, Provare che

$$\int_{\{(x,y):\ x^2+y^2<1,\ y>\max\{\lambda x,\ \lambda^{-1}x\}\}} \frac{1}{y-x}\,dx\,dy < +\infty$$

per ogni $\lambda > 1$ e calcolare

il limite di tale valore per $\lambda \to 1$.

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Poniamo

$$D_\lambda = \{(x,y):\ x^2+y^2<1,\ y>\max\{\lambda x, \lambda^{-1}x\}\}$$

$$= \{(x,y):\ x^2+y^2<1,\ y>\lambda x \text{ se } x \geq 0,\ y > \lambda^{-1}x \text{ se } x \leq 0\}$$

coord. polari

$$\tilde{D}_\lambda = \{(\rho,\varphi):\ 0<\rho<1,\ \sin\varphi > \lambda\cos\varphi \text{ se } -\frac{\pi}{2}<\varphi<\frac{\pi}{2},\ \sin\varphi > \lambda^{-1}\cos\varphi \text{ se } \frac{\pi}{2}<\varphi<\frac{3}{2}\pi\}$$

$$=]0,1[\times \{\varphi \in]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[:\ tg\,\varphi > \lambda\}$$

$$\cup\ ]0,1[\times \{\varphi \in]\tfrac{\pi}{2}, \tfrac{3}{2}\pi[:\ tg\,\varphi < \tfrac{1}{\lambda}\}$$

$$tg\,\varphi = tg(\varphi - \pi) < \frac{1}{\lambda}$$

$$-\frac{\pi}{2} < \varphi - \pi < \frac{\pi}{2} \implies \frac{\pi}{2} < \varphi < \pi + \operatorname{arctg}\frac{1}{\lambda}$$

quindi

$$\tilde{D}_\lambda =]0,1[\times \left( ]\operatorname{arctg}\lambda, \frac{\pi}{2}[\cup]\frac{\pi}{2}, \pi + \operatorname{arctg}\frac{1}{\lambda}[\right)$$

$$\int_{D_\lambda} \frac{1}{y-x}\,dx\,dy = \int_0^1 d\rho \int_{\operatorname{arctg}\lambda}^{\pi/2} \frac{1}{\sin\varphi - \cos\varphi}\,d\varphi +$$

$$+ \int_0^1 d\rho \int_{\frac{\pi}{2}}^{\pi + \operatorname{arctg}\frac{1}{\lambda}} \frac{1}{\sin\varphi - \cos\varphi}\,d\varphi$$

$\frac{\pi}{4} < \operatorname{arctg}\lambda < \varphi < \frac{\pi}{2} \Rightarrow (\sin\varphi - \cos\varphi)' =$

$= \cos\varphi + \sin\varphi > 0$

e $\sin\varphi - \cos\varphi > \cos(\operatorname{arctg}\lambda)\,(\lambda - 1) > 0$

per ogni $\varphi \in]\operatorname{arctg}\lambda, \frac{\pi}{2}[$

$\Rightarrow \int_{\operatorname{arctg}\lambda}^{\pi/2} \frac{1}{\sin\varphi - \cos\varphi}\,d\varphi < +\infty$

$\frac{\pi}{2} < \varphi < \pi + \operatorname{arctg}\frac{1}{\lambda} < \frac{3}{2}\pi \Rightarrow \sin\varphi - \cos\varphi > 0$

per ogni $\varphi \in]\frac{\pi}{2}, \pi]$ e $(\sin\varphi - \cos\varphi)' =$

$= \cos\varphi + \sin\varphi > 0$ per $\varphi \in]\pi, \frac{3}{2}\pi[$

$\Rightarrow \sin\varphi - \cos\varphi > \cos(\pi + \operatorname{arctg}\frac{1}{\lambda})\cdot$

$\cdot [\operatorname{tg}(\pi + \operatorname{arctg}\frac{1}{\lambda}) - 1]$

$= -\cos(\operatorname{arctg}\frac{1}{\lambda})\,(\frac{1}{\lambda} - 1)$

$= (1 - \frac{1}{\lambda})\cos(\operatorname{arctg}\frac{1}{\lambda}) > 0$

per ogni $\varphi \in [\pi, \pi + \operatorname{arctg}\frac{1}{\lambda}[$

quindi $\sin\varphi - \cos\varphi$ è positiva

e continua su $\left[\frac{\pi}{2}, \pi + \operatorname{arctg}\frac{1}{\lambda}\right]$

$$\Rightarrow \int_{\frac{\pi}{2}}^{\pi + \operatorname{arctg}\frac{1}{\lambda}} \frac{1}{\sin\varphi - \cos\varphi}\, d\varphi < +\infty$$

$$\Rightarrow \iint_{D_\lambda} \frac{1}{y-x}\, dx\, dy < +\infty \qquad \forall \lambda > 1$$

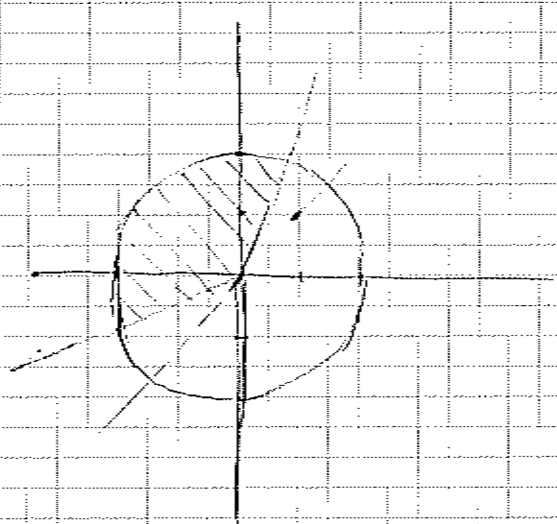
si ottiene che tale integrale

potrebbe anche calcolarsi tramite

la sostituzione trigonometrica

$y = tg\frac{\alpha}{2}$; si ottiene inoltre

che l'integranda è sempre positiva



$x \geq 0 \qquad y - x > y - \lambda x > 0$

$x \leq 0 \qquad y - x = y + |x| \geq y + \lambda^{-1}|x| = y - \lambda^{-1} x > 0$

$$\int_{D_\lambda} \frac{1}{y-x}\, dx\,dy \geq \int_{\operatorname{arctg} \lambda}^{\pi/2} \frac{1}{(\cos\varphi)\,(\operatorname{tg}\varphi - 1)}\, d\varphi$$

$$\operatorname{tg}\varphi = 1 + 2\left(\varphi - \frac{\pi}{4}\right) + o\left(\varphi - \frac{\pi}{4}\right) \qquad \varphi \to \frac{\pi}{4}$$

$$\int_{\operatorname{arctg} \lambda}^{\pi/2} \frac{1}{(\cos\varphi)\, 2\left(\varphi - \frac{\pi}{4}\right)} \cdot (1 + o(1))\, d\varphi \underset{\lambda \to 1^+}{\longrightarrow} +\infty$$

in quanto $\operatorname{arctg} \lambda \longrightarrow \frac{\pi}{4} \quad \lambda \to 1^+$