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Unrectifiability and rigidity in stratified groups

By

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Abstract. In the class of stratified groups endowed with a left invariant Carnot-Carathéodory distance, we give an algebraic characterization of purely unrectifiable groups and we study rigidity properties. The main feature of our approach is the use of a suitable area formula with respect to the Carnot-Carathéodory distance.

1. Introduction. The geometry of stratified groups is the source of several recent investigations connected to Geometric Control Theory, Differential Geometry, PDEs, Sobolev spaces and Geometric Measure Theory. A detailed account of these developments goes beyond the scope of this note, then we limit ourselves to mention some relevant monographs [3], [6], [8], [10], [16], where further references can be found.

A stratified group, commonly known as "Carnot group", is a simply connected nilpotent Lie group with a graded algebra, [6]. Its metric structure is characterized by the first layer of the Lie algebra, where a left invariant scalar product is defined. This structure yields a left invariant "Carnot-Carathéodory distance", see Section 2. Stratified groups endowed with a Carnot-Carathéodory distance constitute a very interesting class of metric spaces to be investigated, due to the richness of their structure. In fact, they are connected by rectifiable curves, satisfy the Poincaré inequality, [7], and possess a one parameter family of dilations which properly scale with the Carnot-Carathéodory distance.

The purpose of developing a metric theory of currents has been successfully achieved in [2], where k-rectifiable sets (see 3.2.14 of [5]) play an important role. In connection with the study of rectifiability in metric spaces, the existence of k-rectifiable sets in stratified groups has been first investigated in [1], where it has been proved that the three dimensional Heisenberg group is purely k-unrectifiable for every k greater than one. In other words, k-rectifiable sets in the Heisenberg group are all $\mathcal{H}_d^k$-negligible, where $\mathcal{H}_d^k$ denotes the k-dimensional Hausdorff measure with respect to the Carnot-Carathéodory distance d of the group. The first result of this note is a simple algebraic characterization of all purely k-unrectifiable stratified groups.

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Theorem 1.1. *Let $\mathbb{M}$ be a stratified group with Lie algebra $\mathcal{M} = W_1 \oplus W_2 \oplus \cdots \oplus W_\iota$. Then $\mathbb{M}$ is purely k-unrectifiable if and only if there do not exist k-dimensional Lie subalgebras contained in the first layer W_1.*

This characterization already contains the leading theme of this note, concerning the strong connection between algebraic and metric structure of a stratified group. According to the notion of rectifiability proposed in [12], a subset E of $\mathbb{M}$ is $\mathbb{G}$-rectifiable if it is the image of a Lipschitz mapping defined on a subset of $\mathbb{G}$, where $\mathbb{G}$ is a subgroup of a possibly different stratified group. In connection with this notion, we more generally establish a criterion for $\mathbb{G}$-unrectifiability in the case $\mathbb{G}$ is a stratified group, see Definition 3.1.

Theorem 1.2. *Let $\mathbb{M}$ and $\mathbb{G}$ be stratified groups with Lie algebras $\mathcal{M}$ and $\mathcal{G}$, respectively. Then $\mathbb{M}$ is purely $\mathbb{G}$-unrectifiable if and only if $\mathcal{M}$ does not contain any Lie subalgebra which is G-isomorphic to $\mathcal{G}$.*

We realize that these theorems are a simple consequence of the area formula for Lipschitz mappings between stratified groups, [9], [12], [17]. Recall that two Lie algebras of stratified groups are G-isomorphic if there exists an algebra isomorphism that respects the grading, see Definition 2.7. It is easy to see that Theorem 1.2 yields Theorem 1.1 in the case $\mathbb{G} = \mathbb{R}^k$, see Remark 3.3.

We emphasize the fact that area formula in stratified groups is a consequence of the remarkable work by Pansu [11], where an extension of the classical Rademacher theorem to stratified groups is obtained. This theorem is one of the key ingredients in the proof of Theorem 3 in [11], which amounts to the following rigidity result, proved in the same paper: two biLipschitz equivalent stratified groups are G-isomorphic. Our approach for finding criteria of unrectifiability also yields a slightly improved version of the previous rigidity result.

Theorem 1.3. *Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups and let $A \subset \mathbb{G}$ be a subset of positive measure. If there exists a biLipschitz mapping $f : A \longrightarrow \mathbb{M}$ such that $f(A) \subset \mathbb{M}$ has positive measure, then $\mathbb{G}$ and $\mathbb{M}$ are G-isomorphic.*

According to results of [13], the previous theorem in the case $\mathbb{G}$ is the Euclidean space $\mathbb{R}^k$ and $\mathbb{M}$ is the Heisenberg group $\mathbb{H}^{2n+1}$ implies that any open subset of $\mathbb{H}^{2n+1}$ cannot be parametrized by any biLipschitz mapping defined on an open subset of $\mathbb{R}^k$ for any $k \geqq 1$. Similarly, another simple application of area formula immediately shows that Nash's embedding theorem for Riemannian manifolds cannot be extended to any noncommutative stratified group endowed with a Carnot-Carathéodory distance, see Remark 3.5.

2. Some preliminary notions.

This introductory section is devoted to the essential notions we will use throughout.

Definition 2.1 (Stratified group). A *stratified group* $\mathbb{G}$ is a simply connected Lie group, whose Lie algebra $\mathcal{G}$ is nilpotent and can be written as a direct sum of subspaces

$V_1 \oplus \cdots \oplus V_\iota$ with the property $[V_1, V_j] = V_{j+1}$ for every $j \geqq 1$, where $V_j = \{0\}$ whenever $j > \iota$. The integer ι is called the *step* of the group.

Throughout the paper we will denote by W_j the factors of the Lie algebra $\mathcal{M}$ of a stratified group $\mathbb{M}$, according to the previous definition, namely, $\mathcal{M} = W_1 \oplus \cdots \oplus W_\upsilon$. More information on stratified groups can be found for instance in [6]. In view of the next definition, we recall that the exponential map $\exp_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism whenever $\mathbb{G}$ is simply connected and nilpotent, [4].

Definition 2.2 (Dilations). Let $\mathbb{G}$ be a stratified group and let $\mathcal{G}$ be its Lie algebra. We define *dilations* on $\mathcal{G}$ by $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$, $\delta_r \left(\sum_{j=1}^{\iota} v_j \right) = \sum_{j=1}^{\iota} r^j v_j$ for every $r > 0$. The exponential map $\exp_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathbb{G}$ transfers dilations from $\mathcal{G}$ to $\mathbb{G}$, hence they are automatically defined on $\mathbb{G}$. We will often denote dilations by $\delta_r^{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}$ or by $\delta_r^{\mathbb{G}} : \mathbb{G} \longrightarrow \mathbb{G}$, depending on their domain.

Definition 2.3. A continuous left invariant distance $d : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty[$ such that $d(\delta_r x, \delta_r y) = r\, d(x, y)$ for every $r > 0$, is called *homogeneous distance*.

Every stratified group possesses an important example of homogeneous distance, called the Carnot-Carathéodory distance. To any point of $p \in \mathbb{G}$ we can associate the subspace $H_p\mathbb{G} = \{X(p) \in T_p\mathbb{G} \mid X \in V_1\}$, where V_1 is the first layer of the Lie algebra $\mathcal{G}$. Here we have identified $\mathcal{G}$ with the space of all left invariant vector fields. An absolutely continuous curve γ whose velocity vector $\gamma'(t)$ is contained in $H_{\gamma(t)}\mathbb{G}$ for a.e. t is called *horizontal curve*. Our hypothesis on the algebra $\mathcal{G}$ implies that iterated commutators of left invariant vector fields of V_1 span any direction of $\mathbb{G}$, hence as a consequence of the well known Chow-Rashevsky theorem any two points of $\mathbb{G}$ are connected by horizontal curves, see Theorem 2.4 at p. 15 of [3]. Taking the infimum among lengths of horizontal curves connecting two points $p, q \in \mathbb{G}$ we obtain the Carnot-Carathéodory distance $d(p, q)$ between them. Note that this distance in the case of stratified groups can be chosen as left invariant by fixing a left invariant Riemannian metric on $\mathbb{G}$ which measures the length of horizontal curves. It is clear that we can choose a smooth metric only on the fibers $H_p\mathbb{G}$ of $\mathbb{G}$. We also point out that continuity, left invariance and homogeneity of homogeneous distances imply that they are all biLipschitz equivalent.

Recall that stratified groups of step one coincide with Euclidean spaces and dilations become the standard multiplication by a scalar number. In this case the Carnot-Carathéodory distance is the Euclidean one. In the sequel the homogeneous distances ρ and d will be fixed on the stratified groups $\mathbb{M}$ and $\mathbb{G}$, respectively.

Remark 2.4. Using dilations introduced in Definition 2.2, and observing that the Jacobian of $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$ is given by r^Q, where $Q = \sum_{j=1}^{\iota} j \dim(V_j)$ and $\mathcal{G} = V_1 \oplus V_2 \oplus \cdots V_\iota$, one can check that the Hausdorff dimension of $\mathbb{G}$ with respect to any homogeneous distance is equal to Q. This also implies that the Q-dimensional Hausdorff measure

$\mathcal{H}_d^Q$ is left invariant, finite and positive on bounded measurable sets, hence it is the Haar measure of the group (up to a constant factor).

Definition 2.5. A *G-linear map* $L : \mathbb{G} \longrightarrow \mathbb{M}$ is a group homomorphism between stratified groups $\mathbb{G}$ and $\mathbb{M}$, having the property $L(\delta_r^{\mathbb{G}} x) = \delta_r^{\mathbb{M}} L(x)$ for every $x \in \mathbb{G}$ and every $r > 0$. An algebra homomorphism $\mathcal{L} : \mathcal{G} \longrightarrow \mathcal{M}$ will be called *G-linear* if it satisfies $\mathcal{L}(\delta_r^{\mathcal{G}} v) = \delta_r^{\mathcal{M}} \mathcal{L}(v)$ for every $v \in \mathcal{G}$ and every $r > 0$.

Note that a G-linear map is automatically continuous, see Proposition 3.11 of [9]. Thus, we can introduce the following definition.

Definition 2.6. Let d and ρ be homogeneous distances on $\mathbb{G}$ and $\mathbb{M}$, respectively. The *homogeneous norm* of a G-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ is defined as follows

$$\|L\| = \sup_{d(x) \leqq 1} \rho(L(x), e_{\mathbb{M}}),$$

where $e_{\mathbb{M}}$ denotes the unnit element of $\mathbb{M}$.

The homogeneity of homogeneous distances implies that for every $x \in \mathbb{G}$ we have

$$\rho(L(x), e_{\mathbb{M}}) \leqq \|L\| \, d(x, e_{\mathbb{G}}), \tag{1}$$

where $e_{\mathbb{G}}$ denotes the unit element of $\mathbb{G}$. Inequality (1) will be used in the proof of Proposition 2.11.

Definition 2.7. Two stratified groups $\mathbb{G}$ and $\mathbb{M}$ are said to be *G-isomorphic* if there exists an invertible G-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$. Equivalently, the corresponding Lie algebras $\mathcal{G}$ and $\mathcal{M}$ are called *G-isomorphic* if there exists an invertible G-linear map $\mathcal{L} : \mathcal{G} \longrightarrow \mathcal{M}$.

Remark 2.8. Let $\exp_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathbb{G}$ and $\exp_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathbb{M}$ denote the exponential mappings of $\mathbb{G}$ and $\mathbb{M}$, respectively. For each G-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ the composition $\mathcal{L} = \exp_{\mathcal{M}}^{-1} \circ L \circ \exp_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{M}$ is a G-linear map of Lie algebras and vice versa. Thus, $\mathbb{G}$ and $\mathbb{M}$ are G-isomorphic if and only if so are their corresponding Lie algebras.

Let A be a measurable subset of $\mathbb{G}$. We denote by $\mathcal{I}(A)$ the subset of all *density points* of A, namely, the set of points $x \in A$ such that

$$\lim_{r \to 0^+} \mathcal{H}_d^Q(A \cap B_{x,r}) / \mathcal{H}_d^Q(B_{x,r}) = 1.$$

It is a general fact that $\mathcal{H}_d^Q(A \setminus \mathcal{I}(A)) = 0$, see for instance [5]. The following important notion of differentiability is due to Pansu, see Section 1.3 of [11].

Definition 2.9 (P-differentiability). Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups endowed with homogeneous distances d and ρ, respectively. We say that a map $f : A \longrightarrow \mathbb{M}$ is *P-differentiable* at $x \in \mathcal{I}(A)$ if there exists a G-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ such that

$$\lim_{y \in A, \; y \to x} \frac{\rho\big(f(x)^{-1} f(y), L(x^{-1} y)\big)}{d(x, y)} = 0 \,. \tag{2}$$

Note that when f is Lipschitz, the G-linear map L in (2) is uniquely defined, see Proposition 2.2 of [9]. The G-linear map of (2) is the *P-differential* of f at x. We will utilize the usual notation $df(x)$ to denote the P-differential, according to the fact that when $\mathbb{G}$ and $\mathbb{M}$ are Euclidean spaces, P-differentiability becomes the classical notion of differentiability.

Theorem 2.10 (Pansu)**.** *Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups and let $f : A \longrightarrow \mathbb{M}$ be a Lipschitz mapping, where $A \subset \mathbb{G}$. Then f is $\mathcal{H}^Q_d$-a.e. P-differentiable.*

The proof of this theorem is due to Pansu, see Théorème 2 of [11]. Its extension to the case of measurable domains is contained in [14], see also [9].

Proposition 2.11 (Chain rule)**.** *Let $\mathbb{G}$, $\mathbb{P}$ and $\mathbb{M}$ be stratified groups and consider $f : A \longrightarrow \mathbb{P}$, assuming that it is P-differentiable at $x \in \mathcal{I}(A)$. Let $g : f(A) \longrightarrow \mathbb{M}$ be P-differentiable at $f(x) \in \mathcal{I}(f(A))$, where $A \subset \mathbb{G}$. Then $g \circ f : A \longrightarrow \mathbb{M}$ is P-differentiable at x, with P-differential $d(g \circ f)(x) = dg(f(x)) \circ df(x)$.*

Proof. Let d, ν and ρ be homogeneous distances on $\mathbb{G}$, $\mathbb{P}$ and $\mathbb{M}$, respectively. We define $h = g \circ f$, $L = dg(y) \circ df(x)$, $y = f(x)$ and fix $\varepsilon > 0$. By hypothesis there exists $\delta > 0$ such that

$$\begin{aligned} \rho(h(x)^{-1}h(u), L(x^{-1}u)) &\leqq \rho(h(x)^{-1}h(u), dg(y)(y^{-1}f(u))) \\ &\quad + \|dg(y)\|\ \nu(df(x)(x^{-1}u), y^{-1}f(u)) \\ &\leqq \varepsilon\ \nu(y, f(u)) + \|dg(y)\|\ \varepsilon\ d(x,u), \end{aligned} \tag{3}$$

whenever $d(x,u)$, $\nu(y, f(u)) \leqq \delta$. The P-differentiability of f at x implies that

$$\nu(y, f(u)) \leqq (\|df(x)\| + 1)\, d(x,u) \leqq \delta$$

whenever $d(u,x) \leqq \delta'$, for some $\delta' \in]0, \delta[$. Replacing the latter inequality in (3) our claim follows. □

Definition 2.12 (Jacobian). Let $L : \mathbb{G} \longrightarrow \mathbb{M}$ be a G-linear map. The *Jacobian* of L is defined by

$$J_Q(L) = \mathcal{H}^Q_\rho(L(B_1))/\mathcal{H}^Q_d(B_1).$$

The next theorem states the area formula for Lipschitz mappings between stratified groups, [9], [12], [17].

Theorem 2.13 (Area formula)**.** *Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups and let $A \subset \mathbb{G}$ be a measurable subset. Then for every Lipschitz mapping $f : A \longrightarrow \mathbb{M}$ there holds*

$$\int_A J_Q(df(x))\, d\mathcal{H}^Q_d(x) = \int_{\mathbb{M}} N(f, A, y)\, d\mathcal{H}^Q_\rho(y). \tag{4}$$

We have denoted by $N(f, A, y)$ the *multiplicity function*, being equal to $+\infty$ if $f^{-1}(y) \cap A$ is not finite and to $\#(f^{-1}(y) \cap A)$ otherwise.

R e m a r k 2.14. Note that the Jacobian of a G-linear map L vanishes whenever L is not injective. This can be seen as follows. In view of Remark 2.4 the Hausdorff dimension of $\mathbb{G}$ is $Q = \sum_{j=1}^{\iota} j \dim(V_j)$. If $L : \mathbb{G} \longrightarrow \mathbb{M}$ is a G-linear map which is not injective, neither is the G-linear map $\mathcal{L} = \exp_{\mathcal{M}}^{-1} \circ L \circ \exp_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{M}$. Therefore, there exists $v = \sum_{j=1}^{\iota} v_j \in \mathcal{G} \setminus \{0\}$ with $v_j \in V_j$ for every $j = 1, \dots, \iota$, such that $\sum_{j=1}^{\iota} \mathcal{L}(v_j) = 0$. The contact property of G-linear maps proved in Corollary 3.15 of [9] shows that $\mathcal{L}(V_1) \subset W_1$. Since $\mathcal{L}$ is also an algebra homomorphism, then $\mathcal{L}(V_j) \subset W_j$ for every $j \geqq 1$. From the fact that $\mathcal{M}$ is a direct sum of W_j and that $\mathcal{L}(v_j) \in W_j$, we conclude that $\mathcal{L}(v_j) = 0$ for every $j = 1, \dots, \iota$. Let $v_{j_0} \neq 0$ for some $j_0 \in \{1, \dots, \iota\}$. It follows that $\mathcal{L}(v_{j_0}) = 0$, then $\dim(L(V_{j_0})) < \dim(V_{j_0})$ and the Hausdorff dimension of the image $L(\mathbb{G})$ is $\sum_{j=1}^{\iota} j \dim(L(V_j)) < \sum_{j=1}^{\iota} j \dim(V_j) = Q$. This shows that $\mathcal{H}^Q(L(\mathbb{G})) = 0$. In view of Definition 2.12, we have proved that $J_Q(L) = 0$.

3. Unrectifiability and rigidity. In this section we give the proofs of theorems stated in the introduction.

D e f i n i t i o n 3.1. Let $\mathbb{G}$ be a stratified group with Hausdorff dimension Q. We say that a metric space (Y, ρ) is *purely* $\mathbb{G}$*-unrectifiable* if for every Lipschitz mapping $f : A \longrightarrow Y$, with $A \subset \mathbb{G}$, we have $\mathcal{H}^Q_\rho(f(A)) = 0$.

R e m a r k 3.2. In the case $\mathbb{G} = \mathbb{R}^k$, a stratified group $\mathbb{M}$ is purely $\mathbb{G}$-unrectifiable if and only if it is *purely k-unrectifiable*, according to 3.2.14 of [5].

P r o o f o f T h e o r e m 1.2. Assume that $\mathbb{M}$ is purely $\mathbb{G}$-unrectifiable. Reasoning by contradiction, if there existed a Lie subalgebra $\mathcal{A} \subset \mathcal{M}$, which is G-isomorphic to $\mathcal{G}$, then we would have an injective G-linear map $\mathcal{L} : \mathcal{G} \longrightarrow \mathcal{M}$. Thus, the G-linear map $L = \exp_{\mathcal{M}} \circ \mathcal{L} \circ \exp_{\mathcal{G}}^{-1} : \mathbb{G} \longrightarrow \mathbb{M}$ is also injective and from Proposition 3.18 of [9] we conclude that $J_Q(L) > 0$. By virtue of (1) we know that every G-linear map is Lipschitz, hence by (4) we immediately have $\mathcal{H}^Q_\rho(L(A)) > 0$, whenever $A \subset \mathbb{G}$ is a measurable subset with positive measure. Thus, $\mathbb{M}$ cannot be purely $\mathbb{G}$-unrectifiable. Conversely, suppose that $\mathcal{M}$ does not contain any Lie subalgebra G-isomorphic to $\mathcal{G}$ and let $f : A \longrightarrow \mathbb{M}$ be a Lipschitz mapping defined on a measurable subset $A \subset \mathbb{G}$. In view of (4), we have

$$\mathcal{H}^Q_\rho(f(A)) \leqq \int_A J_Q(df(x))\, d\mathcal{H}^Q_d(x). \tag{5}$$

Now we notice that the P-differential $df(x) : \mathbb{G} \longrightarrow \mathbb{M}$ cannot be injective. In fact, if this were the case, then the Lie subalgebra $\exp_{\mathcal{M}}^{-1} \circ df(x) \exp_{\mathcal{G}}(\mathcal{G}) \subset \mathcal{M}$ would turn out to be G-isomorphic to $\mathcal{G}$. As we have noticed in Remark 2.14 the non injectivity gives $J_Q(df(x)) = 0$, whence inequality (5) yields $\mathcal{H}^Q_\rho(f(A)) = 0$. This ends the proof.

Remark 3.3. Let us point out that the statement of Theorem 1.1 is contained in that of Theorem 1.2 as a particular case. To see this, it suffices to show that the existence of a k-dimensional Lie subalgebra $\mathcal{A} \subset \mathcal{M}$, which is G-isomorphic to $\mathbb{R}^k$ is necessarily contained in the first layer W_1 of the Lie algebra $\mathcal{M}$. Let $\mathcal{L} : \mathbb{R}^k \longrightarrow \mathcal{A}$ be a G-linear map. Then by virtue of Corollary 3.15 of [9] we obtain that $\mathcal{A} = \mathcal{L}(\mathbb{R}^k) \subset W_1$. Conversely, if $\mathcal{A}$ is a k-dimensional Lie subalgebra of $\mathcal{M}$ contained in W_1, then it is G-isomorphic to $\mathbb{R}^k$. In fact, we have

$$[\mathcal{A}, \mathcal{A}] \subset W_2 \cap \mathcal{A} \subset W_2 \cap W_1 = \{0\}$$

hence $\mathcal{A}$ is abelian, namely, it is G-isomorphic to $\mathbb{R}^k$.

Remark 3.4. By Theorem 1.1 it is easy to see that whenever $\dim(W_1) < k$ the stratified group $\mathbb{M}$ with graded algebra $\mathcal{M} = W_1 \oplus \cdots \oplus W_\iota$ is purely k-unrectifiable. This is the case when we consider $k > 2$ and $\mathbb{M}$ equal to the three dimensional Heisenberg group $\mathbb{H}^3$, whose Lie algebra $\mathfrak{h}_3 = V_1 \oplus V_2$ has the properties $[V_1, V_1] = V_2$, $\dim(V_1) = 2$ and $\dim(V_2) = 1$. Then we conclude that $\mathbb{H}^3$ is purely k-unrectifiable for every $k > 2$. Furthermore, V_1 is not a subalgebra of $\mathfrak{h}^3$, since $[X, Y] \notin V_1$ whenever X, Y are linearly independent vectors of V_1, hence $\mathbb{H}^3$ is also purely 2-unrectifiable, according to results in Section 7 of [1].

Proof of Theorem 1.3. We argue by contradiction, assuming that $\mathbb{G}$ and $\mathbb{M}$ are not G-isomorphic and that there exists a biLipschitz mapping $f : A \longrightarrow \mathbb{M}$, where $A \subset \mathbb{G}$ and $f(A) \subset \mathbb{M}$ have positive measure. Let Q and Q' be the Hausdorff dimensions of $\mathbb{G}$ and $\mathbb{M}$, respectively. Since f is biLipschitz we have $\mathcal{H}^Q_\rho(f(A)) > 0$, in addition for every $r > 0$ we have

$$\mathcal{H}^Q_\rho(f(A \cap B_r)) \leqq \mathrm{Lip}(f)^Q\, \mathcal{H}^Q_d(A \cap B_r) < \infty,$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant of f and $B_r \subset \mathbb{G}$ is the open ball with center at the unit element and radius r. Thus, by uniqueness of the Hausdorff dimension we have $Q = Q'$. Now, we divide A into three disjoint subsets A_0, A_1 and A_2, where A_0 is the subset of points either belonging to $A \setminus \mathcal{I}(A)$ or where f is not P-differentiable, A_1 is the subset of points where the P-differential of f is surjective and A_2 is the subset of points where the P-differential of f is not surjective. As a consequence of Theorem 2.10 and the fact that density points have full measure, we have $\mathcal{H}^Q_d(A_0) = 0$. Let $x \in A_1$ and consider the P-differential $df(x) : \mathbb{G} \longrightarrow \mathbb{M}$. By our assumption the G-linear map $df(x)$ cannot be a G-isomorphism, hence it is not injective. By Remark 2.14 we have $J_Q(df(x)) = 0$ and the area formula (4) gives

$$\mathcal{H}^Q_\rho(Z_1) = \int_{A_1} J_Q\,(df(x))\; d\mathcal{H}^Q_d(x) = 0\,,$$

where $Z_1 = f(A_1)$. The biLipschitz property of f gives $\mathcal{H}^Q_d(A_1) = 0$. Now we define $g = f^{-1} : Z_2 \longrightarrow A_2$, where $Z_2 = f(A_2)$ and consider the subset $Z_2' \subset \mathcal{I}(Z_2)$ where g is P-differentiable. Thus, we have both $\mathcal{H}^Q_\rho(Z_2 \setminus Z_2') = 0$ and $\mathcal{H}^Q_d(A_2 \setminus A_2') = 0$, where we have defined $A_2' = g(Z_2')$. As a consequence of Proposition 2.11 we can P-differentiate the composition $\mathrm{id}_{A_2'} = g \circ f : A_2' \longrightarrow A_2'$, getting

$$\mathrm{id}_{\mathbb{G}} = dg(f(x)) \circ df(x), \tag{6}$$

for any $x \in A_2'$. By virtue of (6) $dg(y)$ is surjective for every $y \in Z_2' = f(A_2')$, then it cannot be injective for every $y \in Z_2' = f(A_2')$. From the previous argument we conclude that $\mathcal{H}_d^Q(g(Z_2')) = \mathcal{H}_d^Q(A_2') = 0$. Joining the fact that $\mathcal{H}_d^Q(A_1) = 0$, we have obtained that $\mathcal{H}^Q(A) = 0$, which conflicts with our assumption $\mathcal{H}_d^Q(A) > 0$.

Remark 3.5. As a final application of area formula, we show that any noncommutative stratified group cannot be embedded into $\mathbb{R}^k$ by a biLipschitz mapping, for every $k \geqq 1$. By contradiction, if we had an open subset Ω of $\mathbb{G}$ and a biLipschitz mapping $f : \Omega \longrightarrow \mathbb{R}^k$, then we would obtain

$$0 < \mathcal{H}_d^Q(\Omega) \leqq \mathrm{Lip}\left((f^{-1})_{|f(\Omega)}\right)^Q \mathcal{H}_{|\cdot|}^Q(f(\Omega)), \tag{7}$$

where Q is the Hausdorff dimension of $\mathbb{G}$ and $\mathcal{H}_{|\cdot|}^Q$ denotes the Q-dimensional Hausdorff measure with respect to the Euclidean distance $|\cdot|$. On the other hand, any G-linear map $L : \mathbb{G} \longrightarrow \mathbb{R}^k$ cannot be injective in that $\mathbb{G}$ is not commutative and L is a group homomorphism, hence $J_Q(L) = 0$. By area formula (4) it follows that $\mathcal{H}_{|\cdot|}^Q(f(\Omega)) = 0$, leading us to a contradiction with (7).

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