- Sia $D = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \geq 1\}$.

Si determinino tutti i $p, q > 0$ tali che si abbia

$$\int_D \frac{1}{|x|^p + |y|^q}\, dx\, dy < +\infty$$

— ^ —

Sia $E = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0 \text{ e } x + y \geq 1\}$.

Allora si osserva che

$$\int_D \frac{1}{|x|^p + |y|^q}\, dx\, dy = 4 \int_E \frac{1}{x^p + y^q}\, dx\, dy$$

$$x^p = \xi^2, \quad y^q = \eta^2$$

$$F(\xi, \eta) = \left(\xi^{2/p}, \eta^{2/q}\right) \Longrightarrow$$

$$JF(\xi, \eta) = \frac{4}{pq}\, \xi^{\frac{2}{p}-1} \eta^{\frac{2}{q}-1}$$

$$\tilde{E} = \{(\xi, \eta) : \xi, \eta \geq 0,\ \xi^{2/p} + \eta^{2/q} \geq 1\}$$

$$\int_E \frac{1}{x^p + y^q}\, dx\, dy = \int_{\tilde{E}} \frac{4}{pq} \frac{\xi^{\frac{2}{p}-1} \eta^{\frac{2}{q}-1}}{\xi^2 + \eta^2}\, d\xi\, d\eta$$

$$\xi = \rho \cos\varphi \qquad \eta = \rho \sin\varphi$$

$$F = \left\{(\rho, \varphi) \in]0, +\infty[\times]0, \tfrac{\pi}{2}[: \rho^{\frac{2}{p}} \cos^{\frac{2}{p}}\varphi + \rho^{\frac{2}{q}} \sin^{\frac{2}{q}}\varphi \geq 1\right\}$$

$$\int_D \frac{1}{|x|^p+|y|^q}\,dx\,dy = \frac{16}{pq}\int_F \rho^{\frac{2}{p}+\frac{2}{q}-3}(\cos\varphi)^{\frac{2}{p}-1}(\sin\varphi)^{\frac{2}{q}-1}\,d\rho\,d\varphi$$

$$(\rho,\varphi)\in F \implies 1 \leq \rho^{\frac{2}{p}}\cos^{\frac{2}{p}}\varphi + \rho^{\frac{2}{q}}\sin^{\frac{2}{q}}\varphi \leq$$

$$\leq \max\{\rho^{\frac{2}{p}},\rho^{\frac{2}{q}}\}\cdot\max\{(\cos\varphi)^{\frac{2}{p}}+(\sin\varphi)^{\frac{2}{q}} : \varphi\in[0,\tfrac{\pi}{2}]\}$$

$$= \psi_{p,q}(\rho)\cdot\sigma_{p,q}$$

$\psi_{p,q} : [0,+\infty[\to [0,+\infty[$ continua

quindi $\rho \in \psi_{p,q}^{-1}([\frac{1}{\sigma_{p,q}},+\infty[)$.

poiché $\psi_{p,q}(\rho) > 0 \ \forall \rho>0$, $\psi_{p,q}(0)=0$

e $\psi_{p,q}(\rho) \to +\infty$ per $\rho\to+\infty \implies$

$$\exists \min \psi_{p,q}^{-1}([\tfrac{1}{\sigma_{p,q}},+\infty[) = \lambda_{p,q} > 0$$

$$\implies F \subset \{(\rho,\varphi)\in\mathbb{R}^2 : 0<\varphi<\tfrac{\pi}{2},\ \rho\geq\lambda_{p,q}\}$$

quindi

$$\int_F \rho^{\frac{2}{p}+\frac{2}{q}-3}(\cos\varphi)^{\frac{2}{p}-1}(\sin\varphi)^{\frac{2}{q}-1}\,d\rho\,d\varphi \leq$$

$$\leq \int_{\lambda_{p,q}}^{+\infty}\left(\int_0^{\pi/2}(\cos\varphi)^{\frac{2}{p}-1}(\sin\varphi)^{\frac{2}{p}-1}\,d\varphi\right)\rho^{\frac{2}{p}+\frac{2}{q}-3}\,d\rho$$

$$\int_0^{\pi/2} (\cos\varphi)^{\frac{2}{p}-1} (\sin\varphi)^{\frac{2}{q}-1} d\varphi =$$

$$= \int_0^{\pi/4} \frac{(\cos\varphi)^{\frac{2}{p}-1}}{(\sin\varphi)^{1-\frac{2}{q}}} d\varphi + \int_{\pi/4}^{\pi/2} \frac{(\sin\varphi)^{\frac{2}{q}-1}}{(\cos\varphi)^{1-\frac{2}{p}}} d\varphi$$

$$= I + II$$

$I < +\infty$ in quanto $\sin\varphi = \varphi + o(\varphi)$

$II < +\infty$ in quanto $\cos\varphi = \frac{\pi}{2} - \varphi + o\left(\frac{\pi}{2} - \varphi\right)$

$$\int_{\lambda_{p,q}}^{+\infty} \frac{1}{\rho^{3-\frac{2}{p}-\frac{2}{q}}} d\rho < +\infty \quad \text{se} \quad 3 - \frac{2}{p} - \frac{2}{q} > 1$$

ovvero se

$\frac{1}{p} + \frac{1}{q} < 1$ si ha

convergenza;

consideriamo ora

$$\rho^{\frac{2}{p}} (\cos\varphi)^{\frac{2}{p}} + \rho^{\frac{2}{q}} (\sin\varphi)^{\frac{2}{q}} \geq$$

$$\geq \min\left\{\rho^{\frac{2}{p}}, \rho^{\frac{2}{q}}\right\} \min\left\{(\cos\varphi)^{\frac{2}{p}} + (\sin\varphi)^{\frac{2}{q}} : \varphi \in \left[0, \frac{\pi}{2}\right]\right\}$$

$$= \phi_{p,q}(\rho) \, \sigma_{p,q} \geq 1$$

$$\iff \rho \in \phi_{p,q}^{-1}\left(\left[\frac{1}{\sigma_{p,q}}, +\infty\right[\right)$$

a priori non so se $\phi_{p,q}^{-1}\left(\left[\frac{1}{\delta_{p,q}}, +\infty\right[\right)$ è un intervallo, ma la condizione

$\phi_{p,q}(\rho) \longrightarrow +\infty \quad \rho \to +\infty$

$\Longrightarrow \exists \ \mu_{p,q} > 0$ tale che

$$[\mu_{p,q}, +\infty[\ \subset \phi_{p,q}^{-1}\left(\left[\frac{1}{\delta_{p,q}}, +\infty\right[\right)$$

quindi $\forall \rho \geq \mu_{p,q} \quad \forall \varphi \in [0, \frac{\pi}{2}]$

si ha $\rho^{\frac{2}{p}}(\cos\varphi)^{\frac{2}{p}} + \rho^{\frac{2}{q}}(\sin\varphi)^{\frac{2}{q}} \geq 1 \Longrightarrow$

$[\mu_{p,q}, +\infty[\ \subset F \Longrightarrow$

$$\int_{\mu_{p,q}}^{+\infty}\left(\int_0^{\pi/2}(\cos\varphi)^{\frac{2}{p}-1}(\sin\varphi)^{\frac{2}{q}-1}\,d\varphi\right)\rho^{\frac{2}{p}+\frac{2}{q}-3}\,d\rho$$

$$\leq \int_F \rho^{\frac{2}{p}+\frac{2}{q}-3}(\cos\varphi)^{\frac{2}{p}-1}(\sin\varphi)^{\frac{2}{q}-1}\,d\varphi\,d\rho$$

con analogo ragionamento il primo integrale diverge per $\frac{1}{p}+\frac{1}{q} \geq 1$