
Generalized Inverses and Applications

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Fredholm's method to solve a particular integral equation in 1903, was probably the first written work on generalized inverses. In 1906, Moore formulated the generalized inverse of a matrix in an algebraic setting, which was published in 1920, and in the thirties von Neumann used generalized inverses in his studies of continuous geometries and regular rings. Kaplansky and Penrose, in 1955, independently showed that the Moore "reciprocal inverse" could be represented by four equations, now known as Moore-Penrose equations. A big expansion of this area came in the fifties, when C.R. Rao and J. Chipman made use of the connection between generalized inverses, least squares and statistics. Generalized inverses, as we know them presently, cover a wide range of mathematical areas, such as matrix theory, operator theory, c^* -algebras, semi-groups or ring Fermigs. They appear in numerous applications that include areas such as linear estimation, differential and difference equations, Markov chains, graphics, cryptography, coding theory, incomplete data recovery and robotics. The aim of this mini-symposium, is to gather researchers involved in the study of generalized inverses and to encourage the exchange of ideas.

Analytic Perturbations of Generalized Inverses

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Mon 12:15, Room Galilei

We investigate analytic perturbations of the reduced resolvent of a finite dimensional linear operator (also known as Drazin inverse in linear algebra literature). By analytic perturbations we mean the perturbed operator depends analytically on a perturbation parameter. Our approach is based on spectral theory of linear operators as well as on a new notion of group reduced resolvent. It allows to treat regular and singular perturbations in a unified framework. We produce an algorithm for computing the coefficients of the Laurent series of the perturbed reduced resolvent. In particular, the regular part coefficients can be calculated by simple recursive formulae. Finally, we apply these results to the perturbation analysis of Moore-Penrose generalized inverses.

Joint work with J. B. Lasserre (LAAS-CNRS)

On group invertibility and representations for the group inverse of partitioned matrices

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Mon 15:00, Room Galilei

In recent papers [1],[2], necessary and sufficient conditions were derived for a partitioned matrix to have several generalized inverses, including inner, reflexive and Moore-Penrose inverse, with Banachiewicz-Schur form. We recall that, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $A^\#$ denotes a generalized inverse of A , then the Schur generalized complement of A in M is defined as $S = D - CA^\#B$, and we say that the generalized inverse of M has the Banachiewicz-Schur form when it is expressible in the form

$$M^\# = \begin{pmatrix} A^\# + A^\#BS^\#CA^\# & -A^\#BS^\# \\ -S^\#CA^\# & S^\# \end{pmatrix}.$$

In this talk, firstly, we address the problem of developing conditions under which the Drazin inverse of a partitioned matrix can be obtained by a formula which involves the Banachiewicz-Schur form. Conditions for the existence of the group inverse of partitioned matrices satisfying the rank formula $\text{rank}(M) = \text{rank}(A^D) + \text{rank}(S^D)$ are given. (Joint work with M.F. Martínez-Serrano).

Next, we study the group invertibility and give representations for the group inverse of a type of block matrices with applications in graph theory. (Joint work with J. Robles and J.Y. Vélez-Cerrada).

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Representations and additive properties of the Drazin inverse

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Tue 15:25, Room Galilei

The theory of Drazin inverses has seen a substantial growth over the past decades. Beside being of great theoretical interest it has found applications in many diverse areas, including statistics, numerical analysis, differential equations, Markov chains, population models, cryptography, control theory etc. One of the topics on the Drazin inverse that is of considerable interest concerns explicit representations for the Drazin inverse of a 2×2 block matrix and explicit representations for the Drazin inverse of the sum of two matrices. Until now, there has been no explicit formula for the Drazin inverse of $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in terms of A^d and D^d with arbitrary A, B, C and D . In the recent years, the representation and characterization of Drazin inverses of matrices or operators on a Hilbert space have been considered by many authors.

Using an additive result for the Drazin inverse, we derive formulae for the Drazin inverse of a 2×2 block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, under conditions weaker than those assumed in papers published before.

Also, we present some additive properties of the generalized Drazin inverse in a Banach algebra and find an explicit expression for the generalized Drazin inverse of the sum $a + b$ in terms of a, a^d, b, b^d under various conditions.

A cancellation property of the Moore-Penrose inverse of triple products

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Mon 17:10, Room Galilei

We study the matrix equation

$$C(BXC)^\dagger B = X^\dagger \quad (*)$$

where X^\dagger is the Moore-Penrose inverse, and we derive conditions for the consistency of (*). Singular vectors of B and C are used to obtain all solutions. Applications to compliance matrices in molecular dynamics, to mixed reverse-order laws of generalized inverses and to weighted Moore-Penrose inverses are given.

Joint work with Harald Wimmer

New results concerning multiple reverse-order law

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Mon 17:35, Room Galilei

In this paper we present new results related to the mixed-type reverse order law for the Moore-Penrose inverse of the various products of multiple bounded Hilbert space operators. Some finite dimensional results are extended to infinite dimensional settings.

- [1] D. S. Djordjevic, N. Č. Dinčić, Reverse order law for the Moore-Penrose inverse, *J. Math. Anal. Appl.* 361(1), pp. 252-261, 2010.
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[3] Y. Tian, Some mixed-type reverse-order laws for the Moore-Penrose inverse of a triple matrix product, *Rocky Mountain Journal of Mathematics* 37(4), pp. 1327-1347, 2007.

Joint work with Dragan S. Djordjević (University of Niš)

On deriving the Drazin inverse of a modified matrix

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Mon 16:45, Room Galilei

Let A be an $n \times n$ complex matrix. The Drazin inverse of A is the unique matrix A^D satisfying the relations:

$$A^D A A^D = A^D, \quad A^D A = A^D A, \quad A^{k+1} A^D = A^k,$$

where k is the index of A . The concept of Drazin inverse plays an important role in various fields like Markov chains, singular differential and difference equations, iterative methods, etc.

A challenge in this area is to establish formulas for computing the Drazin inverse of a modified matrix in terms of the Drazin inverse of the original matrix. These formulas will be of great interest in various applications. They can be useful when the matrix can be expressed as the sum of a matrix with a convenient structure and an additive perturbation, in updating problems, etc.

This problem has been largely studied for invertible matrices. Starting from the well-known formula of Sherman-Morrison-Woodbury given for the regular case:

$$(A + UV^*)^{-1} = A^{-1} - A^{-1}U(I + V^*A^{-1}U)^{-1}V^*A^{-1},$$

where the matrix A and the Schur complement, $I + V^*A^{-1}U$, are invertible, an intensive research has been developed.

In the context of generalized inverses, some analogous formulas have been developed for the Moore-Penrose inverse and for the Drazin inverse under specific conditions. In this paper, we focus on deriving formulas for the Drazin inverse of a modified matrix in terms of the Drazin inverse of the original matrix and the generalized Schur complement, which extend results given in the literature.

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Joint work with M.F. Martínez-Serrano (Technical University of Madrid)

Generalized inverses on the solution of the Toeplitz pencil Conjecture

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A 1981 conjecture by Bumby, Sontag, Sussmann, and Vasconcelos [1] says that the polynomial ring $\mathbb{C}[x]$ is a so called Feedback Cyclization (FC) ring. Two exceptional cases of that conjecture remained unsolved. In 2004 Schmale and Sharma [3] showed that one of these cases would follow from the truth of a simple looking conjecture they formulated for Toeplitz matrices. In [2] the authors show that the Toeplitz pencil conjecture stated in [1] is equivalent to a conjecture for $n \times n$ Hankel pencils, and it is shown to be implied by another conjecture, which is called root conjecture, for matrices up to size 8×8 . In this work we establish how the generalized inverse theory on matrices over rings can be applied to solve this problem.

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On the calculation of different type of generalized inverses for a rectangular matrix using the Kronecker canonical form

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Mon 15:25, Room Galilei

In several significant applications, in control and systems' modelling theory, the methodology of generalized inverses (for instance, the Drazin and the Moore-Penrose inverses) and the Matrix Pencil approach have been extensively used for the study of generalized (descriptor) linear systems with rectangular (or square) constant coefficients, see for instance [1-4]. In this new paper, we extend the recent results of [5]. Analytically, three main directions are discussed and presented: (I) Using the complex Kronecker canonical form, we determine the $\{1, 2\}$ -generalized inverse of a rectangular matrix. (II) Under some interesting additional conditions, the Moore-Penrose inverse of a rectangular matrix is derived using also the matrix pencil approach. (III) Finally, we prove - quite straightforwardly - that the

is no connection between Drazin inverses and the Kronecker canonical form.

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- [2] S.L. Campbell, Singular systems of differential equations, Pitman (Advanced Publishing Program), Vol. II, 1982, UK.
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Joint work with Athanasios A. Pantelous (Department of Mathematical Sciences, University of Liverpool) and Grigoris I. Kalogeropoulos (Department of Mathematics, University of Athens, Greece)

On Generalized Inverses and Green's Relations

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Tue 16:45, Room Galilei

We study generalized inverses on semigroups by means of Green's relations. We first define the notion of inverse along an element and study its properties. Then we show that the classical generalized inverses (group inverse, Drazin inverse and Moore-Penrose inverse) belong to this class. Finally, we prove continuity results for the inverse along an element, in topological rings and Banach algebras.

Recent results on generalized inverses

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Tue 17:10, Room Galilei

We present recent results on generalized inverse of elements in rings with involution. Particularly, the characterizations of partial isometries, EP and star-dagger elements in rings with involution are discussed. We also give several characterizations of Moore-Penrose-invertible normal and Hermitian elements in rings with involution and the proofs are based on ring theory only.

Joint work with D. S. Djordjević (University of Niš)

The generalized inverse of the rectangular Vandermonde matrix

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Mon 11:50, Room Galilei

A Vandermonde matrix is defined in terms of scalars

$\lambda_1, \lambda_2, \dots, \lambda_m$ by

$$V_{nm} = V_n(\lambda_1, \lambda_2, \dots, \lambda_m) = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_m & \cdots & \lambda_m^{n-1} \end{bmatrix}.$$

This particular general family of matrices plays a significant role in different areas of mathematics and applied sciences, see [2], [4] etc. Following the existing literature, the most important applications of the Vandermonde matrix are appeared in approximation problems such as interpolation, least squares and moment problems.

Explicit formulas for solving Vandermonde systems and computing the inverse it are well known, see [1], [3-5] etc.

In this paper, we will discuss and present analytically the generalized inverse of the Rectangular Vandermonde matrix. This general class of Vandermonde matrix has been also appeared in control theory, see for instance [2] for more details.

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- [1] A. Eisenberg and G. Fedele, On the inversion of the Vandermonde matrix, *Applied Mathematics and Computation*, 174, pp. 1384-1396.
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Joint work with Athanasios D. Karageorgos and Grigoris I. Kalogeropoulos (Department of Mathematics, University of Athens, Greece)

Additive Drazin inverses

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Mon 15:50, Room Galilei

We will address to the representation of the Drazin inverses, over a general (associative, with unity) ring, of the block matrix $M = \begin{bmatrix} a & c \\ b & 0 \end{bmatrix}$, in which the (2,2) block is zero. We aim for results in terms of "words" in the three blocks a , b and c , and their g -inverses, such as inner or Drazin inverses. The search for a formula for this Drazin inverse is closely related to the "additive problem" of finding the D-inverse of a sum $(a+b)^d$ in terms of words in a and b , and their g -inverses. As a special case, we shall examine the existence and representation of the group inverse of M .

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Joint work with R.E. Hartwig (North Carolina State University, USA)

Magic generalized inverses

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Mon 11:00, Room Galilei

We consider singular fully-magic matrices in which the numbers in all the rows and columns and in the two main diagonals sum to the same number. Our interest focuses on such magic matrices for which the Moore–Penrose inverse and/or Drazin inverse may also be fully-magic, building on results in [1,2,3,4]. Examples include the matrices for some of the fully-magic squares considered by Heinrich Cornelius Agrippa von Nettesheim (1486–1535), Albrecht Dürer (1471–1528), and Bernard Frénicle de Bessy (c. 1605–1675).

[1] Stephen J. Kirkland & Michael Neumann, Group inverses of M-matrices associated with nonnegative matrices having few eigenvalues, *Linear Algebra and its Applications*, 220, pp. 181–213, 1995.

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Joint work with Ka Lok Chu (Dawson College), S. W. Drury (McGill University) & Götz Trenkler (Universität Dortmund)

Nonnegative Drazin-projectors

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Tue 15:50, Room Galilei

In [1] a characterization of nonnegative matrices with nonnegative Drazin inverse was developed. Later, in [2] the authors gave a characterization of nonnegative matrices A such that $AA^\#$ is a nonnegative matrix, where $A^\#$ denotes the group inverse of the square matrix A . In the last paper only the case of matrices with index 1 was studied. The product $AA^\#$ will be called the group-projector of the matrix A .

In this work, firstly, a necessary and sufficient condition to obtain matrices A with nonnegative group projector is presented. The main contribution of this result is that the nonnegativity condition on the matrix A is removed. Next, the case of the matrix A with index greater than 1 is also analyzed. In this situation, an extended result for the nonnegativity of the Drazin-projector of A (that is, $AA^D \geq O$, where A^D represents the Drazin inverse of A) is obtained.

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Joint work with Alicia Herrero (Instituto de Matemática Multidisciplinar, Universidad Politécnica de Valencia, Spain) and Francisco J. Ramírez (Instituto Tecnológico de Santo Domingo, Dominican Republic)

Condition numbers for the LS and Tikhonov regularization of discrete ill-posed problems

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Mon 11:25, Room Galilei

One of the most successful methods for solving the linear least-squares (LS) problem $\min_x \|Ax - b\|$ with a highly ill-conditioned or rank deficient coefficient matrix A is the method of Tikhonov regularization. In this talk, we derive the normwise, mixed and componentwise condition numbers and componentwise perturbation bounds for LS and the Tikhonov regularization. Our results are sharper than the known results. Some numerical examples are given to illustrate our results.

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