A Note on Normality of Cones Over Symmetric Varieties

Rocco Chirivì a & Andrea Maffei b

a Università di Pisa, Pisa, Italy
b Università di Roma “La Sapienza,,” Rome, Italy


To cite this article: Rocco Chirivì & Andrea Maffei (2012): A Note on Normality of Cones Over Symmetric Varieties, Communications in Algebra, 40:3, 1179-1187

To link to this article: http://dx.doi.org/10.1080/00927872.2010.531993

Full terms and conditions of use: http://www.tandfonline.com/page/terms-and-conditions

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae, and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand, or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.
A NOTE ON NORMALITY OF CONES OVER SYMMETRIC VARIETIES

Rocco Chirivì¹ and Andrea Maffei²

¹Università di Pisa, Pisa, Italy
²Università di Roma “La Sapienza,” Rome, Italy

Let $G$ be a semisimple and simply connected algebraic group, and let $H^0$ be the subgroup of points fixed by an involution of $G$. Let $V$ be an irreducible representation of $G$ with a nonzero vector fixed by $H^0$. In this article, we prove a property of the normalization of the coordinate ring of the closure of $G \cdot [v]$ in $\mathbb{P}(V)$.

Key Words: Complete symmetric variety; Projective normality.

2000 Mathematics Subject Classification: Primary 14M17; Secondary 14L30.

INTRODUCTION

Let $G$ be a semisimple and simply connected algebraic group over an algebraically closed field $k$ of characteristic zero. Let $\sigma$ be an involution of $G$, $H^\sigma = G^\sigma$ the set of points fixed by $\sigma$, and $H$ the normalizer of $H^\sigma$ in $G$. We denote by $X$ the wonderful compactification of $G/H$ introduced by De Concini and Procesi in [4].

If $V$ is an irreducible representation of $G$, we say that it is spherical if $V^{H^\sigma} \neq \{0\}$. In this case, let $h_V$ be a nonzero vector fixed by $H^\sigma$. The map $g \mapsto g \cdot [h_V]$ from $G$ to $\mathbb{P}(V)$ determines a map $q_V$ from $X$ to $\mathbb{P}(V)$, and we denote its image by $Z_V$. Let also $\mathcal{L}_V$ be the pullback of the line bundle $\mathcal{O}(1)$ through the map $q_V$. Let $A_V$ be the ring $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}_V^n)$ and $B_V$ the projective coordinate ring of $Z_V \subset \mathbb{P}(V)$.

In [2] we have proved that $A_V$ is the integral closure of $B_V$; in that article, following an argument of Brion, this result was deduced by a general argument using the results contained in [3]. In this note we present a new proof which is quite longer than the one given by Brion, but which gives a slightly more precise result; this result was needed in a first version of [6].

NOTATIONS

We need to introduce a certain number of objects; for most of them, we use standard notations. For further details about the results given in this section one may look at [3] and the references there.
Let $G$, $\sigma$, $H^0$, and $k$ be as in the introduction and choose a maximal torus $T$ of $G$ that is $\sigma$ stable and such that the dimension of the subtorus $S$ given by the identity component of $\{ t \in T : \sigma(t) = t^{-1}\}$ is the maximal possible. Choose also a Borel subgroup $B$ containing $T$ and such that the dimension of $\sigma(B) \cap B$ is minimal possible. Let $\Lambda$ be the set of characters of $T$, $\Phi \subset \Lambda$ be the set of roots, and $\Phi^+$ (resp., $\Delta$) the choice of positive roots, (resp., simple roots) determined by $B$. Let also $\Lambda^+$ be the set of dominant weights with respect to these choices and, for $\lambda \in \Lambda^+$, let $V_\lambda$ be the irreducible representation of $G$ of highest weight $\lambda$.

We denote the Lie algebras of $G$ and $H$ by $\mathfrak{g}$ and $\mathfrak{h}$, respectively; further $\mathfrak{g}_x$ is the root space of weight $x$ and $x_\delta$ is a nonzero element in $\mathfrak{g}_x$. Let $t$ be the Lie algebra of $T$, we identify $t^*$ with $\Lambda \otimes \mathbb{R} k$ and $t$ with $\text{Hom}(k^*, T) \otimes_k k$. If $x \in \Phi$, we denote by $x' \in t$ the corresponding coroot. More generally if $x \in t^*$ is not zero, we denote by $x'$ the element of $t$ such that $\langle x', y \rangle = \frac{\langle x, y \rangle}{\kappa(x, x)}$ for all $y \in t$, where $\kappa$ is the dual Killing form on $t^*$.

If $x \in \Phi$ we define the associated restricted root as $\tilde{x} = x - \sigma(x)$, and we set $\tilde{\Phi}$ to be the set of nonzero restricted roots. This is a (possibly not reduced) root system, and $\tilde{\Delta} = \{ \tilde{x} \neq 0 : x \in \Delta \}$ is a simple basis for this system.

There are two possible dominant orders on the set of weights, the dominant order defined by $\Delta$ and the one defined by $\tilde{\Delta}$; if $\lambda, \mu \in \Lambda$; we write $\lambda \leq \mu$ if $\mu - \lambda \in \mathbb{N}[\Delta]$ and $\lambda \leq_\sigma \mu$ if $\mu - \lambda \in \mathbb{N}[\tilde{\Delta}]$.

Let also $\Phi_0 = \{ x \in \Phi : \sigma(x) = x \}$, $\Phi_\lambda = \Phi \setminus \Phi_0$, $\Delta_0 = \{ x \in \Delta : \sigma(x) = x \}$ and $\Delta_\lambda = \Delta \setminus \Delta_0$. If $\Delta' \subset \Delta$, we denote by $\Phi_{\Delta'} \subset \Phi$ the root subsystem generated by $\Delta'$. We also denote by $w_{\Delta}$ the longest element of the Weyl group of $\Phi$.

As recalled in the introduction, an irreducible representation is said to be spherical if it has a nonzero weight vector fixed by $H^0$. In particular, we say that $\lambda \in \Lambda^+$ is spherical, if $V_\lambda$ is spherical and we denote by $\Omega^+$ the set of spherical weights and by $\Omega$ the lattice spanned by $\Omega^+$. Similarly, we say that a dominant weight is quasi spherical if $\mathbb{P}(V_{\lambda})^H$ is not empty, in this case this set is just a single point that we denote by $x_{\lambda}$ (see [5]). We denote by $\Pi^+$ the set of quasi spherical weights and by $\Pi$ the sublattice generated by $\Pi^+$ in $\Lambda$. We have $\Omega \cap \Lambda^+ = \Omega^+$, $\Pi \cap \Lambda^+ = \Pi^+$ and, by a result of Helgason,

$$\Omega = \{ \lambda \in \Lambda : \sigma(\lambda) = -\lambda \text{ and } \langle \tilde{x}, \lambda \rangle \in \mathbb{Z} \text{ for all } x \in \Phi \}.$$ 

In particular, $\Omega$ is the set of weights of $\tilde{\Phi}$.

A wonderful compactification $X$ of the symmetric variety $G/H$ has been constructed in [4] in characteristic zero. The following theorem describes some of the main properties of $X$.

**Theorem 1** (Theorem 3.1 in [4]).

(i) $X$ is a smooth projective $G$-variety.

(ii) $X \setminus G/H$ is a divisor with normal crossings and smooth irreducible components $\{ X_\lambda : \lambda \in \tilde{\Delta} \}$ parametrized in a canonical way by the simple roots of the restricted root system.

(iii) The closures of $G$-orbits are given by the subvarieties $X_J = \bigcap_{\lambda \in J} X_\lambda$ for $J$ any subset of $\tilde{\Delta}$, in particular there exists only one closed orbit namely $X_{\tilde{\Delta}}$. 

For each $\lambda \in \Pi^+$, the map $G/H \rightarrow \mathbb{P}(V)$ defined by $gH \mapsto g \cdot x_\lambda$ extends to a morphism from $X$ to $\mathbb{P}(V)$, and we denote by $\mathcal{L}$ the inverse image of $\theta_{\mathbb{P}(V)}(1)$ through this morphism. These line bundles generate the Picard group of $X$, which, in particular, we will identify with the lattice $\Pi$ ([5], Proposition 8.1). The divisors $X_\lambda$ can be parametrized in such a way that $\theta(X_\lambda) \cong \mathcal{L}_\lambda$. There exists a $G$-invariant section $s_\lambda \in \Gamma(X, \mathcal{L})$ whose divisor is $X_\lambda$.

If $\mu \in \Pi^+$, then the module $V^*_\mu$ appears with multiplicity 1 in $\Gamma(X, \mathcal{L}_\mu)$. For an element $v = \sum_{\lambda \in \Delta} n_\lambda \mathcal{L}_\lambda \geq 0$ the multiplication by $s' = \Pi s^{
u_\lambda}_{\lambda}$ injects $\Gamma(X, \mathcal{L}_{\lambda - v})$ in $\Gamma(X, \mathcal{L}_v)$. If $\lambda - \mu \geq 0$, we denote by $s^{(\lambda - \mu)}V^*_\mu \subset \Gamma(X, \mathcal{L}_v)$ the image of $V^*_\mu$ under the multiplication by $s^{(\lambda - \mu)}$. We have the following theorem.

**Theorem 2** (Theorem 5.10 [4]). Let $\lambda \in \Pi$ then $\Gamma(X, \mathcal{L}_\lambda) = \bigoplus_{\mu \leq \lambda, \mu \in \Pi^+} s^{(\lambda - \mu)}V^*_\mu$.

**PRELIMINARIES AND STATEMENT OF THE THEOREM**

The main objects of this article are the following two rings: Given $\lambda \in \Omega^+$ and a natural number $n$, let $A_n(\lambda) = \Gamma(X, \mathcal{L}_\mu)$, and define $A(\lambda)$ as the graded ring $\bigoplus_{n \in \mathbb{N}} A_n(\lambda)$ and $B(\lambda)$ as the subring of $A(\lambda)$ generated by the module $V^*_\mu \subset A_1(\lambda)$. Further, denote by $B_\lambda(\lambda)$ the homogeneous component $B(\lambda) \cap A_n(\lambda)$ of $B(\lambda)$.

In this article, we prove the following result.

**Theorem 3.** Let $\lambda, \mu \in \Omega^+$ and $\mu \leq \lambda$. Then there exists a positive integer $n$ such that $s^{(\lambda - \mu)}V^*_\mu \subset B_n(\lambda)$.

The following corollary, which is an immediate consequence, was needed in [6].

**Corollary 4.** Let $\lambda, \mu \in \Omega^+$ and $\mu \leq \lambda$. Let $f$ be a highest weight vector in $s^{(\lambda - \mu)}V^*_\mu \subset A_1(\lambda)$. Then there exists a positive integer $n$ such that $f^n \in B(\lambda)$.

Now we can use this result to prove the following proposition.

**Proposition 5.** $A(\lambda)$ is the integral closure of $B(\lambda)$.

This result was already stated in [2]. Its proof is in the following three steps: (1) $A(\lambda)$ is integral over $B(\lambda)$, (2) $A(\lambda)$ is integrally closed, and (3) the two rings have the same quotient field. The last step was proved in [2], but the proof there contains a gap that we fill now.

**Proof.** We know that $A(\lambda)$ is generated in degree one (see [3]), hence the highest weight vectors of $A_1(\lambda)$ generate $A(\lambda)$ as a $G$-algebra. Since these vectors are integral over $B(\lambda)$ by Corollary 4 and Theorem 2, $A(\lambda)$ is integral over $B(\lambda)$. On the other hand, $A(\lambda)$ is integrally closed as proved in [3], so we just need to show that $A(\lambda)$ and $B(\lambda)$ have the same quotient field.

Using again that $A(\lambda)$ is generated in degree one, we have that $Y_A = \text{Spec } A(\lambda) \subset A_1(\lambda)_1^*$ is the cone over $\text{Proj } A(\lambda) \subset \mathbb{P}(A_1(\lambda))$. Recall that, by definition, $Y_B = \text{Spec } B(\lambda) \subset V$ is the cone over $Z_Y \subset \mathbb{P}(V)$. The map $\varphi : Y_A \rightarrow Y_B$ determined by the inclusion $B(\lambda) \subset A(\lambda)$ is given by the restriction to $Y_A$ of the $G$-equivariant projection from $A_1(\lambda)$ to $V$. 


Let \( a \in Y_A \) be a nonzero vector in the line fixed by \( H \), and let \( b = \varphi(a) \) be a nonzero multiple of the vector \( h_v \).

The group \( K = G \times k^* \) acts on \( Y_A \) and \( Y_B \) with the second factor acting with multiplication by scalars. The map \( \varphi \) is \( K \)-equivariant, and the orbits \( K \cdot a \) and \( K \cdot b \) are dense in \( Y_A \) and \( Y_B \), respectively.

We now prove that the points \( a \) and \( b \) have the same stabilizers; in particular this proves that \( \varphi \) is birational and completes the proof that \( A(\lambda) \) and \( B(\lambda) \) have the same quotient field.

We argue by induction on the dimension of \( G \). We assume first that \( G = G_1 \times G_2 \) with \( G_1 \) and \( G_2 \) nontrivial \( \sigma \)-stable connected subgroups. Let \( K_j = G_j \times k^* \) for \( j = 1, 2 \) so that \( K = G \times k^* = (K_1 \times K_2) \cap (G_1 \times G_2 \times \Delta k^*) \). In this case \( V_j \) is the tensor product \( V_{\lambda_j} \otimes V_{\mu_j} \) of two irreducible representations of \( G_1 \) and \( G_2 \), respectively. Also by the description of the sections, we have that \( A_1(\lambda) = A_1(\lambda_1) \otimes A_1(\lambda_2) \). So \( a = a_1 \otimes a_2 \) and \( b = b_1 \otimes b_2 \), where \( a_i, b_i \) are points in the line fixed by \( H \) in \( A_1(\lambda_i) \) and \( V_{\lambda_i} \). By induction, we have \( \text{Stab}_{K_j} a_i = \text{Stab}_{K_j} b_i \) for \( i = 1, 2 \), so

\[
\text{Stab}_K a = (\text{Stab}_{K_1} a_1) \times (\text{Stab}_{K_2} a_2) \cap (G \times \Delta k^*) \\
= (\text{Stab}_{K_1} b_1) \times (\text{Stab}_{K_2} b_2) \cap (G \times \Delta k^*) \\
= \text{Stab}_K b.
\]

So we can assume that \( G \) cannot be written as the product of two groups as above. In this case, we say that the involution is \textit{simple}.

If \( \lambda = 0 \), the statement is trivial. If \( \lambda \) is not zero, in Lemma 2.3 in [2], using a simple dimension argument, it is proved that the points \([a] \in \mathbb{P}(A_1(\lambda)^*)\) and \([b] \in \mathbb{P}(V_j)\) have the same stabilizer in \( G \) and this is equal to \( H \). (In [2] we conclude from this result that \( Y_A \) and \( Y_B \) are birational without any further explanation; we give the complete argument here.)

In particular, the stabilizers of \( a \) and \( b \) are contained in \( H \times k^* \). More precisely if \( \chi_a \) is the character given by the action of \( H \) on the line \( k a \) and \( \chi_b \) is the character given by the action of \( H \) on the line \( k b \), then \( \text{Stab}_k(a) = \{(h, \chi_a(h)^{-1}) \in H \times k^* \} \) and \( \text{Stab}_k(b) = \{(h, \chi_b(h)^{-1}) \in H \times k^* \} \). Now the thesis follows since \( \chi_a(h)b = \varphi(\chi_a(h)a) = \varphi(h \cdot a) = h \cdot b = \chi_b(h)b \) and hence \( \chi_a = \chi_b \). \( \square \)

The proof of Theorem 3 will be by induction on the dimension of the variety \( X \). However, in any dimension, it will remain to analyze some particular cases. To deal with these cases, we need a sharper version of Lemma 3.1 in [3]. If \( \lambda, \mu \in \Omega \) let

\[
m_{\lambda, \mu} : \Gamma(X, \mathcal{L}_\lambda) \otimes \Gamma(X, \mathcal{L}_\mu) \to \Gamma(X, \mathcal{L}_{\lambda+\mu})
\]

be the multiplication map. The proof of Lemma 3.1 in [3] gives the following result.

**Lemma 6.** Let \( \lambda \in \Omega^+ \) and let \( \mu = -w_\lambda \lambda \). Consider the modules \( V_\lambda^* \subset \Gamma(X, \mathcal{L}_\lambda) \) and \( V_\mu^* \subset \Gamma(X, \mathcal{L}_\mu) \). Then \( \mathcal{L}_{\lambda+\mu} V_\lambda^* \subset m_{\lambda, \mu}(V_\lambda^* \otimes V_\mu^*) \).

The induction will be performed using the closure of \( G \) orbits \( X_j \). We recall some basic facts about these varieties. Let \( I \subset \Delta \), set \( J = \Delta \setminus I \), \( \Delta(J) = \Delta_0 \cup \{z \in \Delta | z \notin J\} \), denote by \( G_J \) the semisimple part of the Levi associated to \( \Phi_{\Delta(J)} \), and let
Let \( P_J \) be the parabolic of \( G \) containing \( B \) whose Levi factor is \( G_J \). We denote by \( \sigma_j \) the restriction of \( \sigma \) to the subgroup \( G_J \). Let \( H_j \) be the normaliser of the subgroup of fixed points of \( \sigma_j \) in \( G_j \), and let \( X(J) \) be the wonderful compactification of the symmetric variety \( G_J/H_J \). Notice that the center of \( G_J \) acts trivially on \( X(J) \), so also \( P_J \) acts on this variety through its adjoint semisimple quotient. By [4] §5, we have an equivariant isomorphism

\[ X_J \cong G \times_{P_J} X(J). \]

In particular, we denote the subset \( 1 \cdot X(J) \subset X_J \) by \( F_J \) and the inclusion of \( F_J \) in \( X \) by \( J \).

We want to describe some properties of the inclusion \( J \) proved in [3] §2. Let \( \Lambda_J \) be the lattice of integral weights of \( \Phi_{\Delta J} \), and denote by \( \Lambda_J^+ \subset \Lambda_J \) the monoid of dominant weights with respect to \( \Delta(J) \). For \( \lambda \in \Lambda_J^+ \), we denote by \( V^{(J)}_\lambda \) the irreducible representation of \( G_J \) of highest weight \( \lambda \). Observe that the inclusion \( T_j \hookrightarrow T \) induces a map \( r_j : \Lambda \longrightarrow \Lambda_J \). Let \( \Omega_J \subset \Lambda_J \) be the sublattice generated by spherical weights with respect to \( (G_J, \sigma_j) \), and notice that by definition \( r_j(\Omega) \subset \Omega_J \).

We identify \( \text{Pic}(F_J) \) with the sublattice \( \Pi_J \) of \( \Lambda_J \). This sublattice contains \( \Omega_J \), and we denote by \( \mathcal{L}_J \) a line bundle of \( F_J \) associated to the weight \( \lambda \in \Lambda_J \).

For \( \tilde{\alpha} \in J \), we choose \( s_{\tilde{\alpha}, \tilde{\lambda}} \) to be a nonzero \( G_J \)-invariant section of \( \Gamma(F_J, \mathcal{L}_J) \).

Finally, we recall that we have \( \Gamma(F_J, \mathcal{L}_J) = \bigoplus_{\mu \in \Lambda_J^+} \mathcal{L}_J^{\hat{r}_J^{-\mu}}(V^{(J)}_\mu)^* \) for any \( \lambda \in \Pi_J \).

**Proposition 7** (Lemma 2.5 and Lemma 2.7 in [3]).

(i) If \( \lambda \in \Pi_J \subset \Lambda_J^+ \), then \( f_{\tilde{\mu}}^*(\mathcal{L}_J) \cong \mathcal{L}_J^{r_J(\mu)} \).

(ii) Up to rescaling the sections \( s_{\tilde{\alpha}, \tilde{\lambda}} \) by nonzero constant factors we have \( f_{\tilde{\mu}}(s_{\tilde{\alpha}}) = s_{\tilde{\alpha}, \tilde{\lambda}} \) for all \( \tilde{\alpha} \in J \).

(iii) Let \( \lambda \in \Pi_J \), \( \mu \in \Lambda_J^+ \) with \( \mu \leq \tilde{\alpha} \), and let \( \varphi \) be a (nonzero) lowest weight vector in \( s_{\tilde{\alpha}, \tilde{\lambda}}^*V^{(J)}_\mu \). Then \( f_{\tilde{\mu}}(\varphi) \) is a (nonzero) lowest weight vector in \( s_{\tilde{\mu}}^{\hat{r}_J^{-\mu}}(V^{(J)}_\mu)^* \subset \Gamma(F_J, \mathcal{L}_J^{r_J(\mu)}) \).

### 1. PROOF OF THEOREM 3

We now need to introduce some notations and to recall some results on the dominant order by Stembridge [7]. Given two dominant weights \( \lambda, \mu \in \Lambda_J^+ \), we write \( \lambda \to \mu \) if \( \lambda \) covers \( \mu \) with respect to \( \leq_s \); this means that \( \mu \leq_s \lambda \) and that if \( \eta \leq_s \lambda \), then either \( \eta = \lambda \) or \( \eta = \mu \). Recall that the set \( \Omega_J^+ \) of spherical weights is identified with the set of dominant weights of the restricted root system \( \Phi \). Also the longest element \( w_\lambda \) of the Weyl group of \( \Phi \) and the longest element \( w_\lambda \) of the Weyl group of \( \Phi \) act in the same way on \( \Omega_J^+ \).

In the following, supposing \( \Phi \) to be irreducible, we will denote by \( \tilde{\theta} \) the unique short dominant root. Since \( \Phi \) may be not reduced, i.e., of type \( BC_\ell \), we want to add some explanation about this type. We think \( \Phi \) of type \( BC_\ell \) as the union of a type \( B_\ell \) root system with square root lengths 1, 2 and of a type \( C_\ell \) root system with square root lengths 2, 4; the base \( \tilde{\Delta} = \{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_\ell \} \) is that of \( B_\ell \) with \( \tilde{\alpha}_\ell \) the unique simple root such that \( 2\tilde{\alpha}_\ell \in \Phi \), while the fundamental weights \( \tilde{\omega}_1, \ldots, \tilde{\omega}_\ell \) are those of \( C_\ell \); finally, \( \tilde{\theta} = \tilde{\omega}_1 \) is the shortest dominant root.
Given a weight $\eta \in \Omega$, we let $\text{supp}_\Delta(\eta)$ be the set of restricted simple roots $\check{\alpha}$ such that $\eta(\check{\alpha})$ is nonzero.

Lemma 8. Suppose that $\check{\Phi}$ is irreducible, and let $\lambda, \mu \in \Omega^+$ be such that $\lambda \rightarrow^\sigma \mu$ with $\text{supp}_\Delta(\lambda - \mu) = \Delta$. Then we have the following possibilities (the simple roots and the fundamental weights are numbered as in Bourbaki [1]):

1. $\check{\Phi}$ is of type $A_1$ and $\lambda = m\check{\alpha}_1$, $\mu = (m - 2)\check{\alpha}_1$, $m \geq 2$;
2. $\lambda = \underline{\lambda}$ (with $\underline{\lambda}$ the short dominant root of $\check{\Phi}$) and $\mu = 0$; further, this is the unique possibility if $\check{\Phi}$ is of type $BC_\ell$ with $\ell \geq 2$;
3. $\check{\Phi}$ is of type $B_2$ and $\lambda = \check{\alpha}_1 + \check{\alpha}_2$, $\mu = \check{\alpha}_1$;
4. $\check{\Phi}$ is of type $G_2$ and either $\lambda = \check{\alpha}_2$, $\mu = \check{\alpha}_1$ or $\lambda = \check{\alpha}_1 + \check{\alpha}_2$, $\mu = 2\check{\alpha}_1$;
5. $\check{\Phi}$ is of type $BC_1$ and $\lambda = m\check{\alpha}_1$, $\mu = (m - 1)\check{\alpha}_1$ with $m \geq 1$.

Proof. For reduced root systems, the cases (1), (2), (3), and (4) are, respectively, consequence of cases (a), (b), (c), and (d) of Theorem 2.8 in [7].

So suppose $\check{\Phi}$ is of type $BC_\ell$, and suppose also $\ell \geq 2$. Recall our convention that $\Phi$ being the union of $B_2$ and of $C_2$. In the sequel of this proof, we will add a $B$ to denote the corresponding object of $B_2$. For example, we have $\check{\omega}_2 = 2\check{\omega}_1$.

By $\text{supp}_\Delta(\lambda - \mu) = \Delta$ and $\check{\alpha} \in \Omega^+ \subset \Lambda_\mu$, we have that $\lambda$ and $\mu$ are two dominant weights of the root system $B_\ell$ satisfying the hypothesis of the lemma. So there are two possibilities corresponding to (2) and (3). In case (2), we have $\lambda = \underline{\lambda} = \check{\alpha}_1 + \cdots + \check{\alpha}_\ell = \check{\omega}_\ell = \check{\omega}_1$ and $\mu = 0$; this is our claim about type $BC_\ell$ in (2). In case (3), we have $\lambda = \check{\alpha}_2 + \check{\omega}_\ell$, $\mu = \check{\alpha}_1$; but this is impossible since $\check{\omega}_\ell \notin \Omega$.

For $\ell = 1$, the claim in (5) is trivial using $\check{\alpha}_1 = \check{\omega}_1$.

We can now prove Theorem 3.

Proof of Theorem 3. We proceed by induction on $\dim X$. If $X$ is a point, there is nothing to prove. Also if $\check{\Phi}$ is not simple, we can write $G = G_1 \times G_2$, $G_1$ and $G_2$ being proper subgroups, and there exist two involutions $\sigma_i : G_i \rightarrow G_i$, $i = 1, 2$, in such a way that $\sigma = \sigma_1 \times \sigma_2$ and $X = X(\sigma_1) \times X(\sigma_2)$; in this case Pic($X$) = Pic($X(\sigma_1)$) $\oplus$ Pic($X(\sigma_2)$) and, given $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) \in \text{Pic}(X)$, we have $\Gamma(X, \mathcal{L}) = \Gamma(X(\sigma_1), \mathcal{L}_1) \otimes \Gamma(X(\sigma_2), \mathcal{L}_2)$. So our claim follows by induction on the dimension. Hence we may assume that $X$ is simple (so $\check{\Phi}$ is irreducible) and the claim true for lower dimensional complete symmetric varieties. In what follows, given a weight $\eta \in \Omega^+$, we choose a lowest weight vector $\varphi_\eta$ in $V^*_\eta$.

We proceed in three steps.

First Step. Here we prove our claim assuming also $\text{supp}_\Delta(\lambda - \mu) \neq \Delta$. We use induction on dimension. Let $I = \text{supp}_\Delta(\lambda - \mu)$, $J = \Delta \setminus I$. Consider the variety $X_I$ and the fibration $\pi_I : X_I \rightarrow G/P_I$ with fiber $F_I$. Recall that $F_I$ is the wonderful compactification of the symmetric variety associated to $(G_I, \sigma_I)$. Given $\eta \in \Omega^+$, let $\psi(\eta)$ be a lowest weight vector in $V^*_\eta$.

Clearly, $\dim F_I < \dim X$, hence the claim is true for $F_I$. So there exists $n > 0$ such that $s^{n(\lambda) - (\mu)}(V^*_\eta)^* \subset B_n(F_I, r(\lambda))$. 


where \( r = r_j \) and \( B_\mu(F_j, r(\lambda)) \) is the part of degree \( n \) of the subring \( B(F_j, r(\lambda)) \) of \( \bigoplus_{n \geq 0} \Gamma(F_j, \tau_{n\lambda}) \) generated by \( (V_{r(\lambda)}^j)^* \). In particular, if \( \psi \) is a nonzero lowest weight vector of \( s^{(\lambda)-r(\mu)}(V_{r(\mu)}^j)^* \), then \( \psi^n \in B(F_j, r(\lambda)) \).

Consider the multiplication map

\[
m^j : (\Gamma(F_j, \tau_{r(\lambda)}))^{\otimes n} \supset ((V_{r(\mu)}^j)^*)^{\otimes n} \rightarrow B_\mu(F_j, r(\lambda)).
\]

Since \( G_J \) is linearly reductive and \( m^j \) is \( G_J \) equivariant, there exists a lowest weight vector \( \psi \in ( (V_{r(\mu)}^j)^*)^{\otimes n} \) such that \( m^j(\psi) = (s^{(\lambda)-r(\mu)}(V_{r(\mu)}^j)^*)^n \). We can write

\[
\psi = \sum_{h=1}^N x_{h,1} \varphi_{r(\lambda)} \otimes \cdots \otimes x_{h,n} \varphi_{r(\lambda)}
\]

for some \( x_{h,k} \in U(u^n_j) \subset U(g_j) \subset U(\hat{g}) \), the universal enveloping algebra of the positive unipotent part \( u^n_j \) of the Lie algebra \( g_j \) of \( G_J \). Consider now

\[
\varphi = \sum_{h=1}^N x_{h,1} \varphi_j \otimes \cdots \otimes x_{h,n} \varphi_j.
\]

One can show, as in the proof of Lemma 2.8 in [3], that \( \varphi \) is a lowest weight vector in \( (V_j^r)^{\otimes n} \) of weight \( -n\mu \). Let \( m \) be the multiplication map \( \Gamma(X, \tau_j)^{\otimes n} \supset (V_j^r)^{\otimes n} \rightarrow B_\mu(\lambda) \). Notice that \( m(\varphi) \) is a lowest weight vector of weight \( -n\mu \) provided it is different from zero. So if we show \( m(\varphi) \neq 0 \), we have finished. But

\[
j^j(\varphi) = m^j(\varphi),
\]

where the last equality follows since \( j^j(\varphi_j) = \varphi_{r(\lambda)} \) by Proposition 7.

**Second Step.** Now suppose \( \lambda = m\lambda' \) and \( \mu = m\mu' \) for a positive integer \( m \), and suppose also that \( \lambda', \mu' \) are such that \( \lambda' \rightarrow \mu' \) with \( \operatorname{supp} (\lambda' - \mu') = \overline{\Delta} \).

By Lemma 8 this happens in few situations and, for such values of \( \lambda' \) and \( \mu' \), we explicitly find the integer \( n \) satisfying the claim. We will prove, more precisely, that a power of the lowest weight vector in \( s^{m\lambda'-m\mu'}V_{m\mu'}^* \) lies in \( B_n(m\lambda') \). Our proof relies on Lemma 6 and on the result obtained in the first step. The different possibilities are the following (as above the numbering of simple roots and fundamental weights is as in Bourbaki [1]). In what follows we compare powers of weight vectors, any equation of this sort is intended up to nonzero scalar factor.

1. \( \tilde{\Phi} \) is of type \( A_1 \), \( \lambda' = k\tilde{\alpha}_1 \), \( \mu' = (k-2)\tilde{\alpha}_1 \), \( k \geq 2 \).
In this case we can take \( n = k \) (independently on \( m \)). Indeed we have the following identities:

\[
(s^{\lambda - \mu} \varphi) \cdot (s^{(k-2)\lambda} \varphi) = (s^{2k\mu} \varphi)(\varphi) = (s^{2k\mu} \varphi) = (s^{2k\mu} \varphi)^{k-2}.
\]

Now notice that \( s^{2k\mu} \varphi \in m_{\lambda, \tilde{\lambda}}(V^*_\tilde{\lambda} \otimes V^*_\lambda) \) by Lemma 6, hence our claim.

(2) \( \tilde{\Phi} \) is of type BC\( _1 \), \( \lambda' = k\tilde{\omega}_1 \) and \( \mu' = (k - 1)\tilde{\omega}_1 \).

Proceeding as in the previous case we can show that we can take \( n = 2k \).

(3) \( \lambda' = \tilde{\theta} \) and \( \mu' = 0 \).

In this case we can take \( n = 2 \). Indeed notice that \( -w_\Delta \lambda = -w_\Delta \tilde{\lambda} = \lambda \) being \( \tilde{\theta} \) the unique short (or shortest for BC\( _1 \)) dominant root, so we find \( s^{2\lambda} V^*_0 \subset m_{\lambda, \tilde{\lambda}}(V^*_\tilde{\lambda} \otimes V^*_\lambda) \) by Lemma 6. Hence \( (s^{2\lambda} \varphi)(\varphi)(\varphi) \in B_2(\lambda) \).

(4) The following three cases are still left out:

(i) \( \tilde{\Phi} \) of type BC\( _1 \), \( \lambda' = \tilde{\omega}_1 + \tilde{\omega}_t \), \( \mu' = \tilde{\omega}_t \);

(ii) \( \Phi \) of type G\( _2 \), \( \lambda' = \tilde{\omega}_2 \), \( \mu' = \tilde{\omega}_1 \);

(iii) \( \Phi \) of type G\( _2 \), \( \lambda' = \tilde{\omega}_1 + \tilde{\omega}_2 \) and \( \mu' = 2\tilde{\omega}_1 \).

In all these cases we notice that there exist natural numbers \( k > h > 0 \) such that \( h\mu' \geq k\mu' \) and supp\( _2(\lambda' - k\mu') \neq \tilde{\Delta} \). Indeed we can choose \( h = \ell \) and \( k = \ell + 2 \) in the first case, \( h = 2 \) and \( k = 3 \) in the second case, and \( h = 4 \) and \( k = 5 \) in the third case.

Then by what we have proved in the first step, there exists \( n > 0 \) such that \( s^{2\lambda} V^*_0 \subset B_2(h\lambda) \subset B_2(\lambda) \).

Notice also that \( -w_\Delta (\lambda) = \lambda \) so by Lemma 6 we have \( s^{2\lambda} V^*_0 \subset B_2(\lambda) \).

Hence

\[
s^{2nk} \varphi = \left( s^{(h\lambda - k\mu)} \varphi \right)^{k \cdot (s^{2\lambda} \varphi)} \in B_{2nk}(\lambda).
\]

**Third Step.** Conclusion. Let \( \lambda = \lambda_0 \xrightarrow{\sigma} \lambda_1 \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} \lambda_m = \mu \) be a sequence of covers from \( \lambda \) to \( \mu \). We argue by induction on \( m \). If \( m = 1 \), then the claim is contained in the result of the first step in the case supp\( _2(\lambda - \mu) \neq \tilde{\Delta} \) and in the result of the second step in the case supp\( _2(\lambda - \mu) = \tilde{\Delta} \). So assume \( m > 1 \) and that the statement is true for \( m - 1 \). Let \( \nu = \lambda_{m-1} \), then by induction there exists \( n_1 \) such that \( s^{n_1(\lambda - \nu)} V^*_n \subset B_{n_1}(\lambda) \). Hence the ring generated by this submodule is contained in \( B_\nu(\lambda) \) or more explicitly \( s^{n_1(\lambda - \nu)} B_\nu(n_1 \nu) \subset B_{n_1}(\lambda) \) for any natural \( n \). Now we consider the cover \( \nu \xrightarrow{\sigma} \mu \). By using what we proved in first step, and in the second step, we have that there exists \( n_2 > 0 \) such that \( s^{n_2(\lambda - \nu)} V^*_n \subset B_{n_2}(\lambda) \) and multiplying by \( s^{n_2(\lambda - \nu)} V^*_n \) and using the previous inclusion we obtain \( s^{n_2(\lambda - \nu)} V^*_n \subset B_{n_2}(\lambda) \) where we have set \( n = n_1 n_2 \).

As a final remark we notice that one can easily extend Theorem 3 from weights in \( \Omega^+ \) (i.e., spherical weights) to weights in \( \Pi^+ \) (i.e., quasi spherical weights). For example a line of proof may be adapted from [3] (see the end of proof of Theorem A starting in the first paragraph of p. 109 in [3]).
REFERENCES