An introduction to the geometries of Heisenberg groups

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Contents

Chapter 1

Introduction and first notions

1.1 The abstract Heisenberg group

There is a vast literature on the Heisenberg group and its relations with different areas of Mathematics. Fortunately, it is possible to present it as an abstract model of finite dimensional metric space that contains all the characterizing features.

The best way to present this model is to make a comparison with the classical Euclidean space. *The Heisenberg group could be seen, broadly speaking, as a non-commutative version of the classical Euclidean space.*

Let us list the classical objects that we use in the Euclidean space

1. Sum between elements
2. Mutiplication by a scalar number
3. Distance that is compatible with sum and multiplication by a scalar number
4. Partial derivatives
5. Lebesgue measure that is compatible with the previous objects

We will present the Heisenberg group following this list, since this space has an analogous list of objects, that are still compatible with respect to each other. We will see in the sequel, in which sense this compatibility is meant.

1.2 The underlying space

The Heisenberg group as a set can be seen just as a linear space S of odd dimension $2n+1$, where n is a positive integer. This space is the direct sum of two fixed subspaces S_1 and S_2 of S, where

$$S = S_1 \oplus S_2, \quad \dim S_1 = 2n \quad \text{and} \quad \dim S_2 = 1\,.$$

Recall that the direct sum $S = S_1 \oplus S_2$ means that $S_1 \cap S_2 = \{0\}$. We will denote by x an element of S and by s and t general elements of S_1 and S_2, respectively. Writing $x = s + t$ the fact that $s \in S_1$ and $t \in S_2$ will be understood. This abstract decomposition will allow us to introduce the above listed properties.

1.3 Group operation

The group operation in S is given by a non-degenerate skew-symmetric bilinear form

$$\omega : S_1 \times S_1 \longrightarrow S_2$$

where the non-degeneracy condition coincides with the requirement that

$$\omega(s, \cdot) \quad \text{is a nonvanishing linear mapping whenever } s \neq 0.$$

For any two elements $x, x' \in S$ with $x = s + t$ and $x' = s' + t'$ we define the operation

$$xx' = x + x' + \omega(s, s') .$$

Let us show that this formula defines a noncommutative group operation.

1.3.1 Non-commutativity

The skew-symmetry and non-degeneracy of ω implies that this operation is noncommutative. In fact, for a fixed nonvanishing $s \in S_1$, since $\omega(s, \cdot)$ is a non vanishing linear map, there exists an element $s' \in S_1$ such that $\omega(s, s') \neq 0$. We have

$$ss' = s + s' + \omega(s, s') \neq s's = s' + s - \omega(s, s')$$

1.3.2 Associativity

Let us consider $x, y, v \in S$ with decomposition $x = s_1 + t_1$, $y = s_2 + t_2$ and $u = s_3 + t_3$, with $s_i \in S_1$ and $t_i \in S_2$. We consider

$$(xy)u = [x + y + \omega(s_1, s_2)]u = x + y + u + \omega(s_1, s_2) + \omega(s_1 + s_2, s_3)$$

and

$$x(yu) = x[y + u + \omega(s_2, s_3)] = x + y + u + \omega(s_2, s_3) + \omega(s_1, s_2 + s_3)$$

that coincide since ω is bilinear.

1.3.3 Unit element and inverse element.

Of course, the origin is the unit element

$$x\,0 = x + 0 + \omega(x, 0) = x = 0\,x \quad \text{for all} \quad x \in S.$$

Since ω is skew-symmetric, for all $x = s + t$ we have

$$x(-x) = x - x + \omega(s, -s) = 0$$

then the inverse of x is the opposite element $-x$, hence

$$x^{-1} = -x \quad \text{for all} \quad x \in S.$$

1.3.4 Comments.

We have defined a smooth analytic group operation on S, that is obviously a differentiable manifold, so we have constructed a *noncommutative Lie group*.

1.4 Dilations

Dilations in S replace the multiplication by a positive scalar number. For any $r > 0$ and $x = s + t$, we define

$$\delta_r x = rs + r^2 t \quad \text{and} \quad \delta_r \colon S \longrightarrow S.$$

Why we have this definition of dilation? This notion is compatible with the group operation in the following sense

$$\delta_r(x \cdot y) = (\delta_r x)\,(\delta_r y)$$

for all $x, y \in S$. In fact, defining $x = s + t$ and $y = w + z$, we have

$$\begin{aligned}
&\delta_r(xy) = \delta_r(s + w + t + z + \omega(s, w)) = rs + rw + r^2 t + r^2 z + r^2 \omega(s, w) \\
&rs + rw + r^2 t + r^2 z + \omega(rs, rw) = (\delta_r x)(\delta_r y).
\end{aligned}$$

In the terminology of Lie group theory, this property corresponds to say that δ_r is a *Lie group homomorphism of S*. By a simple verification, we get the one-paramter property

$$\delta_r \circ \delta_{r'} = \delta_{rr'} \quad \text{for all} \quad r, r' > 0.$$

This shows that δ_r is a Lie group isomorphism, namely, it is invertible and $(\delta_r)^{-1} = \delta_{1/r}$.

1.5 Distance and homogeneous norm.

A good distance in S is those continuous distance $d \colon S \times S \longrightarrow [0, +\infty)$ such that the following additional properties hold

1. $d(\delta_r x, \delta_r y) = r\, d(x, y)$
2. $d(xy, xu) = d(y, u)$

for all $x, y, u \in S$ and $r > 0$. We say that d with these properties is a *homogeneous distance*.

We can construct many homogeneous distances in S. Let us see the simplest example.

We fix a scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|\cdot|$ on S such that S_1 and S_2 are orthogonal. We have the minimal constant $C_\omega > 0$ such that

$$|\omega(s, s_1)| \le C_\omega\, |s|\, |s_1| \quad \text{for all} \quad s, s_1 \in S_1.$$

We choose any positive number $\alpha \le 2/C_\omega$ and define for $x = s + t$ the following function

$$\|x\| = \max\{|s|, \sqrt{\alpha |t|}\}.$$

It is clear that by definition

$$\|\delta_r x\| = \max\{|rs|, \sqrt{\alpha |r^2 t|}\} = r\, \|x\|.$$

A more interesting property to prove is the following triangle inequality

$$\|xy\| \le \|x\| + \|y\| .$$

Let us set $x = s + t$ and $y = w + z$, hence in the case $\|xy\| = |s + w|$, we trivially have

$$\|xy\| \le |s| + |w| \le \|x\| + \|y\| .$$

In the case $\|xy\| = \sqrt{\alpha |t + z + \omega(s, w)|}$, we have

$$\|xy\|^2 \le \alpha|t| + \alpha|z| + \alpha C_\omega |s|\,|w| \le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$

that concludes the proof of the triangle inequality.

Definition 1.5.1 We say that a continuous function $\|\cdot\| : S \longrightarrow [0, +\infty)$ that satisfies

1. $\|x\| = 0$ iff $x = 0$
2. $\|\delta_r x\| = r\,\|x\|$
3. $\|xy\| \le \|x\| + \|y\|$

is a *homogeneous norm*.

We have constructed an infinite family of homogeneous norms parametrized by α. The point is that any homogeneous norm defines a homogeneous distance as follows

$$d(x, y) := \|x^{-1}y\| \quad \text{for all} \quad x, y \in S .$$

Conversely, if I have given a homogeneous distance d on S, then setting

$$\|x\| := d(x, 0)$$

a homogeneous norm is also defined. This is a simple verification. It is important to compare this distance with the Euclidean distance on S that arises from the fixed scalar product.

1.5.1 Comparison with the Euclidean distance.

Let us consider $|x|, |y| < \nu$ for a fixed $\nu > 0$, where $x = s + t$ and $y = w + z$ and consider

$$|t - z| \le |-t + z - \omega(s, w)| + C_\omega\,\nu|s - w| \le \frac{\|x^{-1}y\|^2}{\alpha} + C_\omega\,\nu\|x^{-1}y\|$$

We denote by $B_E(x, r)$ the Euclidean ball in S of center x and radius $r > 0$, hence we define

$$\sup_{x,y \in B_E(0,\nu)} \|x^{-1}y\| = M_\nu < +\infty.$$

It follows that

$$|x - y| \le |s - w| + |t - z| \le \left(1 + \frac{M_\nu}{\alpha} + C_\omega\,\nu\right)\|x^{-1}y\| \quad \text{whenever} \quad x, y \in B_E(0, \nu) .$$

Conversely, we have

$$|-t + z - \omega(s, w)| \le |t - z| + C_\omega\,\nu\,|s - w| \le (1 + C_\omega\,\nu)|x - y|,$$

so it follows that

$$\|x^{-1}y\| = \max\{|s-w|, \sqrt{\alpha|-t+z-\omega(s,w)|}\} \le \left(\sqrt{2\nu} + \sqrt{\alpha(1+C_\omega\,\nu)}\right)\sqrt{|x-y|}$$

for all $x, y \in B_E(0,\nu)$. We have shown that for any bounded set K of S, there exists a geometric constant $C_K > 0$ depending on K and on $\|\cdot\|$ such that

$$C_K^{-1}\,|x-y| \le \|x^{-1}y\| \le C_K\,\sqrt{|x-y|} \quad \text{for any } x, y \in B_E(0,\nu). \tag{1.1}$$

Remark 1.5.1 As a consequence of (1.1), the topology induced by d coincides with the topology of S as a linear space.

Definition 1.5.2 When the linear space S is equipped with the group operation and the homogeneous distance described above, we denoted it by $\mathbb{H}^n$.

Definition 1.5.3 The open unit ball of $\mathbb{H}^n$ with respect to $\|\cdot\|$ and centered at 0 will be denoted by $\mathbb{B}$. The open ball of center x and radius r with respect to the same distance will be denoted by $B(x,r)$.

Chapter 2

Partial derivatives and geometric measures in the Heisenberg group

2.1 Bases on the Heisenberg group

A *graded basis* $(e_1, \dots, e_{2n+1})$ of $\mathbb{H}^n$ seen as a linear space, has the propery that $(e_1, \dots, e_{2n})$ spans S_1 and e_{2n+1} spans S_2. When we consider an element $x \in S$ and the decomposition $x = \sum_{j=1}^{2n+1} x_j e_j$ with respect to the graded basis (e_j), we say that $(x_1, \dots, x_{2n+1})$ are *graded coordinates*.

Once we have fixed e_{2n+1} spanning S_2, the mapping $\omega(s,w) \in S_2$ becomes a scalar-valued, $\omega(s,w) = \bar{\omega}(s,w)e_{2n+1}$. The mapping $\bar{\omega}\colon S_1 \times S_1 \longrightarrow \mathbb{R}$ is exactly a *symplectic form* on S_1 since it is nondegenerate.

As an exercise of linear algebra, one can show that there always exist a special basis $(e_1, \dots, e_{2n})$ of S_1, called *symplectic basis*, such that

$$\begin{aligned} &\bar{\omega}(e_i, e_{n+j}) = \delta_{ij} \quad \text{for all } i, j = 1, \dots, n \\ &\bar{\omega}(e_i, e_j) = 0 \quad \text{for all } 1 \le i, j \le 2n \text{ such that } |i-j| \ne n \end{aligned}$$

The group operation with respect to the graded coordinates associated to the symplectic basis takes a simple explicit form, since

$$\omega(x,y) = \omega\left(\sum_j x_j e_j, \sum_j y_j e_j\right) = \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i)$$

then we have

$$xy = x + y + \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i) e_{2n+1}.$$

We use the terminology symplectic basis also for the full basis $(e_1, \dots, e_{2n+1})$, since e_{2n+1} is fixed once for all.

2.1.1 Different isomorphic group operations

If we consider the group operation associated to $\omega_\beta = \beta\,\omega$, where $\beta \neq 0$, with respect to the same symplectic basis, we get the new group operation

$$xy = x + y + \beta \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) e_{2n+1} \quad \text{where } \beta \neq 0$$

that gives rise to the same group, so we have freedom, if necessary, in the choice of β.

2.2 Partial derivatives and left translations

Definition 2.2.1 Any basis of this vector space equips it with a global differentiable structure that makes it an analytic smooth manifold. Then we can consider C^∞ smooth functions on this space, namely, those functions that are C^∞ smooth in $\mathbb{R}^{2n+1}$ when read in any fixed basis of $\mathbb{H}^n$. We denote this space by $C^\infty(\mathbb{H}^n)$.

Of course, one already has the standard partial derivatives on S as a linear space, but we want to have a differentiation that respect to group operation and also dilations. To find the right partial derivative, we have to consider the large space of first order differential operators on S, that in particular contains the partial derivatives. A linear differential operator $L : C^\infty(\mathbb{H}^n) \longrightarrow C^\infty(\mathbb{H}^n)$ with respect to an understood graded basis is represented as $L = \sum_{j=1}^{2n+1} a_j(x)\,\partial_j$ where a_j are smooth function on $\mathbb{H}^n$. The operator L acts on a smooth function u as follows. With respect to the graded basis $(e_1, \ldots, e_{2n+1})$ we have

$$u(x) = u(x_1 e_1 + \cdots + x_{2n+1} e_{2n+1}) \quad \text{and} \quad \frac{d}{dt} u\Big((x_j + t)e_j + \sum_{i \neq j} x_i e_i\Big)_{|t=0} = (\partial_{x_j} u)(x)$$

hence

$$Lu(x) = \sum_{j=1}^{2n+1} a_j(x)\,\partial_{x_j} u(x)\,.$$

Partial derivatives are invariant with respect to sum of vectors of S in the following sense

$$(\partial_{x_j} u)(y + x) = \partial_{x_j}\big(u(y + \cdot)\big)(x) \quad \text{for all } x, y \in \mathbb{H}^n. \tag{2.1}$$

Let us introduce translations in S with respect to the sum of vectors

$$T_x : C^\infty(\mathbb{H}^n) \longrightarrow C^\infty(\mathbb{H}^n), \quad (T_x u)(y) = u(x + y)$$

Then we read formula (2.1) as follows

$$T_y\,\partial_{x_j} = \partial_{x_j}\,T_y\,.$$

This suggests to introduce the operators L_j replacing traslations with *left translations*. For every $x \in \mathbb{H}^n$, we define the left translation with respet to x as follows

$$l_x : \mathbb{H}^n \longrightarrow \mathbb{H}^n,\ y \longrightarrow xy.$$

Notice that l_x is invertible for all x and

$$(l_x)^{-1} = l_{x^{-1}}.$$

Left translations define the mappings

$$l_x^* : C^\infty(\mathbb{H}^n) \longrightarrow C^\infty(\mathbb{H}^n),\ u \longrightarrow u \circ l_x\,.$$

To discover the form of the partial derivative L_j on $\mathbb{H}^n$ that respect the left translations of S, we impose the relationship

$$L_j l_x^* u = l_x^* L_j \tag{2.2}$$

under the condition that $L_j(0) = \partial_{x_j}$. We have

$$L_j u(x) = l_x^*(L_j u)(0) = L_j(l_x^* u)(0) = \partial_{x_j}(u \circ l_x)(0)$$

It follows that

$$L_j u(x) = du \circ dl_x e_j\,.$$

This suggests us to define the vector field X_j associated to L_j as follows

$$X_j(x) = dl_x(0)e_j \quad \text{for all } x \in \mathbb{H}^n. \tag{2.3}$$

We have

$$L_j u(x) = \langle du(x), X_j(x)\rangle$$

We can concretely compute the formula for X_j with respect to the understood graded coordinates. Let us consider $1 \le j \le n$, then

$$X_j(x) = \frac{d}{dt}x(te_j)_{|t=0} = \frac{d}{dt}(x + te_j - tx_{n+j}e_{2n+1})_{|t=0} = e_j - x_{n+j}\,e_{2n+1}$$

and the analogous computation yields

$$X_{j+n}(x) = e_{n+j} + x_j\,e_{2n+1}$$

We can identify the vector fields X_j with L_j replacing e_j with ∂_{x_j}. Then we will use the notation X_j also to denote the partial derivative

$$X_j = \partial_{x_j} - x_{j+n}\partial_{2n+1} \quad \text{and} \quad X_j u(x) = \partial_{x_j}u(x) - x_{j+n}\partial_{x_{2n+1}}u(x)\,.$$

One can easily observe that $x(te_{2n+1}) = x + te_{2n+1}$, hence the same computations yield

$$X_{2n+1} = \partial_{2n+1}.$$

We have found the natural frame of left invariant differential operators on $\mathbb{H}^n$

$$(X_1, \ldots, X_{2n+1}), \quad X_j = \partial_{x_j} - x_{j+n}\partial_{x_{2n+1}}, \quad X_{n+j} = \partial_{x_{j+n}} + x_j\partial_{x_j}$$

$1 \le j \le n$ and $X_{2n+1} = \partial_{x_{2n+1}}$. The left invariance follows by their construction, namely,

$$X_j u(x) = X_j(u \circ l_x)(0)\,.$$

This transformation can be equivalently translated in terms push-forward of vector fields.

Definition 2.2.2 Let $f : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ be any diffeomorphism and let $Z = a_1 e_1 + \cdots + a_{2n+1} e_{2n+1}$ be any vector field on $\mathbb{H}^n$. We define the "image vector field", or *push-forward* as the new vector field

$$f_* Z(y) = df(f^{-1}y) Z(f^{-1}(y)) \quad \text{for all } y \in \mathbb{H}^n.$$

The geometric meaning of the push-forward vector field $f_* Z$ is the following. Let γ be any solution of the equation $\gamma' = Z(\gamma)$, then the curve $f \circ \gamma$ is the solution of $(f \circ \gamma)' = (f_* Z)(f \circ \gamma)$. In other words, the images of the orbits of Z through f are the orbits of $f_* Z$.

This notion allows us to better clarify the notion of left invariance of X_j. In fact, we have

$$(l_x)_*(X_j) = X_j \quad \text{for all } x \in \mathbb{H}^n \text{ and } j = 1, \ldots, 2n+1.$$

This property rigorously defines the notion of left invariance. By (2.3), for all $z \in \mathbb{H}^n$ we have

$$(l_x)_*(X_j)(z) = dl_x(l_{x^{-1}} z) X_j(l_{x^{-1}} z) = dl_x \circ dl_{x^{-1}z}(0) e_j = dl_z(0) e_j = X_j(z) .$$

2.2.1 Derivatives and dilations

We have seen that the differential operators X_j and dilations δ_r respect the group operation. To see their compatibility, we consider the transformation of X_j by δ_r, namely, the push-forward vector field $(\delta_r)_* X_j$. We have

$$(\delta_r)_* X_j(z) = d\delta_r(\delta_{1/r} z) X_j(\delta_{1/r} z) = d\delta_r \circ dl_{\delta_{1/r} z}(0) e_j = \frac{d}{dt}(\delta_r \circ l_{\delta_{1/r} z})(te_j)_{|t=0} .$$

The last composition can be written as follows

$$\delta_r \circ l_{\delta_{1/r} z}(te_j) = \delta_r((\delta_{1/r} z)(te_j)) = z(t\delta_r e_j) .$$

If $1 \le j \le 2n$, then $\delta_r e_j = re_j$ and we have

$$(\delta_r)_* X_j(z) = \frac{d}{dt}(z(tre_j)_{|t=0} = r \frac{d}{dt}(z(te_j)_{|t=0} = r\, X_j(z) .$$

Since $\delta_r e_{2n+1} = r^2 e_{2n+1}$, with the same argument we get

$$(\delta_r)_* X_{2n+1}(z) = r^2 \, X_{2n+1}(z) .$$

The exponent of r is called the *degree* of the vector field, so that $X_1, \ldots, X_{2n}$ have degree one and X_{2n+1} has degree two. Let us define $d_1 = \cdots = d_{2n} = 1$ and $d_{2n+1} = 2$. Thus, we have proved that

$$(\delta_r)_* X_j = r^{d_j} X_j. \tag{2.4}$$

The previous formulae can be tested on smooth functions $u \in C^\infty(\mathbb{H}^n)$ as follows

$$X_j(u \circ \delta_r) = du \circ d\delta_r(X_j) = du \circ (\delta_r)_* X_j \circ \delta_r = r^{d_j}\, du \circ X_j \circ \delta_r = r^{d_j} (X_j u) \circ \delta_r .$$

As a consequence, defining $\delta_r^* u(x) = u(\delta_r x)$, we have proved that $X_j \delta_r^* = r^{d_j} \delta_r^* X_j$ as linear operators on $C^\infty(\mathbb{H}^n)$. The vector fields $X_1, \ldots, X_{2n}$ are called *horizontal vector fields*.

2.3 Haar measure

A measure μ in S that is compatible with the group operation in S has to satisfy

$$\mu(xA) = \mu(A)$$

for every measurable set A and every $x \in S$. These measures are called *Haar measures*.

Remark 2.3.1 Any locally compact Lie group has a unique Haar measure up to multiplication by a positive number.

In the Heisenberg group equipped with any *graded basis* $(e_1, \ldots, e_{2n+1})$ the mapping $y \longrightarrow xy$ has Jacobian equal to one with respect to graded coordinates. In fact, we have

$$l_x(y) = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_{2n} + y_{2n} \\ x_{2n+1} + y_{2n+1} + \omega\big(\sum_{j=1}^{2n} x_j e_j, \sum_{j=1}^{2n} y_j e_j \big) \end{pmatrix} \quad \text{and} \quad Dl_x = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & \\ * & * & \cdots & 1 \end{pmatrix}$$

where the $*$ denotes the places where the entries are the partial derivatives of $\omega\big(\sum_{i=1}^{2n} x_i e_i, t e_j \big)$ with respect to t at $t = 0$ and $j = 1, \ldots, 2n+1$. Thus, the Lebesgue measure $\mathcal{L}^{2n+1}$ yields the Haar measure

$$\mathcal{L}^{2n+1}(l_x(A)) = \int_A J l_x(y) \, d\mathcal{L}^{2n+1}(y) = \mathcal{L}^{2n+1}(A) .$$

Exercise 2.3.1 Show that the right translations $r_x(z) = zx$ preserve the Lebesgue measure in $\mathbb{H}^n$.

The Haar measure of $\mathbb{H}^n$ behaves well also with respect to dilations. In fact, when the graded basis is fixed, dilations read as follows

$$\delta_r \Big(\sum_{j=1}^{2n+1} x_j e_j \Big) = \sum_{j=1}^{2n+1} r^{d_j} e_j .$$

The Jacobian determinant of δ_r is $r^{\sum_{j=1}^{2n+1} d_j} = r^{2n+2}$, hence

$$\mathcal{L}^{2n+1}(\delta_r A) = r^{2n+2} \, \mathcal{L}^{2n+1}(A) \tag{2.5}$$

for every measurable set $A \subset \mathbb{H}^n$.

The number $2n+2$ is the intrinsic dimension of the Heisenberg group, that is greater than its topological dimension. In Subsection 2.4.3, we will see that this number is actually the Hausdorff dimension of the group with respect to its distance d.

2.4 Hausdorff measure

The Hausdorff measure in an important notion of “measure with dimension” that can be used to study lower dimensional objects in the space. Furthermore, it can be introduced in a general metric space.

2.4.1 Hausdorff measure in metric spaces

Let (X, d) be a metric space and denote

$$\operatorname{diam} S = \sup_{x,y \in S} d(x, y) .$$

Fix $\alpha > 0$ and $c_\alpha > 0$ and for every $t > 0$ and $E \subset X$ define

$$\mathcal{H}^\alpha_{d,t}(E) = c_\alpha \inf \left\{ \sum_{j=0}^{\infty} \operatorname{diam}(E_j)^\alpha : E \subset \bigcup_{j \in \mathbb{N}} E_j ,\ \operatorname{diam}(E_j) \le t \right\}$$

we notice that here c_α plays the role of the geometric constant, to be properly chosen, depending on the metric space we consider. We define

$$\mathcal{H}^\alpha_d(E) = \sup_{t>0} \mathcal{H}^\alpha_{d,t}(E) = \lim_{t \to 0^+} \mathcal{H}^\alpha_{d,t}(E)$$

As a consequence of the well known Carathéodory criterion, we have the following theorem.

Theorem 2.4.1 *The set function $\mathcal{H}^\alpha_d$ is a Borel regular outer measure in X.*

In particular, $\mathcal{H}^\alpha_d$ is countably additive on disjoint unions of Borel sets.

Exercise 2.4.1 Show that $\mathcal{H}^\alpha_d(E) < \infty$ implies $\mathcal{H}^\beta_d(E) = 0$ for all $\beta > \alpha$ and $\mathcal{H}^\alpha_d(E) > 0$ implies $\mathcal{H}^\beta_d(E) = +\infty$ for all $\beta < \alpha$.

The previous exercise allows us to introduce the *Hausdorff dimension* of $E \subset X$ as follows

$$\text{H-}\dim(E) = \inf\{\alpha > 0 : \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha > 0 : \mathcal{H}^\alpha(E) = +\infty\}$$

with the convention $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$.

2.4.2 Hausdorff measure in Euclidean spaces

We consider the metric space $\mathbb{R}^n$ equipped with the Euclidean distance $d(x, y) = |x - y|$ and for any positive integer k we set $c_k = 2^{-k} \mathcal{L}^k(\mathbb{B}_k)$ where $\mathbb{B}_k$ is the Euclidean unit ball in $\mathbb{R}^k$. In this setting, we define the standard Hausdorff measure

$$\mathcal{H}^k(E) = \lim_{t \to 0^+} \mathcal{H}^k_t(E) \quad \text{with} \quad \mathcal{H}^k_t(E) = c_k \inf \left\{ \sum_{j=0}^{\infty} \operatorname{diam}(E_j)^\alpha : E \subset \bigcup_{j \in \mathbb{N}} E_j ,\ \operatorname{diam}(E_j) \le t \right\} .$$

Exploiting the *isodiametric inequality* we have the following fact.

Theorem 2.4.2 *Under the previous assumptions, the Lebesgue measure $\mathcal{L}^n$ coincides with the Hausdorff measure $\mathcal{H}^n$ in $\mathbb{R}^n$.*

2.4.3 Hausdorff measure in Heisenberg groups

If we consider the Heisenberg group $\mathbb{H}^n$ as a metric space equipped with homogeneous distance $d(x,y) = \|x^{-1}y\|$, it is natural to look for its Hausdorff dimension with repsect to d. Let us consider the open unit ball $\mathbb{B}$ of $\mathbb{H}^n$ with respect to d and fix $\varepsilon \in (0,1)$. By the properties of the Lebesgue measure, we have seen that

$$\mathcal{L}^{2n+1}(B(x_j,\varepsilon)) = \mathcal{L}^{2n+1}(l_{x_j}\delta_\varepsilon\mathbb{B}) = \varepsilon^Q\,\mathcal{L}^{2n+1}(\mathbb{B}) = \omega_Q\,\varepsilon^Q. \tag{2.6}$$

Then we can consider a maximal family of disjoint balls $\mathcal{F} = \{B(x_j,\varepsilon) : j = 1,\dots,N\}$ contained in $\mathbb{B}$, that has to be finite, due to (2.6). The maximality means that there exists no collection of disjoint balls of radius ε and contained in $\mathbb{B}$ that strictly contains $\mathcal{F}$. The existence of this set can be proved by the classical Zorn's lemma. By the maximality of this family, we have

$$\bigcup_{j=1}^{N} B(x_j,\varepsilon) \subset \mathbb{B} \subset \bigcup_{j=1}^{n} B(x_j,2\varepsilon)\,.$$

Let us denote by Q the number $2n+2$ and by ω_Q the volume of the unit ball $\mathcal{L}^{2n+1}(\mathbb{B})$. The previous inclusions yield

$$\omega_Q\sum_{j=1}^{N}\varepsilon^Q \le \omega_Q \le 2^Q\omega_Q\sum_{j=1}^{N}\varepsilon^Q$$

that yields

$$2^{-Q}\varepsilon^{-Q} \le N \le \varepsilon^{-Q}\,.$$

This immediately leads to

$$\mathcal{H}^Q_{d,4\varepsilon}(\mathbb{B}) \le c_Q\,N\omega_Q(2\varepsilon)^Q \le 2^Q\,\omega_Q\,c_Q$$

hence $\mathcal{H}^Q(\mathbb{B}) < +\infty$. This implies that for an arbitrary $\varepsilon > 0$ there exist a family $\{E_j\}$ that covers $\mathbb{B}$ with $\mathrm{diam}(E_j) \le \varepsilon$ for all j and such that

$$c_Q\sum_{j=0}^{\infty}\mathrm{diam}(E_j)^Q < \varepsilon + \mathcal{H}^Q_\varepsilon(\mathbb{B})$$

We pick $x_j \in E_j$ for all j, observing that $E_j \subset B(x_j, 2\mathrm{diam}(E_j))$, hence

$$\omega_Q \le \sum_{j=0}^{\infty}\mathcal{L}^{2n+1}(E_j) \le \omega_Q 2^Q\sum_{j=0}^{\infty}\mathrm{diam}(E_j)^Q$$

that implies

$$2^{-Q}c_Q \le c_Q\sum_{j=0}^{\infty}\mathrm{diam}(E_j)^Q < \varepsilon + \mathcal{H}^Q_\varepsilon(\mathbb{B})$$

and letting $\varepsilon \to 0^+$, we have finally establiahed that

$$0 < \mathcal{H}^Q(\mathbb{B}) < +\infty\,.$$

This precisely shows that H-dim $\mathbb{H}^n = Q = 2n+2$. In fact, whenever $\alpha > Q$ any $B(0,k) = \delta_k\mathbb{B}$ has vanishing $\mathcal{H}^\alpha_d$-measure, hence $\mathbb{H}^n = \bigcup_{k\ge1}B(0,k)$ satisfies $\mathcal{H}^\alpha_d(\mathbb{H}^n) = 0$.

Remark 2.4.1 It is certainly a natural question to find the Hausdorff dimension of sumanifolds in the Heisenberg group. Here we meet at the same time the problem of establishing who are the "good submanifolfds" that respect the geometry of the Heisenberg group.

Exercise 2.4.2 Let us consider the Heisenberg group $\mathbb{H}^n$ equipped with its homogeneous distance d and a graded basis $(e_1, \ldots, e_{2n+1})$ is fixed. Consider any linear subspace Z of the form $V \oplus S_2$, where V is a linear subspace of S_1. Show that the Hausdorff dimension of Z is $\dim V + 2$.

It can be shown that any C^1 smooth hypersurface in the Heisenberg group has Hausdorff dimension $2n + 1$, where $2n$ is the topological dimension of the hypersurface in $\mathbb{H}^n$.

Chapter 3

Heisenberg algebra, connectivity and sub-Riemannian distance

3.1 Lie algebra.

Any vector field $X = \sum_{j=1}^{n} a_j(x)e_j$ on a linear space U of dimension n can be identified with a first order differential operator $X = \sum_{j=1}^{n} a_j(x)\partial_{x_j}$, where a basis $(e_1, \ldots, e_n)$ of U is understood. Then looking at the vector fields $X = \sum_j a_j(x)\partial_{x_j}$ and $Y = \sum_j b_j(x)\partial_{x_j}$ as differential operators, we can consider their composition

$$XY - YX$$

that is a priori a second order differential operator. On the other hand, an easy computation shows that all second order terms vanish, getting

$$XY - YX = \sum_{j,l=1}^{n} \Big(a_j(x)\partial_{x_j} b_l(x) - b_j(x)\partial_{x_j} a_l(x)\Big)\partial_{x_l} \tag{3.1}$$

so we still have a vector field, that is called the *commutator of X and Y*. The commutator, or Lie product, of X and Y is denoted by

$$[X, Y] := XY - YX.$$

This operation satisfies the so-called Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{3.2}$$

for all X, Y, Z vector fields (or linear differential operators) on U. This is just a direct algebraic verification. The commutator $[X, Y]$ is obviously skew-symmetric

$$[X, Y] = -[Y, X]$$

so it defines a Lie algebra structure on the space of all vector fields on U.

Definition 3.1.1 A vector space $\mathfrak{g}$ (real and finite dimensional) is a *Lie algebra* if it is equipped with a skew-symmetric mappings $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ that satisfies (3.2).

A Lie algebra is not an associative algebra. The Jacobi identity somehow replaces the associativity.

3.2 Heisenberg algebra.

We have previously found a basis of left invariant vector fields in $\mathbb{H}^n$. For all $1 \le j \le n$, we have

$$X_j = \partial_{x_j} - x_{j+n}\partial_{x_{2n+1}}, \quad X_{n+j} = \partial_{x_{j+n}} + x_j\partial_{x_j} \quad \text{and} \quad X_{2n+1} = \partial_{2n+1}. \tag{3.3}$$

We have

$$\begin{cases} [X_i, X_{n+j}] = 2\delta_{ij}\, X_{2n+1} & \text{for all } i, j = 1, \ldots, n \\ [X_i, X_j] = 0 & \text{for all } 1 \le i, j \le 2n \text{ such that } |i - j| \ne n \end{cases}. \tag{3.4}$$

Definition 3.2.1 Let $\mathfrak{h}^n$ to be the linear space of vector fields $X_1, \ldots, X_{2n+1}$. Relations (3.4) show that any couple of elements $X, Y \in \mathfrak{h}^n$ satisfy $[X, Y] \in \mathfrak{h}^n$. We say that $\mathfrak{h}^n$ equipped with this product is the *Heisenberg algebra*.

In fact, the Jacobi identity for elements of $\mathfrak{h}^n$ holds, since this property holds for any triple of vector fields. This shows that $\mathfrak{h}^n$ is indeed a *Lie algebra*.

Notice that we can define relations (3.4) for any basis of a $2n + 1$ dimensional linear space and this will define an abstract Heisenberg algebra.

Definition 3.2.2 For any $X \in \mathfrak{h}^n$ there exist a unique solution $t \to \gamma(t)$ of the Cauchy problem

$$\begin{cases} \dot\gamma(t) = X(\gamma(t)) \\ \gamma(0) = 0 \end{cases}$$

that we denote by $\exp(tX)$, called the "exponential of X".

Remark 3.2.1 One can check that the solutions $t \to \exp(tX)$ extend to $\mathbb{R}$ for any choice of X.

3.3 Horizontal structure and connectivity in the Heisenberg group

The vector fields $X_1, \ldots, X_{2n}$ of (3.3) span at any point of $\mathbb{H}^n$ the space of *horizontal directions.* Precisely, at any $x \in \mathbb{H}^n$ we define the *horizontal subspace* as follows

$$H_x\mathbb{H}^n = \Big\{ \sum_{j=1}^{2n} \lambda_j\, X_j(x) : \lambda_j \in \mathbb{R} \Big\}.$$

This "horizontal structure" allows us to select those curves that move only along these directions.

Definition 3.3.1 A *horizontal curve* in $\mathbb{H}^n$ is an absolutely continuous curve $\gamma : [a, b] \longrightarrow \mathbb{H}^n$, with respect to the underlying Euclidean metric of $\mathbb{H}^n$, such that for a.e. $t \in [a, b]$, we have

$$\dot\gamma(t) = \sum_{j=1}^{2n} \lambda_j(t)\, X_j(\gamma(t)) \tag{3.5}$$

where all λ_j are integrable on (a, b).

Condition (3.5) can be written in the following equivalent way

$$\dot\gamma(t) \in H_{\gamma(t)}\mathbb{H}^n.$$

Remark 3.3.1 Obvious examples of horizontal curves are the exponentials $\exp(tX)$, whenever X is a linear combinations of $X_1, \ldots, X_{2n}$.

Can we define horizontal curves through a differential constraint?

The answer is positive. It suffices to write (3.5) explicitly as follows

$$\begin{cases} \dot\gamma_j(t) = \lambda_j(t) & 1 \le j \le n \\ \dot\gamma_{n+j}(t) = \lambda_{n+j}(t) & 1 \le j \le n \\ \dot\gamma_{2n+1}(t) = \sum_{j=1}^n \left(-\lambda_j(t)\gamma_{n+j}(t) + \lambda_{n+j}(t)\gamma_j(t)\right) \end{cases}.$$

Joining these equations we get a unique condition

$$\dot\gamma_{2n+1}(t) = \sum_{j=1}^n \Big(-\dot\gamma_j(t)\gamma_{n+j}(t) + \dot\gamma_{n+j}(t)\gamma_j(t)\Big). \tag{3.6}$$

This is a "contact condition" or *contact equation* for the curve γ.

Remark 3.3.2 The contact condition (3.6) tells us that we can "lift" any curve $c : [a,b] \longrightarrow \mathbb{H}^n$ taking values in the hyperplane $\{x \in \mathbb{H}^n :\ x_{2n+1} = 0\}$ to a horizontal curve in $\mathbb{H}^n$ as follows

$$\gamma(t) = \left(c(t), \tau_0 + \sum_{j=1}^n \int_a^t \Big(-\dot c_j(\tau)c_{n+j}(\tau) + \dot c_{n+j}(\tau)c_j(\tau)\Big)\, d\tau \right) \tag{3.7}$$

where $\tau_0 \in \mathbb{R}$ can be arbitrarily chosen.

Can we claim to connect any couple of points of $\mathbb{H}^n$ by a horizontal curve?

The answer to this question is yes. To prove this fact, we start with the following proposition.

Proposition 3.3.1 *If γ is a horizontal curve and $x \in \mathbb{H}^n$, then the left translated $\gamma_x = l_x \circ \gamma$ is also horizontal.*

Proof. It suffices to differentiate, getting

$$\frac{d}{dt}\gamma_x(t) = dl_x \circ \dot\gamma = \sum_{j=1}^{2n} \lambda_j(t)\, dl_x \circ X_j(\gamma(t)) = \sum_{j=1}^{2n} \lambda_j(t)\, X_j(l_x \circ \gamma(t))$$

where the last equality follows from the left invariance of X_j. □

Remark 3.3.3 By Proposition 3.3.1, for $X = \sum_{j=1}^{2n} \alpha_j X_j$ and $x \in \mathbb{H}^n$, the curve $t \to x\exp(tX)$ is horizontal. We call these curves *horizontal lines* through x. Next, we also see that not all horizontal curves are necessarily horizontal lines.

Proposition 3.3.2 *For each $y \in \mathbb{H}^n \setminus \{0\}$, there exists a horizontal curve that connects the origing with y.*

Proof. We preliminary observe that for every $h = (h_1, \ldots, h_{2n}, 0) \in \mathbb{H}^n$, with respect to a symplectic basis, the curve $\gamma(t) = \tau e_{2n+1} + th$ is horizontal for any $\tau \in \mathbb{R}$. In fact, we have

$$\sum_{j=1}^{2n} h_j X_j(\tau e_{2n+1} + th) = \sum_{j=1}^{n} \Big[h_j(e_j - th_{n+j}e_{2n+1}) + h_{n+j}(e_{n+j} + th_j e_{2n+1})\Big] = \sum_{j=1}^{2n} h_j e_j = \dot\gamma.$$

For $\tau = 0$, the previous fact allows us to connect any point $y = (y_1, \dots, y_{2n}, 0)$ with the origin through the horizontal curve $\gamma(t) = ty$. Let us now consider a point $y = (y_1, \dots, y_{2n+1})$ with $y_{2n+1} > 0$. We first choose an arbitrary $r > 0$ and consider

$$c_r(t) = r(\cos t, 0, \dots, 0, 1 + \sin t, 0, \dots, 0) \in \mathbb{H}^n .$$

Using (3.7), we have the horizontal curve

$$\gamma_r(t) = \left(r\cos t, 0, \dots, 0, r + r\sin t, 0, \dots, r^2 \int_{-\pi/2}^{t} (1 + \sin\tau)d\tau \right).$$

Therefore, choosing $r_0 = \sqrt{y_{2n+1} / \int_{-\pi/2}^{3\pi/2} (1 + \sin\tau)d\tau}$, we have $\gamma_{r_0} : [-\pi/2, 3\pi/2] \longrightarrow \mathbb{H}^n$ horizontal and such that $\gamma_{r_0}(0) = 0$ and $\gamma_{r_0}(2\pi) = (0, \dots, 0, y_{2n+1})$. Exploiting our first observation, we know that the curve

$$\tilde{\gamma}(t) = y_{2n+1}e_{2n+1} + t(y_1, \dots, y_{2n}, 0)$$

is also horizontal and connects $(0, \dots, 0, y_{2n+1})$ to $(y_1, \dots, y_{2n+1})$. Joining γ_{r_0} and $\tilde{\gamma}$, we get a horizontal curve joining the origin with y with $y_{2n+1} > 0$. The case $y_{2n+1} < 0$ can be obtained in analogous way. Here the main observation is that setting

$$r_1 = \sqrt{|y_{2n+1}| / \int_{-\pi/2}^{3\pi/2} (1 + \sin\tau)d\tau}$$

the curve $\gamma_{r_1} : [-\pi/2, 3\pi/2] \longrightarrow \mathbb{H}^n$ previously defined is horizontal and connects the origin with $(0, \dots, 0, |y_{2n+1}|)$. Reparametrizing the curve in the opposite direction, we get $\bar{\gamma} : [-\pi/2, 3\pi/2] \longrightarrow \mathbb{H}^n$, $\bar{\gamma}(t) = \gamma_{r_1}(\pi - t)$ starts from $(0, \dots, 0, |y_{2n+1}|)$ and finaly reaches the origin. Thus, we consider

$$(y_{2n+1}e_{2n+1})\bar{\gamma}(t) = y_{2n+1}e_{2n+1} + \bar{\gamma}(t)$$

that is horizontal by Proposition 3.3.1. This curves connects the origin with $(0, \dots, 0, y_{2n+1})$. □

Theorem 3.3.1 (Connectivity) *Any $x, y \in \mathbb{H}^n$ can be connected by a horizontal curve.*

Proof. Let γ be a horizontal curve connecting the origin with $x^{-1}y$. This curves exists by the previous proposition. Finally, Proposition 3.3.1 shows that $l_x \circ \gamma$, connecting x with y, is also horizontal. □

Remark 3.3.4 This connectivity theorem is a very special instance of a more general connectivity theorem on manifolds. The famous sufficient condition on a set of vector fields $X_1, \dots, X_m$ to get this connectivity is the so-called *Lie bracket generating condition*: the linear span of all commutators of X_i has to yield all of the tangent space at every point.

3.4 Sub-Riemannian distance

Definition 3.4.1 For any horizontal curve $\gamma : [a, b] \longrightarrow \mathbb{H}^n$, with velocity $\dot{\gamma} = \sum_{j=1}^{2n} \lambda_j X_j$, we define the length

$$l(\gamma) = \int_a^b \left(\sum_{j=1}^{2n} \lambda_j(\tau)^2 \right)^{1/2} d\tau .$$

Exercise 3.4.1 Show that $l(\gamma) = l(\tilde{\gamma})$ whenever $\tilde{\gamma}$ is a reparametrization of γ.

In view of the previous connectivity, we can introduce the following definition.

Definition 3.4.2 For any $x, y \in \mathbb{H}^n$, we define $\mathcal{H}_{x,y}$ to be the set of horizontal curves connecting x with y. By Theorem 3.3.1, $\mathcal{H}_{x,y} \neq \emptyset$ and we can define

$$\rho(x, y) = \inf\{l(\gamma) : \gamma \in \mathcal{H}_{x,y}\}$$

that is called the *sub-Riemannian distance* between x and y, in short *SR-distance*.

One can check by standard methods the validity of the following result.

Theorem 3.4.1 *The SR-distance ρ is actually a distance on $\mathbb{H}^n$ and it is continuous with respect to the topology of $\mathbb{H}^n$ as a linear space.*

Theorem 3.4.2 *The SR-distance is a homogeneous distance.*

Proof. In view of the previous theorem, we have to check the left invariance and the homogeneity. Let $x, y, w \in \mathbb{H}^n$ and consider any $\gamma \in \mathcal{H}_{x,y}$. In view of Proposition 3.3.1, it follows that $l_w \circ \gamma \in \mathcal{H}_{wx,wy}$ and converse holds using $l_{w^{-1}}$. By definition of ρ, this proves that $\rho(wx, wy) = \rho(x, y)$. Let us consider $\gamma_r = \delta_r \circ \gamma$ and a differentiability point t for γ, with $\dot{\gamma}(t) = \sum_{j=1}^{2n} \lambda_j X_j(\gamma(t))$. We have

$$\dot{\gamma}_r(t) = d\delta_r \circ \dot{\gamma}(t) = \sum_{j=1}^{2n} \lambda_j(t)\, d\delta_r X_j(\gamma(t)) = \sum_{j=1}^{2n} \lambda_j(t)\, ((\delta_r)_* X_j)(\gamma_r(t))\,.$$

In view of (2.4), we have

$$\dot{\gamma}_r(t) = r \sum_{j=1}^{2n} \lambda_j(t)\, X_j(\gamma_r(t))\,. \tag{3.8}$$

This shows that $\gamma_r \in \mathcal{H}_{\delta_r x, \delta_r y}$ and clearly $l(\gamma_r) = r\, l(\gamma)$. An analogous argument can be carried out for a curve in $\mathcal{H}_{\delta_r x, \delta_r y}$ and using $\delta_{1/r}$, eventually getting $\rho(\delta_r x, \delta_r y) = r\, \rho(x, y)$ for any $x, y \in \mathbb{H}^n$. □

Since d and ρ are two *homogeneous distances*, there exists $C > 0$ such that

$$C^{-1} d(x, y) \le \rho(x, y) \le C d(x, y) \quad \text{for all} \quad x, y \in \mathbb{H}^n\,.$$

This follows by homogeneity and the fact that

$$\max_{\rho(x,0)=1} d(x, 0) \quad \text{and} \quad \max_{d(x,0)=1} \rho(x, 0)$$

are positive numbers.

Chapter 4

Homogeneous differentiability and Lipschitz functions

4.1 Horizontal gradient and Pansu differentiability

Given a smooth function $u \in C^\infty(\mathbb{H}^n)$ and the frame (3.3) of left invariant vector fields $X_1, \ldots, X_{2n+1}$ spanning $\mathfrak{h}^n$. We define the *horizontal gradient* of u at x as

$$\nabla_h u(x) = \big(X_1 u(x), \ldots, X_{2n} u(x)\big)\,.$$

This is a gradient along the horizontal directions, so the vertical direction e_{2n+1} is missing. We have somehow a "degenerate gradient".

Can we associate to this gradient a notion of differentiability?

The answer is yes: we have first to introduce a notion of "linear mapping" from $\mathbb{H}^n$ to $\mathbb{R}$ and then to use any homogeneous distance in $\mathbb{H}^n$.

4.1.1 Homogeneous homomorphisms

Definition 4.1.1 A mapping $L : \mathbb{H}^n \longrightarrow \mathbb{R}$ is an h-homomorphism if we have

1. $L(xy) = L(x) + L(y)$
2. $L(\delta_r x) = r\, L(x)$

for all $x, y \in \mathbb{H}^n$ and $r > 0$.

The analogy between h-homomorphisms of $\mathbb{H}^n$ and linear functions of $\mathbb{R}^n$ is evident.

Exercise 4.1.1 Let L be an h-homomorphism. Show that for all $x, y \in \mathbb{H}^n$ we have

$$L(x^{-1}) = -L(x) \quad \text{and} \quad L(y) - L(x) = L(x^{-1}y) = L(yx^{-1})\,.$$

Proposition 4.1.1 *Any h-homomorphism vanishes on S_2 and it is also a linear mapping.*

Proof. Let $\beta \in \mathbb{R} \setminus \{0\}$. Since the product $(\beta e_{2n+1}) \cdots (\beta e_{2n+1})$ iterated k times yields $k\beta e_{2n+1}$, due to the skew-symmetry of ω in the expression of the group operation, we have

$$kL(\beta e_{2n+1}) = L(\beta e_{2n+1} \cdots \beta e_{2n+1}) = L(k\beta e_{2n+1}) .$$

On the other hand, by homogeneity of L, we also have

$$\sqrt{k}\, L(\beta e_{2n+1}) = L(k\beta e_{2n+1}) = kL(\beta e_{2n+1}) .$$

For instance $k = 4$ in the preceding equality implies that $L(\beta e_{2n+1}) = 0$. We consider $x = s_1 + t_1$, $y = s_2 + t_2$ with $s_1, s_2 \in S_1$ and $t_1, t_2 \in S_2$. Since $L(\omega(s_1, s_2)) = 0$ and $(x+y)\omega(s_1, s_2) = x+y+\omega(s_1, s_2)$, we have

$$L(x + y) = L(x + y) + L(\omega(s_1, s_2)) = L(x + y + \omega(s_1, s_2)) = L(xy) = L(x) + L(y) .$$

We also have $L(rx) = L(rs_1 + rt_1) = L((rs_1)(rt_1)) = L(rs_1) = L((rs_1)(r^2 t_1)) = L(rs_1 + r^2 t_1) = rL(x)$ for all $r > 0$. If $\lambda < 0$, then $L(\lambda x) = L(|\lambda| x^{-1}) = |\lambda| L(x^{-1}) = -|\lambda| L(x) = \lambda L(x)$. This ends the proof. □

Corollary 4.1.1 *Let $(e_1, \ldots, e_{2n+1})$ be a graded basis and let $L : \mathbb{H}^n \longrightarrow \mathbb{R}$ be an h-homomorphism. Then there exist $\beta_j \in \mathbb{R}$ for $j = 1, \ldots, 2n$ such that*

$$L\Big(\sum_{j=1}^{2n+1} x_j e_j\Big) = \sum_{j=1}^{2n} \beta_j x_j . \tag{4.1}$$

Remark 4.1.1 Clearly, L is also continuous, due to its linearity. In addition, since it is Lipschitz continuous with respect to the Euclidean distance, it is also locally Lipschitz continuous with respect to d, due to the estimates (1.1).

Next, we will show that L is Lipschitz on $\mathbb{H}^n$ with respect to d. To see this fact, we have to use more intrinsic arguments, namely group operation and dilations.

Definition 4.1.2 We define the mapping

$$F(a_1, \ldots, a_{2n+2}) = (a_1 e_1) \cdots (a_{2n} e_{2n}) \Big((a_{2n+1} e_1)(a_{2n+2} e_{n+1})(a_{2n+1} e_1^{-1})(a_{2n+2} e_{n+1}^{-1}) \Big)$$

from $\mathbb{R}^{2n+2}$ to $\mathbb{H}^n$. By direct computation one observes that

$$F(a_1, \ldots, a_{2n+2}) = (a_1 e_1) \cdots (a_{2n} e_{2n}) \Big(2 a_{2n+1} a_{2n+2} e_{2n+1} \Big) .$$

Exercise 4.1.2 Show that F is surjective and homogeneous, namely, $F(ra) = \delta_r F(a)$ for all $r > 0$.

Proposition 4.1.2 *Every h-linear mapping $L : \mathbb{H}^n \longrightarrow \mathbb{R}$ is Lipschitz continuous on $\mathbb{H}^n$.*

Proof. There exists a bounded set $\tilde{B}$ of $\mathbb{R}^{2n+2}$ such that $F(\tilde{B}) \supset \bar{B}$, hence for all $F(a)$ with $a \in \tilde{B}$, by the properties of h-homomorphisms, we have

$$|L(F(a))| \le \Big(\sum_{j=1}^{2n} |a_j| |L(e_j)| \Big) + 2|a_{2n+1} L(e_1)| + 2|a_{2n+2} L(e_{n+1})| \le \sup_{a \in \tilde{B}} |a| \max_{1 \le j \le 2n+1} |L(e_j)| = C_0 < +\infty .$$

For any distinct elements $x, y \in \mathbb{H}^n$, setting $\lambda = d(x, y) > 0$, we have

$$|L(x) - L(y)| = |L(y^{-1} x)| = d(x, y) |L(\delta_{1/\lambda}(y^{-1} x))| \le C_0\, d(x, y) ,$$

concluding the proof. □

4.1.2 Homogeneous differentiability

In the sequel, Ω will dentoe an open set of $\mathbb{H}^n$.

Definition 4.1.3 Let $f : \Omega \longrightarrow \mathbb{R}$ and let $x \in \Omega$. We say that f is *h-differentiable* at x if there exists an h-linear mapping $L : \mathbb{H}^n \longrightarrow \mathbb{R}$ such that

$$f(xv) = f(x) + L(v) + o(\|v\|) \quad \text{as} \quad v \to 0.$$

L is the h-differential and it is denoted by $d_h f(x)$, since it is uniquely defined.

This notion of differentiability does not require differentiability in the sense of smooth manifolds, although one can easily show that all horizontal partial derivatives $X_j u$ exist in the pointwise sense

$$X_j u(x) = \lim_{t\to 0} \frac{u(x\exp(tX_j)) - u(x)}{t}\,. \tag{4.2}$$

Remark 4.1.2 Notice that the existence of the limit (4.2) for some j might hold also for a nonsmooth function in the standard sense. in the standard sense.

Exercise 4.1.3 Let $X = \sum_{j=1}^{2n+1} \alpha_j X_j$ and $x \in \mathbb{H}^n$. Verify that

$$x\exp(tX) = x\delta_t(\exp X) \quad \text{iff} \quad \alpha_{2n+1} = 0\,.$$

Definition 4.1.4 We denote by $C^1_h(\Omega)$ the space of all functions $f\colon \Omega \longrightarrow \mathbb{R}$ such that f is everywhere h-differentiable and the mapping $x \longrightarrow d_h f(x)$ is continuous.

Proposition 4.1.3 *For every $f \in C^1(\Omega)$, we have $f \in C^1_h(\Omega)$ and there holds*

$$\lim_{t\to 0} \frac{f(x\delta_t w) - f(x)}{t} = \langle \nabla_h f(x), w\rangle \tag{4.3}$$

as $t \to 0^+$, uniformly with respect to both x and w that vary in compact sets.

Proof. Since f is C^1 smooth, we have

$$\frac{f(x\delta_t w) - f(x)}{t} = \frac{1}{t}\int_0^1 \frac{d}{ds}\big(f\circ l_x)(s\delta_t w)\big)\,ds = \int_0^1 d(f\circ l_x)(s\delta_t w)(w_1 + tw_2)\,ds$$

where $w_1 = \sum_{j=1}^{2n} \beta_k\, e_j$ and $w_2 = \beta_{2n+1}\, e_{2n+1}$. The limits of the integrals

$$\int_0^1 d(f\circ l_x)(s\delta_t w)(w_1)\,ds \quad \text{and} \quad \int_0^1 t\,d(f\circ l_x)(s\delta_t w)(w_2)\,ds$$

as $t \to 0$ are $d(f\circ l_x)(0)(w_1)$ and 0 and they are uniform with respect to both x and w, varying in compact sets. Finally, we observe that

$$d(f\circ l_x)(0)(w_1) = df(x)\circ dl_x(0)\left(\sum_{j=1}^{2n}\beta_j e_j\right) = \sum_{j=1}^{2n}\beta_j X_j f(x)\,.$$

Being $\nabla_h f(x) = (X_1 f(x), \ldots, X_{2n} f(x))$ identified with $\nabla_h f(x) = (X_1 f(x), \ldots, X_{2n} f(x), 0)$ our claim is achieved. □

Remark 4.1.3 B. Franchi, R. Serapioni and F. Serra Cassano have shown a concrete example of function that is $C^1_H(\mathbb{H}^1)$ but it is not locally Lipschitz in the Euclidean sense. Indeed this function is is smooth outside a negligible set, hence a.e. differentiable in the classical sense. It is also possible to show that there exist C^1_h functions that are not differentiable in the classical sense on a set of positive measure.

4.1.3 Intrinsic regular hypersurfaces submanifolds

The notion of C_h^1 smoothness for functions has an associated notion of C_h^1 smoothness for subsets. B. Franchi, R. Serapioni and F. Serra Cassano in their 2001 paper on rectifiability of h-finite perimeter sets in the Heisenberg group have introduced the following notion of intrinsic regular hypersurface, along with the associated implicit function theorem.

Definition 4.1.5 We say that $\Sigma \subset \mathbb{H}^n$ is an h-regular hypersurface of $\mathbb{H}^n$ if for every $x \in \Sigma$ there exists an open set U containing x and a functions $f \in C_h^1(U)$ such that $\nabla_h f(z) \neq 0$ for all $z \in U$ and $f^{-1}(f(x)) = U \cap \Sigma$.

The point of this definition is that we have an associated implicit function theorem analogous to the classical one.

Theorem 4.1.1 *Let $f \in C_h^1(\Omega)$, where $\Omega \subset \mathbb{H}^n$ is an open set. Assume that $X_1 f(x) > 0$ at some $x \in \Omega$. Then there exists an open set U containing x such that*

$$f^{-1}(f(x)) \cap U = \{n(\varphi(n)e_1) : n \in V\}, \tag{4.4}$$

where $N = \mathrm{span}\{e_2, \dots, e_{2n+1}\}$, V is an open set of N and $\varphi : V \longrightarrow \mathbb{R}$ is continuous.

Sets having the form given in (4.4) are called "intrinsic graphs" since they are represented by the group operation.

4.2 Sobolev spaces

On an open set Ω of $\mathbb{H}^n$ *equipped with the Lebesgue measure*, we define the *anisotropic Sobolev space*

$$W_h^{1,p} = \left\{u \in L^p(\Omega) : X_j u \in L^p(\Omega), 1 \leq j \leq m\right\}, \tag{4.5}$$

where $1 \leq p \leq \infty$ and $X_j u$ is meant in the distributional sense, namely, as those function satisfying

$$\int_\Omega u\, X_j \varphi = -\int_\Omega X_j u\, \varphi \tag{4.6}$$

for every $\varphi \in C_c^\infty(\Omega)$. Formula (4.6) can be proved in the case u is smooth. This is a consequence of the divergence theorem and the fact that $\mathrm{div} X_j = 0$ for all j. The last condition can be written as a condition for the *formal adjoint X_j^**,

$$X_j^* = -X_j.$$

If we define

$$||u||_{1,p} = ||u||_{L^p(\Omega)} + ||\nabla_h u||_{L^p(\Omega)},$$

then we make $W_h^{1,p}(\Omega)$ a Banach space.

4.2.1 Smooth approximation

The classical Meyers-Serrin theorem "H=W"asserts that the Sobolev space is the closure of smooth functions in the topology of the Sobolev norm. This fact also true for the anisotropic Sobolev spaces and holds in the more general class of Carnot-Carathèodory spaces. The main reference for this resul are the papers by B. Franchi, R. Serapioni and F. Serra Cassano, Houston J. Math (1996) and by N. Garofalo and D.M. Nhieu, CPAM (1996). Precisely, for any $1 \le p < \infty$ we have

$$W_h^{1,p}(\Omega) = W_h^{1,p}\text{-closure}\left(C^1(\Omega) \cap W_h^{1,p}(\Omega)\right) .$$

The main point here is that the classical smoothing procedure still works, up to a subtle observation, according to the classical work by Friedrichs (1944).

From the paper by N. Garofalo and D.M. Nhieu (1996), we state the key point.

Lemma 4.2.1 (Friedrichs lemma) *For any first order differential operator $Y = \sum_{j=1}^n b_j(x)\,\partial_{x_j}$ in $\mathbb{R}^n$ and any symmetric Euclidean mollifier $\phi_\varepsilon = \varepsilon^{-n}\phi(\frac{\cdot}{\varepsilon})$ and every $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ we have*

$$Y(\phi_\varepsilon * u) = \phi_\varepsilon * Yu + R_\varepsilon u$$

in the distributional sense. We have defined

$$R_\varepsilon u(x) = \int_{\mathbb{R}^n} u(x + \varepsilon h)\,\tilde{K}_\varepsilon(x, h)\,dh$$

and $\tilde{K}_\varepsilon(x, h) = \varepsilon^{-1} \sum_{j=1}^n \partial_{h_j}[(b_j(x + \varepsilon h) - b_j(x))\,\phi(h)]$.

4.3 Lipschitz functions

The following theorem has been proved in general Carnot-Carathéodory spaces by N. Garofalo and D. M. Nhieu, J. Analyse Math. (1998).

Theorem 4.3.1 *Any Lipschitz function $f : \Omega \longrightarrow \mathbb{R}$ belongs to $W^{1,\infty}_{h,loc}(\Omega)$.*

Sketch. Consider $\varphi \in C_c^\infty(\Omega)$ along with its support K contained in Ω and define

$$f(x, t, w) = \frac{f(x\delta_t w) - f(x)}{t},$$

that for any fixed w and sufficiently small t is well defined in Ω whenever $x \in K$, hence we consider the integral

$$I_t = \int_\Omega f(x, t, w)\,\varphi(x)\,dx\,.$$

The change of variable $z = x\delta_t w$ preserves the measure, hence

$$I_t = \int_\Omega \frac{f(z)\varphi(z\delta_t w^{-1})}{t}\,dz - \int_\Omega \frac{f(x)\varphi(x)}{t}\,dx = \int_\Omega f(x)\,\frac{\varphi(z\delta_t w^{-1}) - \varphi}{t}\,dz.$$

This leads to the existence of the limit

$$\lim_{t\to 0} I_t = -\int_\Omega f(x)\,\langle \nabla_h\varphi(x), w\rangle\,dx\,.$$

In the special case $w = e_j$ with $j = 1, \dots, 2n$, we get

$$\langle \nabla_h\varphi(x), w\rangle = X_j\varphi(x).$$

The Lipschitz continuity yields $|f(x,t,e_j)| \le L_f$ for any t sufficiently small. We have

$$\|f(x,t,e_j)\,\mathbf{1}_K\|_{L^\infty(\Omega)} \le L_f\,.$$

By the weak*-compactness of bounded sequences in $L^\infty(K)$, there exists an infinitesimal sequence (t_k) and $g_j \in L^\infty(K)$ such that

$$\lim_{k\to\infty} I_{t_k} = \int_\Omega g_j(x)\,\varphi(x)\,dx$$

that gives

$$\int_\Omega g_j(x)\,\varphi(x)\,dx = -\int_\Omega f(x)\,X_j\varphi(x)\,dx.$$

By the arbitrary choice of φ, we have that g_j is uniquely defined in Ω up to a negligible set, hence $X_j f$ exist for $j = 1, \dots, 2n$ and belong to $L^\infty_{loc}(\Omega)$. □

Joining Theorem 4.3.1 with Lemma 4.2.1 one reaches the following result.

Theorem 4.3.2 *If $f : \Omega \longrightarrow \mathbb{R}$ is Lipschitz continuous, then $f_\varepsilon = f * K_\varepsilon$ locally uniformly converges to f and $X_j f_\varepsilon$ a.e. converges to $X_j f$ as $\varepsilon \to 0^+$ and for all $j = 1, \dots, 2n$, where $X_j f$ are the distributional horizontal derivatives of f.*

The pointwise convergence of $X_j f_\varepsilon$ holds exactly at Lebesgue points of both $X_j f$ and f.