Esercizio I° compitino 18/12/2001

N. 1

$$f(x,y) = xy\, e^{-4x^2-y^2}$$

a) determinare p.ti critici e studiarne la natura

$$f_x = (y - 8x^2 y)\, e^{-4x^2-y^2}$$

$$f_y = (x - 2xy^2)\, e^{-4x^2-y^2}$$

$$f_{xx} = (-16xy - 8xy + 64x^3 y)\, e^{-4x^2-y^2}$$

$$f_{xy} = (1 - 8x^2 - 2y^2 + 16x^2y^2)\, e^{-4x^2-y^2}$$

$$f_{yy} = (-4xy - 2xy + 4xy^3)\, e^{-4x^2-y^2}$$

p.ti critici

$$\begin{cases} y(1-8x^2) = 0 \\ x(1-2y^2) = 0 \end{cases}$$

$$\Rightarrow (0,0),\ \pm\left(\tfrac{1}{\sqrt{8}}, \tfrac{1}{\sqrt{2}}\right),\ \pm\left(\tfrac{1}{\sqrt{8}}, -\tfrac{1}{\sqrt{2}}\right)$$

$$\begin{cases} f_{xx} = 8xy(-3+8x^2)\, e^{-4x^2-y^2} \\ f_{xy} = (1-8x^2-2y^2+16x^2y^2)\, e^{-4x^2-y^2} \\ f_{yy} = 2xy(-3+2y^2)\, e^{-4x^2-y^2} \end{cases}$$

simmetria per $(x,y) \longrightarrow (-x,-y)$

$$f_{xx}\left(\pm\left(\tfrac{1}{\sqrt{8}},\tfrac{1}{\sqrt{2}}\right)\right) = -4/e$$

$$f_{xy}\left(\pm\left(\tfrac{1}{\sqrt{8}},\tfrac{1}{\sqrt{2}}\right)\right) = 1/e$$

$$f_{yy}\left(\pm\left(\tfrac{1}{\sqrt{8}},\tfrac{1}{\sqrt{2}}\right)\right) = -1/e$$

$$f_{xx}\left(\pm\left(\tfrac{1}{\sqrt{8}},-\tfrac{1}{\sqrt{2}}\right)\right) = 4/e$$

$$f_{xy}\left(\pm\left(\tfrac{1}{\sqrt{8}},-\tfrac{1}{\sqrt{2}}\right)\right) = 1/e$$

$$f_{yy}\left(\pm\left(\tfrac{1}{\sqrt{8}},-\tfrac{1}{\sqrt{2}}\right)\right) = 1/e$$

$$D^2f\left(\pm\left(\tfrac{1}{\sqrt{8}},\tfrac{1}{\sqrt{2}}\right)\right) = \begin{pmatrix} -4/e & 1/e \\ 1/e & -1/e \end{pmatrix}$$

$$Jf\left(\pm\left(\tfrac{1}{\sqrt{8}},\tfrac{1}{\sqrt{2}}\right)\right) = \frac{3}{e^2}$$

$$\mathrm{Tr}\, Df\left(\pm\left(\tfrac{1}{\sqrt{8}},\tfrac{1}{\sqrt{2}}\right)\right) = -\frac{5}{e}$$

λ_1, λ_2 autovalori $\Rightarrow \lambda_1+\lambda_2<0,\ \lambda_1 \cdot \lambda_2 > 0$

$\Rightarrow \lambda_1, \lambda_2 < 0$

$$Jf\left(\pm\left(\tfrac{1}{\sqrt{8}},-\tfrac{1}{\sqrt{2}}\right)\right) = \frac{3}{e^2} \quad \Rightarrow \quad \lambda_1, \lambda_2 > 0$$

$$\mathrm{Tr}\, D^2f\left(\pm\left(\tfrac{1}{\sqrt{8}},-\tfrac{1}{\sqrt{2}}\right)\right) = \frac{5}{e}$$

quindi $\pm\left(\frac{1}{\sqrt8},\frac{1}{\sqrt2}\right)$ pt' max loc.

$\pm\left(\frac{1}{\sqrt8},-\frac{1}{\sqrt2}\right)$ pt' min loc.

inoltre

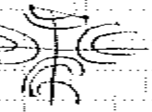
$$f \longrightarrow 0 \qquad (x,y)\to\infty$$

$\Rightarrow$ i pti visti sono di massimo e minimo globali

$$D^2f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\lambda_1+\lambda_2 = 0 \qquad \lambda_1\lambda_2 = -1$

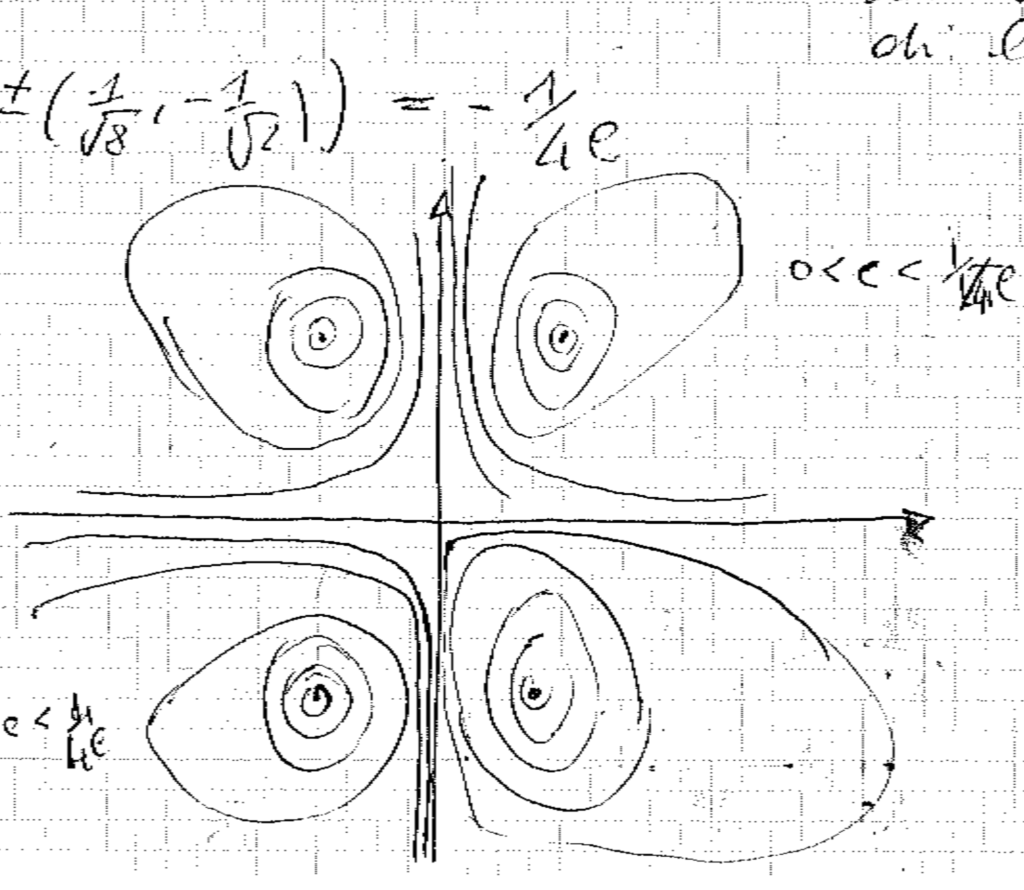
$\lambda_1 = 1, \quad \lambda_2 = -1$

punto di sella, quindi la forma è del tipo 

$$f\left(\pm\left(\frac{1}{\sqrt8},\frac{1}{\sqrt2}\right)\right) = \frac{1}{4e}$$

b) si tracciano le curve di livello

$$f\left(\pm\left(\frac{1}{\sqrt8},-\frac{1}{\sqrt2}\right)\right) = -\frac{1}{4e}$$



SIMMETRIE: $f(x,-y) = -f(x,y) = f(-x,y)$
permette di ridurre lo studio su $\{x \cdot y > 0\}$
$f(x,y) = f(-x,-y)$ permette di ridurre ulteriormente lo studio a $\{x, y > 0\}$

- Le curve di livello sono curve chiuse?

Sia $c \in \,]0, \frac{1}{4e}[$.

Ragionando in termini generali

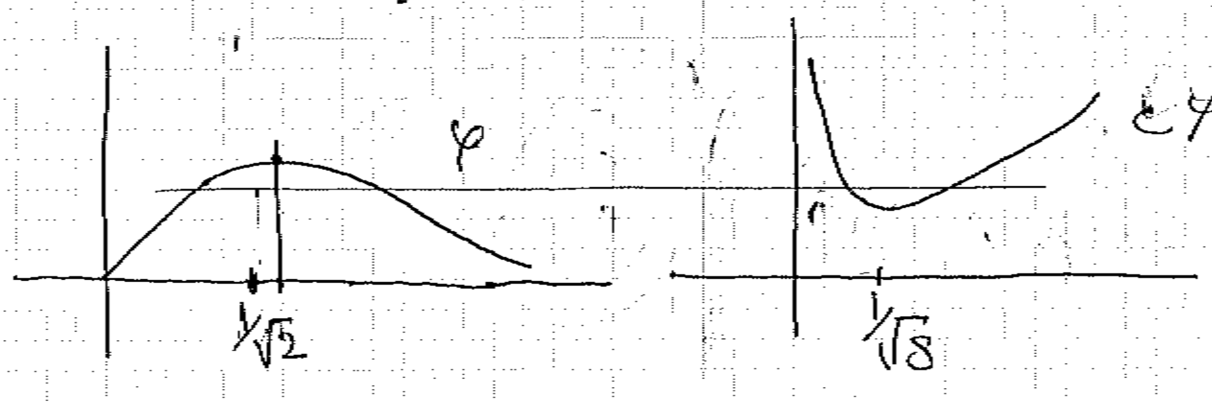
$f^{-1}(c)$ è chiuso, non vuoto e limitato
non vuoto perché $\operatorname{Im} f = [-\frac{1}{4e}, \frac{1}{4e}]$
limitato perché se non lo fosse
$\exists (x_n) \subset f^{-1}(c) \quad x_n \to \infty \quad , \quad f(x_n) = c$
assurdo essendo $f \to 0 \quad x \to \infty$

inoltre $f^{-1}(c)$ non interseca i punti critici ovvero

$$\nabla f(x) \neq 0 \quad \forall x \in f^{-1}(c)$$

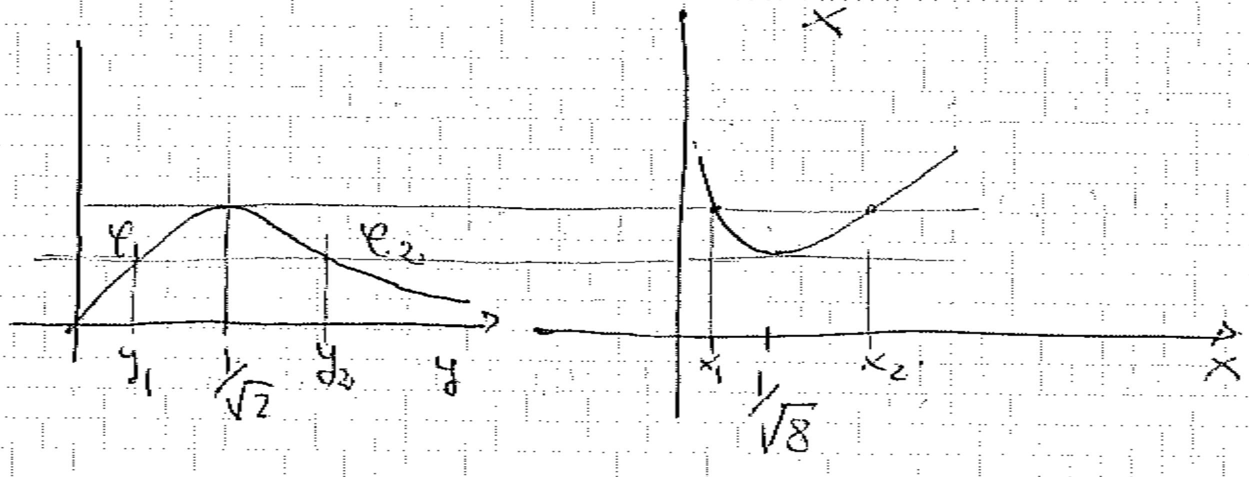
quindi $f^{-1}(c)$ è una curva regolare chiusa.

Non sappiamo però la sua forma, quindi studiamo



ovvero $xy\, e^{-4x^2-y^2} = c \iff$

$$\varphi(y) = y\, e^{-y^2} = c\, \frac{e^{4x^2}}{x} = c\, \psi(x)$$



$$y_2(x) = \varphi_2^{-1}\big(c\, \psi(x)\big) \qquad x_1 \le x \le x_2$$

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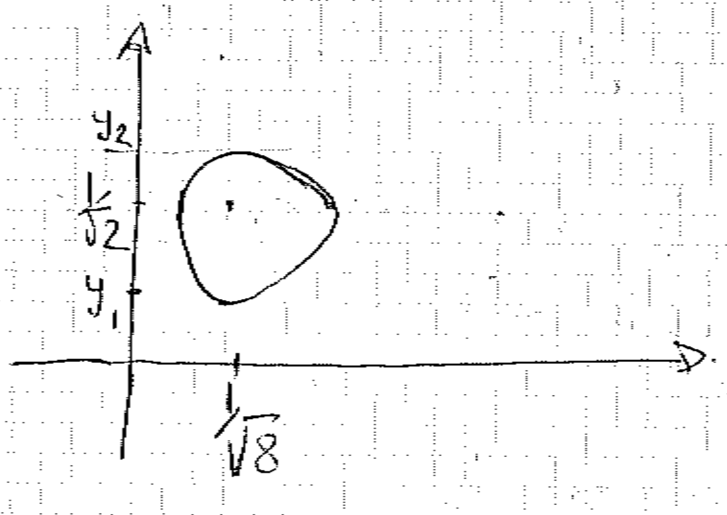


le curve sono tutte chiuse

c) Si calcoli lo sviluppo di Taylor di ordine 4 di f in zero.

Poniamo $h(x,y) = 4x^2+y^2$

$$e^{-h(x,y)} = 1 - h(x,y) + o(h(x,y))$$

$$f(x,y) = xy - xy\,h(x,y) + xy\,o(h(x,y))$$

$$xy\,o(h(x,y)) = o(h(x,y))$$

$$|xy\,o(h(x,y))| < \varepsilon\,(4x^2+y^2)\,|xy| \leq 4\,\varepsilon\,|(x,y)|^4$$

$$f(x,y) = xy - xy^3 - 4x^3y + o(|(x,y)|^4)$$

d) $D = \{ h \leq 1/4 \}$, α

$\max_D f$, $\min_D f$

pti max e min $\notin D$

$0 \in D$ non è max o min loc.

quindi pti max e min $\in \partial D$

$$\partial D = \{ h = 1/4 \}$$

$$f_{|\partial D}(x,y) = xy\,e^{-1/4}$$

$$y = \pm\sqrt{1/4 - 4x^2}$$

$$\varphi(x) = x\sqrt{\tfrac{1}{4} - 4x^2} = \frac{x}{2}\sqrt{1-16x^2}$$

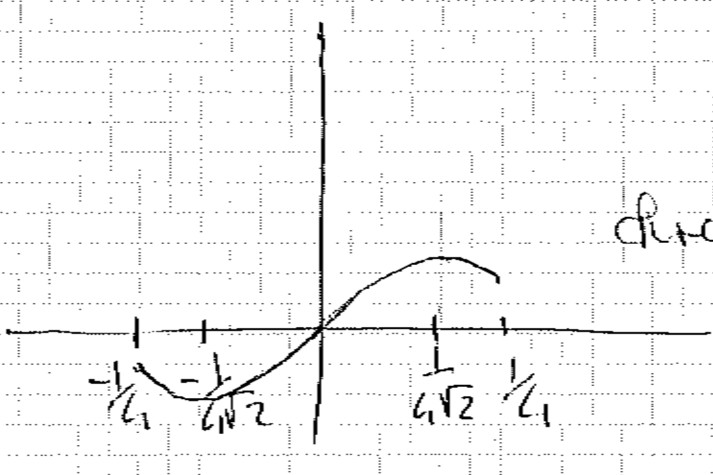
ove $|x| \leq \frac{1}{4}$

$$\varphi'(x) = \frac{1}{2}\left(\sqrt{1-16x^2} - \frac{16x^2}{\sqrt{1-16x^2}}\right)$$

$$= \frac{1}{2\sqrt{1-16x^2}}\left(1-16x^2-16x^2\right)$$

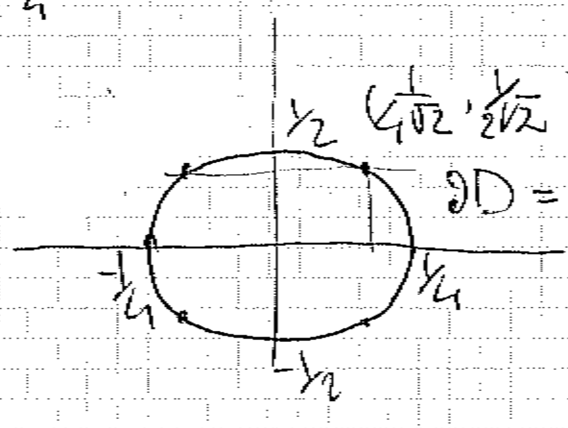
$$= \frac{1-32x^2}{2\sqrt{1-16x^2}}$$

$$\varphi'(x) = 0 \iff x = \pm\frac{1}{4\sqrt{2}}$$



dire dispari

$$\begin{cases} f\left(x, \sqrt{\tfrac{1}{4} - 4x^2}\right) & \text{ha massimo in } x = \frac{1}{4\sqrt{2}} \\ |x| \leq 1/4 \end{cases}$$



$\partial D = \{4x^2 + y^2 = \frac{1}{4}\}$

per la simmetria avremo che

$$f\left(\left(\frac{1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)\right) = f\left(\left(-\frac{1}{4\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)\right)$$
$$= \max_D f = \frac{1}{16} e^{-1/4};$$

$$f\left(\left(-\frac{1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)\right) = f\left(\left(\frac{1}{4\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)\right) = -\frac{1}{16} e^{-1/4}$$
$$= \min_D f.$$