions for the LLT
- Necessary and sufficient conditions for the LLT with rate
- The general LLT for random variables in the domain of attraction of a stable distribution

Outline

Almost Sure Local Limit Theorems (ASLLT)

- Motivation: $\longrightarrow$ Almost Sure Central Limit Theorem (ASCLT)
- ASLLT for random sequences in the domain of attraction of the normal law
    - ASLLT with rate (random sequences with moment $2+\epsilon$)
    - ASLLT for random sequences with second moment
- ASLLT for random sequences in the domain of attraction of a stable law with $\alpha < 2$ (without second moment)

The DeMoivre–Laplace theorem

Theorem

Let $g_n(k)$ be the probability of getting k heads in n tosses of a coin which gives a head with probability p. Then

$$\lim_{n\to\infty} \frac{g_n(k)}{\left(\frac{1}{\sqrt{2npq}} e^{-\frac{(k-np)^2}{2npq}}\right)} = 1,$$

uniformly for k such that $\left|\frac{k-np}{\sqrt{npq}}\right|$ remains bounded.

- Abraham DeMoivre proved it only for a fair coin ($p = 1/2$) in *Approximatio ad Summam Terminorum Binomii* $(a+b)^n$ *in Seriem expansi* (1733).
- Pierre-Simon Laplace proved it in full generality in *Théorie Analytique des probabilités* (1812).

In the DeMoivre Theorem $\longrightarrow p = E[X_1]$ and $pq = VarX_1$.

We could expect

Theorem

Let $(X_n)_{n \geqslant 1}$ be a sequence of i.i.d. random variables, with $E[X_1] = \mu$, $VarX_1 = \sigma^2$. Then

$$P(S_n = k) \approx \frac{1}{\sqrt{2\pi n\sigma}} e^{-\frac{(k-n\mu)^2}{2n\sigma^2}}$$

if $\left|\frac{k-n\mu}{\sqrt{npq}}\right|$ is bounded.

A particular case

But certainly this cannot be true in general: if

$$X_1 = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2} \end{cases}$$

If k is odd, then $P(S_{2n} = k) = 0$.

If k is even, say $k = 2h$, we should obtain (with $2n$ in place of n and $2h$ in place of k, $\mu = 0$, $\sigma^2 = 1$).

$$P(S_{2n} = 2h) \approx \frac{1}{2\sqrt{\pi n}} e^{-\frac{h^2}{n}},$$

with $\frac{h}{\sqrt{n}}$ bounded.

In particular

$\longrightarrow$ for (h_n) such that $\frac{h_n}{\sqrt{n}} \to_n \frac{x}{\sqrt{2}}$:

$$P(S_{2n} = 2h_n) \approx \frac{1}{2\sqrt{\pi n}} e^{-\frac{x^2}{2}}. \tag{1}$$

On the contrary

Theorem

If $\frac{h_n}{\sqrt{n}} \to_n \frac{x}{\sqrt{2}}$

$$\lim_{n\to\infty} \sqrt{n} P(S_n = 2h_n) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}}. \tag{2}$$

A tentative statement

Hence a general theorem should roughly state that

$$P(S_n = k) \approx \frac{c}{\sqrt{2\pi n\sigma}} e^{-\frac{(k-n\mu)^2}{2n\sigma^2}}$$

if k is a value of S_n and $\left|\frac{k-n\mu}{\sqrt{npq}}\right|$ is bounded, for a suitable c. But what is c? Comparing (1) and (2), we notice that the main difference is a factor of 2 in the second member of (2); where does it come from? We also notice that in this case the support of S_{2n} is concentrated on even integers, and two successive even integers differ by 2. So one guesses that $c = 2$ in our case, and in general c is maybe connected with the gap between successive values of S_n.

Preliminaries

$\mathcal{L}(a, \lambda) := a + \lambda\mathbb{Z} = \{a + \lambda k, k \in \mathbb{Z}\}$.

Definition

A random variable X has a **lattice distribution** if there exist two constant a and $\lambda > 0$ such that $P(X \in \mathcal{L}(a, \lambda)) = 1$.

$\phi = \mathbb{E}[e^{itX}] =$ the characteristic function of X.

Link between lattice distribution and the behaviour ϕ :

Theorem

There are only three possibilities:
(i) there exists a $t_0 > 0$ such that $|\phi(t_0)| = 1$ and $|\phi(t)| < 1$ for every $0 < t < t_0$. In this case X has a lattice distribution.
(ii) $|\phi(t)| < 1$ for every $t \neq 0$ (non lattice distribution).
(iii) $|\phi(t)| = 1$ for every $t \in \mathbb{R}$: In this case X is constant a.s. (degenerate distribution).

The proof shows in particular that

Corollary

In case (i) of the preceding Theorem, we have

$$\frac{2\pi}{t_0} = \max\{\lambda > 0 : \exists\, a \in \mathbb{R},\ P(X \in \mathcal{L}(a,\lambda)) = 1\}.$$

Hence

Definition

In case (i) of the preceding Theorem, the number

$$\Lambda = \frac{2\pi}{t_0} = \max\{\lambda > 0 : \exists\, a \in \mathbb{R},\ P(X \in \mathcal{L}(a,\lambda)) = 1\}$$

is called the *(maximal) span* of the distribution of X.

Some examples

- (i) Let

$$X_1 = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

  Then $\phi(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right) = \cos t$, and $|\phi(t)| = 1$ if and only if $t = n\pi$, $n \in \mathbb{Z}$. Hence $t_0 = \pi$, and the maximal span of the distribution is $\frac{2\pi}{t_0} = 2$.

- (ii) Let X_1 have standard gaussian law. Then $\phi(t) = e^{-\frac{t^2}{2}}$. In this case $|\phi(t)| = 1$ only for $t = 0$.

- (iii) If $X_1 = c$ (c some constant) we have $\phi(t) = e^{itc}$, and $|\phi(t)| = 1$ for every t.

Gnedenko's Local Limit Theorem (1948)

$(X_n)_{n\geqslant 1}$ sequence of i.i.d random variables, $\mathbf{E}[X_i] = \mu$, $\mathbf{Var} X_i = \sigma^2$ (finite) with lattice distribution. $\Lambda =$ maximal span.

$$S_n = X_1 + \cdots + X_n.$$

$$P(X_i \in \mathcal{L}(a, \Lambda)) = 1 \Longrightarrow P(S_n \in \mathcal{L}(na, \Lambda) = 1.$$

Theorem

We have

$$\lim_{n\to\infty} \sup_{N\in\mathcal{L}(na,\Lambda)} \left| \frac{\sqrt{n}}{\Lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \right| = 0.$$

Gnedenko's Local Limit Theorem (1948)

Some heuristics $(\mu = 0)$

By the CLT

$$\begin{aligned}
P\Big(S_n = N\Big) &\approx P\Big(N - \frac{\Lambda}{2} \leqslant S_n \leqslant N + \frac{\Lambda}{2}\Big) \\
&= P\Big(\frac{N}{\sqrt{n}\sigma} - \frac{\Lambda}{2\sigma\sqrt{n}} \leqslant \frac{S_n}{\sigma\sqrt{n}} \leqslant \frac{N}{\sqrt{n}\sigma} - \frac{\Lambda}{2\sigma\sqrt{n}}\Big) \\
&\approx \int_{\frac{N}{\sqrt{n}\sigma} - \frac{\Lambda}{2\sigma\sqrt{n}}}^{\frac{N}{\sqrt{n}\sigma} + \frac{\Lambda}{2\sigma\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt = \int_{\frac{N}{\sqrt{n}} - \frac{\Lambda}{2\sqrt{n}}}^{\frac{N}{\sqrt{n}} + \frac{\Lambda}{2\sqrt{n}}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}\, dt \\
&\approx \frac{\Lambda}{\sqrt{n}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{N^2}{2n\sigma^2}}.
\end{aligned}$$

Gnedenko's Local Limit Theorem (necessary and sufficient conditions for the LLT)

Actually, the complete formulation of Gnedenko's result is

Theorem

With the same assumptions as above, in order that

$$\lim_{n\to\infty} \sup_{N\in\mathcal{L}(na,\lambda)} \left| \frac{\sqrt{n}}{\lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \right| = 0,$$

it is necessary and sufficient that $\lambda = \Lambda$.

Necessary and sufficient conditions for the Local Limit Theorem with rate

The following result completes the theory

Theorem

With the same assumptions as in Theorem 8, in order that

$$\sup_{N \in \mathcal{L}(na,\lambda)} \left| \frac{\sqrt{n}}{\lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \right| = O(n^{-\alpha}),$$

with

$$0 < \alpha < \frac{1}{2}$$

it is necessary and sufficient that the following conditions are satisfied

(i) $\lambda = \Lambda$;

(ii) if F *denotes the distribution function of* X_1, *then, as* $u \to \infty$, $\int_{|x| \geq u} x^2 F(dx) = O(u^{-2\alpha})$.

Local Limit Theorem in the nonlattice case

$(X_n)_{n\geq 1}$ sequence of i.i.d. random variables . $\phi=$ characteristic function with $|\phi(t)| < 1$ for every $t \neq 0$.

Remark

In the nonlattice case, most characteristic functions verify the Cramer's condition, i.e.

$$\limsup_{t\to\infty} |\phi(t)| < 1.$$

Remark

There exist characteristic functions of nonlattice random variables, that do not verify Cramer's condition.

An example is

$$\phi(t) = \prod_{k=1}^{\infty} \cos\left(\frac{t}{k!}\right).$$

$|\phi(t)| = 1 \iff \frac{t}{k!}$ *is a multiple of* π *for each integer* k*: impossible unless* $t = 0$.

But

$$1 - \phi(2\pi N!) \to 0, \qquad N \to \infty.$$

Local Limit Theorem in the nonlattice case

The following result holds

Theorem

Let $(X_n)_{n\geqslant 1}$ *be sequence of i.i.d. nonlattice random variables, with* $\mathbf{E}[X_1] = \mu$, $\mathbf{Var} X_1 = \sigma^2 < \infty$. *If* $\frac{x_n}{\sqrt{n}} \to x$ *and* $a < b$,

$$\lim_{n\to\infty} \sqrt{n} P(S_n - n\mu \in (x_n + a, x_n + b)) = (b-a)\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

The heuristics are as before.

Local Limit Theorem in the nonlattice case

With some further properties on $|\phi|$:

Theorem

If $|\phi|$ is integrable, then $\frac{S_n - n\mu}{\sigma\sqrt{n}}$ has a density f_n; moreover f_n tends undiformly to the standard normal density

$$\eta(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

The general Local Limit Theorem for random variables in the domain of attraction of a stable distribution: preliminaries

$(X_n)_{n \geqslant 1}$ i.i.d. with (common) distribution F (not necessarily lattice) and partial sums $S_n = X_1 + \cdots + X_n$.
G = distribution.

Definition

The **domain of attraction** of G is the set of distributions F having the following property: there exists two sequences (a_n) and (b_n) of real numbers, with $b_n \to_n \infty$, such that

$$\frac{S_n - a_n}{b_n} \xrightarrow{\mathcal{L}} G$$

as $n \to \infty$.

The general Local Limit Theorem for random variables in the domain of attraction of a stable distribution: stable distributions

G possesses a domain of attraction iff G is *stable*, i.e.

Definition

A non-degenerate distribution G is **stable** if it satisfies the following property:
let X_1 and X_2 be independent variables with distribution G; for any constants $a > 0$ and $b > 0$ the random variable $aX_1 + bX_2$ has the same distribution as $cX_1 + d$ for some constants $c > 0$ and d.

The general Local Limit Theorem for random variables in the domain of attraction of a stable distribution: alternative definition of stable distributions

Definition

G is stable if its characteristic function can be written as

$$\varphi(t;\mu,c,\alpha,\beta) = \exp\left[\ it\mu - |ct|^{\alpha}\left(1 - i\beta\,\mathrm{sgn}(t)\Phi\right)\ \right]$$

where $\alpha \in (0,2]$, $\mu \in \mathbb{R}$, $\beta \in [-1,1]$; $sgn(t)$ is just the sign of t and

$$\Phi = \begin{cases} \tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi}\log|t| & \text{if } \alpha = 1. \end{cases}$$

The parameter α is the **exponent** of the distribution.

Remark

The normal law is stable with exponent $\alpha = 2$.

The general Local Limit Theorem for random variables in the domain of attraction of a stable distribution

Theorem

Let X_n have lattice distribution with maximal span Λ. In order that, for some choice of constants a_n and b_n

$$\lim_{n\to\infty} \sup_{N\in\mathcal{L}(na,\lambda)} \left| \frac{b_n}{\lambda} P(S_n = N) - g\left(\frac{N - a_n}{b_n}\right) \right| = 0,$$

where g is the density of some stable distribution G with exponent $0 < \alpha \le 2$,
it is necessary and sufficient that
(i) the common distribution F of the X_n belongs to the domain of attraction of G;
(ii) $\lambda = \Lambda$ (i.e. maximal).

Almost Sure Local Limit Theorems (ASLLT): the motivation (starting from the Almost Sure Central Limit Theorem)

$(X_n)_{n\geqslant 1}$ i.i.d with $\mathbf{E}[X_1] = \mu$, $\mathbf{Var}X_1 = \sigma^2$.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

Central Limit Theorem

Theorem

For every $x \in \mathbb{R}$

$$\mathbf{E}[\boxed{1_{\{Z_n \leqslant x\}}}] = P(Z_n \leqslant x) \underset{n}{\longrightarrow} \Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

Almost Sure Local Limit Theorems (ASLLT): the motivation (starting from the Almost Sure Central Limit Theorem)

Almost Sure Central Limit Theorem

Theorem

$P-$a. s., for every $x \in \mathbb{R}$

$$\frac{1}{\log n}\sum_{h=1}^{n}\frac{1}{h}\boxed{1_{\{Z_h \leqslant x\}}} \underset{n}{\longrightarrow} \Phi(x).$$

By analogy (for the case of Gnedenko's Theorem)

$$\kappa_n \in \mathcal{L}(na, \Lambda) \text{ such that } \frac{\kappa_n - n\mu}{\sqrt{n}} \to \kappa.$$

Gnedenko's Theorem $\Longrightarrow$

$$\mathbf{E}[\boxed{\sqrt{n}1_{\{S_n=\kappa_n\}}}] = \sqrt{n}P(S_n = \kappa_n) \to \Lambda\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\kappa^2}{2\sigma^2}}.$$

Comparing with the case of the Central Theorem:

<u>Tentative</u> Almost Sure Local Limit Theorem

Theorem

$P\!-\!a.\ s.$,

$$\frac{1}{\log n}\sum_{h=1}^{n}\frac{1}{h}\Big(\boxed{\sqrt{h}1_{\{S_h=\kappa_n\}}}\Big)$$

$$=\frac{1}{\log n}\sum_{h=1}^{n}\frac{1}{\sqrt{h}}1_{\{S_h=\kappa_h\}}\underset{n}{\longrightarrow}\Lambda\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{\kappa^2}{2\sigma^2}}.$$

Some history

- In 1951 Chung and Erdös proved

Theorem

Let $(X_n)_{n\geqslant 1}$ be a centered Bernoulli process with parameter p. Then

$$\frac{1}{\log n}\sum_{h=1}^{n}\frac{1}{\sqrt{h}}1_{\{S_h=0\}}\xrightarrow[n]{}\frac{1}{\sqrt{2\pi p(1-p)}},\qquad a.s.$$

This is a particular case of our tentative ASLLT: just take $\kappa_n = np$.

Some history

- In 1993 Csáki, Földes and Révész proved

Theorem

Let $(X_n)_{n\geqslant 1}$ be i.i.d. centered and with finite third moment. Then

$$\frac{1}{\log n}\sum_{h=1}^{n}\frac{1}{p_h}1_{\{a_h\leqslant S_h\leqslant b_h\}}\xrightarrow[n]{}1, \qquad a.s.$$

where $p_n = P(a_n \leqslant S_n \leqslant b_n)$.

This generalizes the Chung–Erdös Theorem: just take $a_n = b_n = 0$ and recall Gnedenko's Theorem.

The ASLLT for random sequences in the domain of attraction of the normal law

$(X_n)_{n\geqslant 1}$ i.i.d. having lattice distribution F with maximal span Λ; $\mathbf{E}[X_1] =: \mu$, $\mathbf{Var}X_1 =: \sigma^2$ (finite). We assume $\mu = 0$ and $\sigma^2 = 1$ (no loss of generality).

Definition

$(X_n)_{n\geqslant 1}$ satisfies an ASLLT if

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{\sqrt{h}} 1_{\{S_h=\kappa_h\}} \overset{a.s.}{=} \Lambda \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\kappa^2}{2\sigma^2}},$$

for any sequence of integers $(\kappa_n)_{n\geqslant 1}$ in $\mathcal{L}(na, \Lambda)$ such that

$$\lim_{n\to\infty} \frac{\kappa_n - n\mu}{\sqrt{n}} = \kappa.$$

ASLLT with rate

Theorem

Let $\epsilon > 0$ *and assume that* $\mathbf{E}\left[|X_1^{2+\epsilon}|\right] < \infty$. *Then* $(X_n)_{n\geqslant 1}$ *satisfies an ASLLT. Moreover, if the sequence* $(\kappa_n)_{n\geqslant 1}$ *verifies the stronger condition*

$$\frac{\kappa_n - n\mu}{\sqrt{n}} = \kappa + O_\delta\left((\log n)^{-1/2+\delta}\right)$$

then

$$\sum_{h=1}^{n} \frac{1}{\sqrt{h}} 1_{\{S_h=\kappa_h\}} = \Lambda \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\kappa^2}{2\sigma^2}} + O_\delta\left((\log n)^{-1/2+\delta}\right).$$

Remark

If $\mathbf{E}\left[|X_1^{2+\epsilon}|\right] < \infty$ *for some positive* ϵ, *then the condition of Gnedenko's Theorem with rate, i.e.*

$$\sup_{N \in \mathcal{L}(na,\lambda)} \left| \frac{\sqrt{n}}{\lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \right| = O(n^{-\alpha}),$$

$$0 < \alpha < \frac{1}{2}$$

is satisfied with $\alpha = \epsilon/2$. *In fact*

$$\int_{|x| \geqslant u} x^2 F(dx) = \int_{|x| \geqslant u} |x|^{2+\epsilon} |x|^{-\epsilon} F(dx) \leqslant \mathbf{E}[|X_1^{2+\epsilon}|] u^{-\epsilon}.$$

Key ingredients for the proof

- (i) a suitable correlation inequality;
- (ii) Gnedenko's Theorem with rate;
- (iii) the notion of **quasi orthogonal system.**

Correlation inequality

$$Y_h = \sqrt{h}\big(1_{\{S_h=\kappa_h\}} - P(S_h = \kappa_h)\big).$$

Proposition

Assume that

$$r(n) := \sup_{N\in\mathcal{L}(na,\Lambda)} \left|\frac{\sqrt{n}}{\Lambda}P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(N-n\mu)^2}{2n\sigma^2}}\right| = O(n^{-\alpha}),$$

$$0 < \alpha < \frac{1}{2}$$

Then there exists a constant C such that, for all integers m, n with $1 \leq m < n$

$$\left|\mathbf{Cov}(Y_m, Y_n)\right| \leq C\left(\frac{1}{\sqrt{\frac{n}{m}} - 1} + \sqrt{\frac{n}{n-m}}\cdot\frac{1}{(n-m)^{\alpha}}\right).$$

Kac–Salem–Zygmund definition of quasi–orthogonal system

Definition

A sequence of functions $\Psi := (f_n)_{n \geqslant 1}$ defined on a Hilbert space $\mathcal{H}$ is said **quasi–orthogonal** if the quadratic form on ℓ^2: $(x_n) \mapsto \sum_{h,k} \langle f_h, f_k \rangle x_h x_k$ is bounded (as a quadratic form).

Weber's criterion for quasi–orthogonality

Lemma

In order that $\Psi := (f_n)_{n \geqslant 1}$ be a quasi–orthogonal system, it is sufficient that

$$\sup_h \sum_k |\langle f_h, f_k \rangle| < \infty.$$

The notion of quasi orthogonal system

Remark

If $\mathcal{H} = L^2(T)$, *where* $(T, \mathcal{A}, \mu)$ *is some probability space, then* $\sum_{h,k} \langle f_h, f_k \rangle x_h x_k = \sum_{h,k} \big(\int f_h f_k \, d\mu \big) x_h x_k$. *By Rademacher–Menchov Theorem, it is seen that the series* $\sum_n c_n f_n$ *converges if for instance* $c_n = n^{-\frac{1}{2}} (\log n)^{-b}$ *with* $b > \frac{3}{2}$.

Main steps of the proof

(i) Any $\rho > 1$ fixed.

The basic correlation inequality $\Longrightarrow$

$Z_j = \sum_{\rho^j \leqslant h < \rho^{j+1}} \frac{Y_h}{h}$ is quasi–orthogonal.

(ii) By the preceding remark

$\sum_j \frac{Z_j}{\sqrt{j}(\log j)^b}$ converges as soon as $b > \frac{3}{2}$.

(iii) By Kronecker's Lemma

$$\frac{1}{\sqrt{n}(\log n)^b} \sum_{j=1}^{n} Z_j = \frac{1}{\sqrt{n}(\log n)^b} \sum_{j=1}^{n} \sum_{\rho^j \leqslant h < \rho^{j+1}} \frac{Y_h}{h}$$

$$= \frac{1}{\sqrt{n}(\log n)^b} \sum_{1 \leqslant h < \rho^{n+1}} \frac{Y_h}{h} \xrightarrow[n]{} 0.$$

Main steps of the proof

(iv) The preceding relation yields easily (details omitted)

$$\frac{\sqrt{\log t}}{(\log\log t)^b}\Big(\frac{1}{\log t}\sum_{h\leqslant t}\frac{Y_h}{h}\Big)=\frac{1}{\sqrt{\log t}(\log\log t)^b}\sum_{h\leqslant t}\frac{Y_h}{h}\underset{t}{\longrightarrow}0;$$

since $\frac{\sqrt{\log t}}{(\log\log t)^b}\longrightarrow_t\infty$, this implies

$$\frac{1}{\log t}\sum_{h\leqslant t}\frac{Y_h}{h}=\frac{1}{\log t}\sum_{h\leqslant t}\frac{1_{\{S_h=\kappa_h\}}}{\sqrt{h}}-\frac{1}{\log t}\sum_{h\leqslant t}\frac{\sqrt{h}P(S_h=\kappa_h)}{h}\underset{t}{\longrightarrow}0$$

(v) Last, by Gnedenko's Theorem

$$\frac{1}{\log t}\sum_{h\leqslant t}\frac{\sqrt{h}P(S_h=\kappa_h)}{h}\underset{t}{\longrightarrow}\Lambda\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\kappa^2}{2\sigma^2}},$$

and the result follows.

The second part of the Theorem is proved similarly.

ASLLT for random sequences with second moment

When only the second moment is available, the ASLLT with rate doen't hold (see remark). M. Weber proved the ASLLT afresh:

1. with the additional assumption (v_k, $k \in \mathbb{Z}$ are the elements of the lattice)

$$P(X = v_k) \wedge P(X = v_{k+1}) > 0 \quad \text{for some } k \in \mathbb{Z}. \qquad (3)$$

With (3), a basic correlation inequality holds, again for $Y_h = \sqrt{h}(1_{\{S_h = \kappa_h\}} - P(S_h = \kappa_h))$:

Proposition

Assume that (3) holds. Then there exists a constant C (depending on the sequence (κ_n)) such that, for all integers m, n with $1 \le m < n$

$$\left|\mathbf{Cov}(Y_m, Y_n)\right| \le C\Big(\frac{1}{\sqrt{\frac{n}{m}} - 1} + \sqrt{\frac{n}{n-m}} \cdot \frac{1}{n-m}\Big).$$

2. in the general case

ASLLT for random sequences with second moment

Main ingredients similar to the previous ones:

1. the above correlation inequality;
2. Gnedenko's Theorem (without rate);
3. the notion of **quasi orthogonal system.**

ASLLT for random sequences in the domain of attraction of a stable law with $0 < \alpha < 2$.

Recent work by R. Giuliano and Z. Szewczak (work in progress).

- $(X_n)_{n\geq 1}$ sequence of i.i.d. random variables; common distribution F is in the domain of attraction of a stable distribution G (having density g) with exponent α $(0 < \alpha < 2)$.
- (a_n) and (b_n) as before; $b_n = L(n)n^{1/\alpha}$, L slowly varying in Karamata's sense.
- X_1 with lattice distribution; $\Lambda =$ maximal span.
- F and G symmetric $\Longrightarrow a_n = 0$.

IMPORTANT

Since X_1 doesn't possess second moment, the discussion of the preceding section doesn't work.

But we can use similar ingredients as before:

1. a correlation inequality;
2. General LLT Theorem (for stable distributions);
3. Gaal–Koksma LLN.

The correlation inequality when $0 < \alpha < 2$

Proposition

- *For every pair (m, n) of integers, with $1 \le m < n$,*

$$b_m b_n \Big| P(S_m = \kappa_m, S_n = \kappa_n) - P(S_m = \kappa_m) P(S_n = \kappa_n) \Big|$$
$$\le C \Big\{ \Big(\frac{n}{n-m} \Big)^{1/\alpha} \frac{L(n)}{L(n-m)} + 1 \Big\}.$$

- *For every pair (m, n) of integers, with $1 \le m < n$,*

$$b_m b_n \Big| P(S_m = \kappa_m, S_n = \kappa_n) - P(S_m = \kappa_m) P(S_n = \kappa_n) \Big|$$
$$\le C \cdot L(n) \Bigg\{ n^{1/\alpha} \Big(\frac{1}{e^{(n-m)c}} + \frac{1}{e^{nc}} \Big) + \frac{\frac{m}{n}}{\big(1 - \frac{m}{n}\big)^{1+1/\alpha}} +$$
$$+ \left(\frac{\frac{m}{n}}{\big(1 - \frac{m}{2n}\big)^2} \right)^{1/\alpha} \Bigg\}.$$

The Gaal–Koksma LLN

Theorem

Let $(Z_n)_{n\geqslant 1}$ be a sequence of centered random variables with finite variance. Suppose that there exists a constant $\beta > 0$ such that, for all integers $m \geq 0$, $n > 0$,

$$E\Big[\Big(\sum_{i=m+1}^{m+n} Z_i\Big)^2\Big] \leq C\big((m+n)^\beta - m^\beta\big), \tag{4}$$

for a suitable constant C independent of m and n. Then, for each $\delta > 0$,

$$\sum_{i=1}^{n} Z_i = O(n^{\beta/2}(\log n)^{2+\delta}), \quad P - a.s.$$

Theorem

Let $\alpha > 1$ *and assume that there exists* $\gamma \in (0,2)$ *such that*

$$\sum_{k=a}^{b} \frac{L(k)}{k} \leq C(\log^\gamma b - \log^\gamma a).$$

Then $(X_n)_{n \geqslant 1}$ *satisfies an ASLLT, i.e.*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{b_n}{n} 1_{\{S_n = \kappa_n\}} = \Lambda g(\kappa).$$

Main steps

We apply the above theorem to the sequence

$$Z_j := \sum_{h=\rho^{j-1}}^{\rho^j - 1} \frac{Y_h}{h}.$$

where

$$Y_n = b_n\Big(1_{\{S_n=\kappa_n\}} - P(S_n = \kappa_n)\Big).$$

We obtain that

$$\frac{1}{n} \sum_{1\leqslant h\leqslant \rho^n} \frac{Y_h}{h} \to 0,$$

whence (as before)

$$\frac{1}{\log n} \sum_{1\leqslant h\leqslant n} \frac{Y_h}{h} \to 0.$$

References

[1] J. Aaronson and M. Denker,Characteristic functionsof random variables attracted to 1–stable Laws, Ann. Prob.(1998),26, 399–415

[2] D. R. Bellhouse and C. Genest, Maty's Biography of Abraham De Moivre, Translated, Annotated and Augmented, *Statist. Sci.*, Volume 22, Number 1 (2007), 109-136. available on the Web at *Project Euclid*.

[3] G.A. Brosamler. An almost everywhere central limit theorem. *Math. Proc. Camb. Phil. Soc.* 104 (1988) 561–574.

[4] K. L. Chung and P. Erdös, Probability limit theorems assuming only the first moment., *Mem. Amer. Math. Soc.*, (1951), no. 6, 19pp.

References

[5] E. Csáki, A. Földes and P. Révész (1993), On almost sure local and global central limit theorems, *Prob. Th. Rel. Fields* , 97, 321– 337.

[6] M. Denker and S. Koch, Almost sure local limit theorems, *Stat. Neerland.*,(2002), 56, 143–151.

[7] R. Durrett, *Probability: Theory and Examples*, Brooks/Cole (2005).

[8] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, (1971).

References

[9] R. Giuliano–Antonini and M. Weber, Almost sure local limit theorems with rate, *Stoch. Anal. Appl.*, 29,(2011), 779–798.

[10] B. V. Gnedenko, On a local limit theorem of the theory of probability *Uspekhi Mat. Nauk.*, **3**, (1948), 187–194.

[11] B.V. Gnedenko, *A course in the theory of probability*, 5th edition, Nauka, Moscow (1967), English translation of 4th edition: Chelsea, New York

[12] I. A. Ibragimov and Y. V. Linnik,*Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff Publishing Groningen, The Netherlands (1971).

[13] M. Kac, R. Salem and A. Zygmund, A gap theorem, Trans. Amer. Math. Soc., 63 (1948), 235–243.

[14] E. Lukacs, *Characteristic Functions* (second ed.). New York: Hafner Pub. Co.(1970).

[15] W. Philipp and W. Stout, *Almost sure invariance principles for partial sums of weakly dependent random variables*(1975), Memoirs Ser. No. 161 (Memoirs of the American Mathematical Society)

[16] P. Schatte, On strong versions of the central limit theorem, *Math. Nachr.*, 137 (1988),249–256.

References

[17] M. Weber, *Entropie metrique et convergence presque partout* , Travaux en Cours [Works in Progress], 58. Hermann, Paris, (1998), vi+151 pp.

[18] M. Weber,*Dynamical Systems and Processes*, (IRMA Lectures in Mathematics and Theoretical Physics 14), European Mathematical Society, Zürich, (2009), 773 pp.

[19] M. Weber, A sharp correlation inequality with applications to almost sure local limit theorem, *Probab. Math. Statist.*, 31,(2011) 79–98.