The coarea formula for real-valued Lipschitz maps on stratified groups

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Abstract

We establish a coarea formula for real-valued Lipschitz maps on stratified groups when the domain is endowed with a homogeneous distance and level sets are measured by the Q-1 dimensional spherical Hausdorff measure. The number Q is the Hausdorff dimension of the group with respect to its Carnot-Carathéodory distance. We construct a Lipschitz function on the Heisenberg group which is not approximately differentiable on a set of positive measure, provided that the Euclidean notion of differentiability is adopted. The coarea formula for stratified groups also applies to this function, where the Euclidean one clearly fails. This phenomenon shows that the coarea formula holds for the natural class of Lipschitz functions which arises from the geometry of the group and that this class may be strictly larger than the usual one.

Contents

Mathematics Subject Classification: 28A75 (22E25)
Key words and phrases: coarea formula, stratified groups, Lipschitz maps

Introduction

It is well known that the coarea formula in Euclidean spaces is an important tool of Geometric Measure Theory with several applications to Analysis. Let $u : A \longrightarrow \mathbb{R}$ be a Lipschitz function, where $A \subset \mathbb{R}^n$ is measurable. The coarea formula reads as follows

$$\int_A h(x)\, |\nabla u(x)| dx = \int_{\mathbb{R}} \int_{u^{-1}(y)} h(x)\, \mathcal{H}^{n-1}_{|\cdot|}(x)\, dy\,, \tag{1}$$

where $\mathcal{H}^{n-1}_{|\cdot|}$ denotes the Hausdorff measure with respect to the Euclidean distance and $h : A \longrightarrow [0,\infty]$ is a measurable function, see for instance [12]. The purpose of the present paper is to obtain a version of (1) for real-valued Lipschitz maps on stratified groups endowed with either the Carnot-Carathéodory distance or any homogeneous distance. The interest in developing tools of Geometric Measure Theory on stratified groups has several motivations that arise from subelliptic PDEs, Harmonic Analysis, Sobolev spaces with respect to Hörmander vector fields and the geometry of Carnot-Carathéodory spaces. Some relevant books are the following [6], [13], [19], [30], [39]. There are several recent contributions in this stream, we cite the papers [1], [4], [5], [10], [14], [15], [16], [17], [18], [22], [23], [24], [25], [26], [31], [34], [35], [36], [40], but surely the list could be enlarged.

Recall that a stratified group is a simply connected nilpotent Lie group with a graded Lie algebra. These groups encompass a wide family of geometries, where the Euclidean one fits into the commutative case. Two nonisomorphic stratified groups cannot be biLipschitz equivalent even locally. In fact, their metric properties might be considerably diverse, [27], [33], [37]. It turns out that results which are valid for arbitrary stratified groups actually hold for a huge family of geometries which are different from each other. However, it may happen that some facts only hold for some class of stratified groups and not for anyone. This is one of the reasons why the aim to establish results and formulae for arbitrary stratified groups is often a delicate matter.

Let us look more closely to the object of the present paper. The coarea formula for functions of bounded variations can be stated in the general framework of metric spaces, as it has been proved in [29]. A general version on Carnot-Carathéodory spaces can be found in [14], [18] and [31]. In this case the notion of perimeter measure with respect to a set of vector fields plays a crucial role, [8]. Another intrinsic surface measure can be naturally considered in stratified groups, which possess a precise Hausdorff dimension, denoted by Q. This is the spherical Hausdorff measure $\mathcal{S}^{Q-1}$, which is built with respect to a homogeneous distance of the group. Here a natural question comes up: when we consider Lipschitz functions with respect to the Carnot-Carathéodory distance can we replace in the coarea formula the perimeter measure of lower level sets $\{u < t\}$ with the restriction of $\mathcal{S}^{Q-1}$ to the level sets $\{u = t\}$? This problem has been raised in [31], Remark 4.8, where it is also pointed out that this formulation is meaningless on general Carnot-Carathéodory spaces, where the Hausdorff dimension might change in different regions of the space. A first partial answer to this question has been given in [25], where the coarea formula is obtained in the case of Heisenberg groups through rectifiability of the perimeter measure in this class of groups, [15]. In Theorem 3.5 of the present paper we answer this question for arbitrary stratified groups without relying on any rectifiability result for the perimeter measure. In fact the rectifiability of perimeter measure is presently known only for stratified groups of step two, [17]. We emphasize the importance of the distance with respect to which the Lipschitz property is assumed. The case of Riemannian Lipschitz functions, i.e. where the Riemannian distance of the group is considered, deals with more regular maps and the corresponding coarea formula can be obtained using a blow-up of the measure associated to C^1 hypersurfaces on stratified groups, see the relevant works [21], [26] and [32].

In the present paper we will always deal with Lipschitz functions with respect to the Carnot-Carathéodory distance of the group. This class of functions is considerably larger than that of Riemannian Lipschitz functions and it includes also highly irregular maps which naturally arise from the Carnot-Carathéodory geometry of the group. This class of "intrinsic" Lipschitz functions has also a natural notion of differentiability, which is associated to the algebraic structure of the group. This is the notion of H-differentiability, introduced in Definition 1.12, which gives in turn the notion of C^1_H function, namely, an H-differentiable function with continuous H-differential. This definition was successfully introduced by Pansu in [33], in order to study rigidity properties. However, even C^1_H functions on noncommutative stratified groups might be extremely different from the usual C^1 functions. Through a technique used by Kirchheim and Serra Cassano, [22], we also show the existence of a C^1_H function which is not approximately differentiable in the Euclidean sense on a set of positive measure. Note that even the more general Euclidean coarea formula for Sobolev functions cannot be applied to this case, because it is well known that Sobolev functions are a.e. approximately differentiable. Concerning the coarea formula for Euclidean Sobolev mappings we refer the reader to the recent result by Malý, Swanson and Ziemer [28] and references therein. The intrinsic coarea formula we obtain in Theorem 3.5 also applies to the singular function described above, where the Euclidean coarea formula clearly has no meaning.

Our strategy relies on several recent results of Geometric Measure Theory both in stratified groups and in general metric spaces. We first consider the "weak coarea formula" for functions of bounded variations and we seek more regularity of level sets due to the Lipschitz property of the function. By a version of the Whitney extension theorem on stratified groups, proved in [17], we can assume that the Lipschitz function is C^1_H. The implicit function theorem on stratified groups, proved by Franchi, Serapioni and Serra Cassano in [16], imply that the regular part of the level set, where the H-differential does not vanish, can be represented locally by a continuous parametrization. In order to establish a relation between the perimeter measure of lower level sets and the spherical Hausdorff measure of the corresponding level sets we dilate and rescale a.e. level set at every regular point. Here the difficulty is that this process essentially amounts to an intrinsic differentation of the implicit map which locally parametrizes the level set and that is only continuous. The singular part of the level set, where the H-differential vanishes, might be highly singular. To get rid of this part we utilize other two results. The first, is a weak Sard-type theorem, proved in [25], by which the singular part of a.e. level set is $\mathcal{S}^{Q-1}$-negligible. The second, is a general result due to Ambrosio, according to which the perimeter measure on a general k-Ahlfors regular metric space supporting a weak $(1,1)$-Poincaré inequality is absolutely continuous with respect to $\mathcal{S}^{k-1}$, see [1]. It is well known that stratified groups have these properties with $k = Q$, see for instance [20]. Thus, we can conclude that also the perimeter measure of the singular part is vanishing.

Let A be a measurable set of a stratified group $\mathbb{G}$, let $u : A \longrightarrow \mathbb{R}$ be a Lipschitz function and let $h : A \longrightarrow [0, \infty]$ be a measurable function. Then the following coarea formula holds

$$\int_A h(x)\, |\nabla_H u(x)|\, dx = \int_{\mathbb{R}} \int_{u^{-1}(s)\cap A} h(x)\; \theta^g_{Q-1}(\nabla_H u(x))\; d\mathcal{S}^{Q-1}(x)\, ds\,. \tag{2}$$

The symbol ∇_H denotes the H-gradient of u (Definition 1.14) and $\theta^g_{Q-1}(\nabla_H u(p))$ is the metric factor (Definition 1.7). This function takes into account the anisotropy of the distance. Even in a finite dimensional space (corresponding to a commutative stratified group) the choice of a Banach norm different from the Euclidean norm makes the metric factor nontrivially depending on the direction of the gradient. However, in the class of rotational groups endowed with the associated Carnot-Carathéodory distance the metric factor $\theta^g_{Q-1}(\nu)$ does not depend on the horizontal direction ν. Furthermore, this class of groups includes Euclidean spaces, Heisenberg groups and more general H-type groups, see [26] for more information. Recently, Franchi, Serapioni and Serra Cassano have introduced a homogeneous distance d_∞ on arbitrary stratified groups, [17]. It is possible to check that this distance yields a constant metric factor, see Proposition 1.9. As a consequence, either on rotational groups with Carnot-Carathéodory distance or on arbitrary stratified group with the distance d_∞, we can define the geometric constant $\alpha_{Q-1} = \theta^g_{Q-1}$ and $\mathcal{S}^{Q-1}_{\mathbb{G}} = \alpha_{Q-1}\mathcal{S}^{Q-1}$, so that the coarea formula becomes

$$\int_A h(x)\, |\nabla_H u(x)|\, dx = \int_{\mathbb{R}} \int_{u^{-1}(s)\cap A} h(x)\; d\mathcal{S}^{Q-1}_{\mathbb{G}}(x)\, ds\,, \tag{3}$$

which resembles the Euclidean one. The possibility to replace the spherical Hausdorff measure $\mathcal{S}^{Q-1}_{\mathbb{G}}$ with the corresponding Hausdorff measure $\mathcal{H}^{Q-1}_{\mathbb{G}}$ is another nontrivial challenge. An interesting development of (2) would be its extension to Q-Ahlfors regular Carnot-Carathéodory spaces. Unfortuately, in this setting the implicit function theorem of [16] and the weak Sard-type theorem of [25] are not available.

Let us give a brief overview of the paper. In Section 1 we recall basic facts on stratified groups and introduce the important concepts of H-finite perimeter sets and of H-differentiability. In Section 2 we obtain an explicit formula for the perimeter measure of lower level sets in terms of the spherical Hausdorff measure and of the metric factor. Section 3 is devoted to the proof of coarea formula for Lipschitz functions and it is also presented its simpler form when suitable homogeneous distances are chosen. In Section 4 we construct special examples of C^1_H functions, which are not approximately differentiable in the Euclidean sense on a set of positive measure.

Acknowledgements. I am indebted to Bernd Kirchheim and Francesco Serra Cassano for having shown me their recent paper. I thank Luigi Ambrosio for fruitful discussions.

1 Preliminaries

In this section we introduce stratified groups and present their basic properties. Through their "sub-Riemannian" metric structure we recall the well known notions of H-finite perimeter set and of H-differentiability.

1.1 Basic notions on stratified groups

A *stratified group* is a simply connected nilpotent Lie group $\mathbb{G}$ endowed with a graded Lie algebra $\mathcal{G}$, which is decomposed into a direct sum of subspaces V_j subject to the condition $V_{j+1} = [V_j, V_1]$ for every $j \in \mathbb{N} \setminus \{0\}$, and $V_j = \{0\}$ whenever j is greater than some positive integer. We denote by ι the maximum integer such that $V_\iota \neq \{0\}$ and we call it the *nilpotence degree* of the group or the *step* of the group. Recall that for arbitrary subspaces $V, W \subset \mathcal{G}$ we define $[V, W]$ =span$\{[X, Y] \mid X \in V,\ Y \in W\}$.

The assumption that $\mathbb{G}$ is simply connected and nilpotent ensures that the exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism. This structure allows us to define dilations on the group, i.e. maps $\delta_t : \mathcal{G} \longrightarrow \mathcal{G}$ with $\delta_t(\sum_{j=1}^{\iota} v_j) = \sum_{j=1}^{\iota} t^j v_j$, where $t > 0$ and $\sum_{j=1}^{\iota} v_j \in \mathcal{G}$ is the unique representation of a vector of $\mathcal{G}$, provided that $v_j \in V_j$ for every $j = 1, \ldots, \iota$. This notion of dilation is motivated by the fact that the composition $\exp \circ \delta_t \circ \exp^{-1} : \mathbb{G} \longrightarrow \mathbb{G}$ is a group homomorphism. We will use the same symbol to denote dilations which are read on the group. More information on stratified groups can be found for instance in [9], [13] and [21].

The grading of the algebra allows us to choose a privileged basis of $\mathcal{G}$, where single bases of subspaces V_j are joined together in an ordered way. This is what we call an *adapted basis* to the grading of $\mathcal{G}$. We will denote this basis as follows

$$(X_1, X_2, \ldots, X_m, Y_{m+1}, \ldots, Y_q) \tag{4}$$

where only the basis $(X_1, \ldots, X_m)$ of V_1 is emphasized by a different notation, due to its privileged role. The space V_1 defines the *horizontal subbundle* of the group. We introduce the *horizontal space* $H_p\mathbb{G} = \{X(p) \mid X \in V_1\}$ at $p \in \mathbb{G}$ and consider the disjoint union of all these subspaces with the relevant vector bundle topology. We denote by $H\mathbb{G}$ the horizontal subbundle. Smooth sections of the horizontal subbundle are called *horizontal vector fields*. The metric structure on the group is given by the following class of left invariant Riemannian metrics.

Definition 1.1 (Graded metric) We say that a left invariant Riemannian metric g on $\mathbb{G}$ is a *graded metric* if all subspaces $\{V_j \mid j = 1, \ldots, \iota\}$ are orthogonal each other. We denote by v_g the Riemannian volume on $\mathbb{G}$ associated to g.

Definition 1.2 (Graded coordinates) We consider an adapted basis (4) which is orthonormal with respect to a graded metric g. A system of *graded coordinates* associated to this basis is represented by the diffeomorphism $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ defined by

$$F(\xi) = \exp\Big(\sum_{i=1}^{m} \xi_i X_i + \sum_{i=m+1}^{q} \xi_i Y_i\Big).$$

Note that in the previous definition we have use the fact that the exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ for simply connected nilpotent groups is a diffeomorphism. We also point out that whenever an adapted basis (4) is fixed, it is automatically defined the unique left invariant metric g which makes this basis orthonormal and it is also a graded metric. In the sequel, whenever a graded basis and a system of graded coordinates are fixed the unique graded metric that makes this basis orthonormal will be also understood.

Remark 1.3 Let us fix a system of graded coordinates $F : \mathbb{R}^q \longrightarrow \mathbb{G}$. One can check that the Riemannian volume measure coincides with the Lebesgue measure. Precisely, we have that $F_\sharp \mathcal{L}^q = v_g$, where $\mathcal{L}^q$ denotes the q-dimensional Lebesgue measure on $\mathbb{R}^q$ and $F_\sharp \mathcal{L}^q$ is the push-forward measure, defined by $F_\sharp \mathcal{L}^q(A) = \mathcal{L}^q\big(F^{-1}(A)\big)$ for any measurable set $A \subset \mathbb{G}$. This fact justifies our notation $v_g(A) = |A|$, where $A \subset \mathbb{G}$ is a measurable set, and the symbol dx when the integration is considered with respect to the Riemannian volume measure v_g.

Definition 1.4 (Coordinate dilations) Let F be a system of graded coordinates. The associated *coordinate dilations* $\Lambda_t : \mathbb{R}^q \longrightarrow \mathbb{R}^q$, with $t > 0$, are defined by

$$\Lambda_t(\xi) = \sum_{j=1}^{q} t^{d_j}\, \xi_j\, e_j, \tag{5}$$

where (e_j) denotes the canonical basis of $\mathbb{R}^q$ and $d_j = \dim V_j$.

The Carnot-Carathéodory distance between two points p and p' is obtained by taking the infimum among lengths of absolutely continuous curves a.e. tangent to the horizontal subbundle and which connect p with p'. The length of connecting curves is computed by a graded metric g, hence the Carnot-Carathéodory distance ρ is left invariant and it has the important scaling property $\rho(\delta_t p, \delta_t p') = t\rho(p,p')$, where $p, p' \in \mathbb{G}$. Any continuous distance which satisfies the previous scaling property and it is left invariant will be called *homogeneous distance.* The possibility to define a Carnot-Carathéodory distance depends on the existence of connecting curves tangent to the horizontal subbundle, the so-called *horizontal curves.* By definition of stratified group we have that the Lie algebra generated by V_1 coincides with $\mathcal{G}$, then the well known Chow theorem implies that any two points are connected by at least one horizontal curve, see for instance [6]. The Chow theorem also ensures that the Carnot-Carathéodory distance is continuous with respect to the topology of the group and it also induces the same topology. It is also worth to remark that any two homogeneous distances d_1 and d_2 are biLipschitz equivalent in the following sense

$$c_1\, d_1(p,p') \le d_2(p,p') \le c_2\, d_1(p,p'),$$

where $c_1, c_2 > 0$ are geometrical constants and $p, p' \in \mathbb{G}$. This immediately follows by homogeneity, left invariance and continuity of homogeneous distances, similarly to the classical argument for norms of finite dimensional spaces.

Definition 1.5 (Metric balls) Let d be a homogeneous distance of $\mathbb{G}$. The open ball of center p and radius t is defined by $B_{p,t} = \{r \in \mathbb{G} \mid d(p,r) < t\}$. The closed ball with the same center and the same radius is $D_{p,t} = \{r \in \mathbb{G} \mid d(p,r) \le t\}$. When the center of the ball is the unit element we simply write B_t and D_t, respectively.

The left translations on $\mathbb{G}$ will be denoted by l_p, where $p \in \mathbb{G}$ and $l_p(x) = px$ for any $x \in \mathbb{G}$. Sometimes we will also use the notation pf to denote the composition $l_p \circ f$. By properties of homogeneous distances metric balls can be written as $l_p \delta_t B_1 = B_{p,t}$.

As a consequence of the last formula and of (5) one easily sees that $|B_{p,t}| = t^Q |B_1|$, since the jacobian of Λ_t is t^Q, where $Q = \sum_{j=1}^{\iota} j\ \dim V_j$, and $F_\sharp \mathcal{L}^q = v_g$. This scaling property implies that the Hausdorff dimension of the group with respect to an arbitrary homogeneous distance is exactly Q and that the corresponding Q-dimensional Hausdorff measure $\mathcal{H}^Q$ is finite on compact sets. Then the measure $\mathcal{H}^Q$ is proportional to the volume measure v_g, due to the left invariance of both measures.

Definition 1.6 (Hausdorff measures) Let d be a homogeneous distance of $\mathbb{G}$ and let $a \ge 0$. For each subset $E \subset \mathbb{G}$ we define the *a-dimensional spherical Hausdorff measure*

$$\mathcal{S}^a(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_{j=1}^{\infty} \frac{\operatorname{diam}(D_{x_i,t_i})^a}{2^a} \mid E \subset \bigcup_{i=1}^{\infty} D_{x_i,t_i}\,,\ t_i \le \varepsilon \Big\}$$

and the *a-dimensional Hausdorff measure* of E as

$$\mathcal{H}^a(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_{j=1}^{\infty} \frac{\operatorname{diam}(F_i)^a}{2^a} \mid E \subset \bigcup_{i=1}^{\infty} F_i\,,\ \operatorname{diam}(F_i) \le \varepsilon \Big\}$$

where $\{F_i\}$ are subsets of $\mathbb{G}$ and $\operatorname{diam}(A) = \sup_{(x,y)\in A\times A} d(x,y)$ for any $A \subset \mathbb{G}$.

In the case we want to emphasize the use of a particular homogeneous distance d we will use the symbol $\mathcal{H}^a_d$. We will indicate by $|\cdot|$ the Euclidean norm in $\mathbb{R}^q$. The same symbol will also denote the Riemannian norm for vectors of the tangent bundle of the group. Note that we do not introduce any geometrical constant in the definitions of Hausdorff and spherical Hausdorff measures. In fact, we will work in codimension one, where these geometrical constants are replaced by the metric factor, which will be introduced in the following definition.

Definition 1.7 (Metric factor) Let B_1 be the open unit ball with respect to a fixed homogeneous distance d of $\mathbb{G}$. Consider a vector $\nu \in V_1 \setminus \{0\}$ together with its orthogonal hyperplane $\tilde{\mathcal{L}}(\nu)$ in $\mathcal{G}$ and define $\mathcal{L}(\nu) = \exp \tilde{\mathcal{L}}(\nu) \subset \mathbb{G}$. We fix a system of graded coordinates $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ and define

$$\theta^g_{Q-1}(\nu) = \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}(\nu) \cap B_1)\right) . \tag{6}$$

The map $\nu \longrightarrow \theta^g_{Q-1}(\nu)$ is the *metric factor* of the homogeneous distance d with respect to the direction ν and the Riemannian metric g.

Remark 1.8 The notion of metric factor does not depend on the choice of graded coordinates, as it has been proved in Lemma 1.10 of [26]. Notice also that $\theta^g_{Q-1}(\cdot)$ is uniformly bounded from above and from below by positive constants.

In Theorem 5.1 of [17] a homogeneous distance d_∞ has been explixitly constructed in every stratified group. Its explicit formula is stated in $\mathbb{R}^q$ with respect to graded coordinates:

$$d_\infty(x, 0) = \max_{j=1,\ldots,\iota} \{\varepsilon_j \, |(x^{m_{j-1}+1}, \ldots, x^{m_j})|^{1/j}\} \tag{7}$$

with the left invariant property $d_\infty(x, y) = d_\infty(x^{-1} \cdot y, 0)$ and for every $j = 1, \ldots, \iota$ the number $\varepsilon_j \in]0, 1[$ is suitable dimensional constant depending only on the group. The integer m_j is equal to zero if $j = 0$ and it corresponds to the sum $\sum_{k=1}^j n_k$, with $n_k = \dim(V_k)$, if $j = 1, \ldots, \iota$. In the next proposition we prove that the metric factor with respect to this distance becomes a constant function of the horizontal direction ν.

Proposition 1.9 Let θ^g_{Q-1} represent the metric factor with respect to the distance d_∞. Then there exists $\alpha_{Q-1} > 0$ such that for every $\nu \in H^1 \setminus \{0\}$ we have $\theta^g_{Q-1}(\nu) = \alpha_{Q-1}$.

Proof. For any couple of horizontal vectors $\nu, \mu \in H \setminus \{0\}$ we can find an isometry $\tau : \mathcal{G} \longrightarrow \mathcal{G}$ such that $\tau(\nu) = \mu$ and $\tau(\tilde{\mathcal{L}}(\nu)) = \tilde{\mathcal{L}}(\mu)$, where $\tilde{\mathcal{L}}(\nu)$ and $\tilde{\mathcal{L}}(\mu)$ are the orthogonal spaces to ν and μ, respectively. We read these orthogonal spaces in $\mathbb{G}$ defining $\mathcal{L}(\nu) = \exp\left(\tilde{\mathcal{L}}(\nu)\right)$ and $\mathcal{L}(\mu) = \exp\left(\tilde{\mathcal{L}}(\mu)\right)$. Let $(W_1, \ldots, W_q)$ be a graded basis of $\mathcal{G}$, let $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ be the associated system of graded coordinates and define $I(x) = \sum_{j=1}^q x^j \, W_j \in \mathcal{G}$ for every $x \in \mathbb{R}^q$. By the expression of d_∞ it is easy to see that the unit ball $\tilde{B}_1 \subset \mathbb{R}^q$ with respect to d_∞ is preserved under Euclidean isometries, i.e. $\tilde{\tau}(\tilde{B}_1) = \tilde{B}_1$ whenever $\tilde{\tau}$ is an Euclidean isometry. Thus, taking into account that $F = \exp \circ I$ and that $\tilde{\tau} = I^{-1} \circ \tau \circ I : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is an Euclidean isometry

$$\begin{aligned} F^{-1}(\mathcal{L}(\mu)) \cap \tilde{B}_1 &= I^{-1}(\tilde{\mathcal{L}}(\mu)) \cap \tilde{B}_1 = I^{-1}(\tau(\tilde{\mathcal{L}}(\nu))) \cap \tilde{B}_1 \\ &= \tilde{\tau}\left(I^{-1}(\tilde{\mathcal{L}}(\nu)) \cap \tilde{\tau}^{-1}(\tilde{B}_1)\right) = \tilde{\tau}\left(I^{-1}(\tilde{\mathcal{L}}(\nu)) \cap \tilde{B}_1\right) = \tilde{\tau}\left(F^{-1}(\mathcal{L}(\nu)) \cap \tilde{B}_1\right) , \end{aligned}$$

then, by Definition 1.7 it follows that

$$\begin{aligned} \theta^g_{Q-1}(\mu) &= \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}(\mu)) \cap \tilde{B}_1)\right) = \mathcal{H}^{q-1}_{|\cdot|}\left(\tilde{\tau}\left(F^{-1}(\mathcal{L}(\nu)) \cap \tilde{B}_1)\right)\right) \\ &= \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}(\nu)) \cap \tilde{B}_1)\right) = \theta^g_{Q-1}(\nu). \end{aligned}$$

This concludes the proof. □

1.2 H-perimeter and H-differentiability

We assume throughout the paper that Ω is an open subset of a stratified group $\mathbb{G}$. The space of smooth horizontal vector fields of Ω is denoted by $\Gamma(H\Omega)$ and the one of compactly supported horizontal vector fields by $\Gamma_c(H\Omega)$. Note that a horizontal vector field φ can be written as $\sum_{j=1}^m \varphi_j X_j$, where $(X_1, X_2, \ldots, X_m)$ is a basis of the first layer $V_1 \subset \mathcal{G}$. The *H-divergence* of φ is defined as $\sum_{j=1}^m X_j \varphi_j$ and it is denoted by $\mathrm{div}_H \varphi$. This definition is independent of the choice of the basis of V_1. All these definitions allow us to define H-BV functions on stratified groups. These functions were introduced in general Carnot-Carathéodory spaces by Capogna, Danielli and Garofalo, [8].

Definition 1.10 (H-BV functions) We say that a function $u \in L^1(\Omega)$ is a function of H-*bounded variation* (in short, an *H-BV function*) if

$$|D_H u|(\Omega) := \sup \left\{ \int_\Omega u(x)\ \mathrm{div}_H \varphi(x)\ dx \,\middle|\, \varphi \in \Gamma_c(H\Omega),\ |\varphi| \leq 1 \right\} < \infty ,$$

We denote by $BV_H(\Omega)$ and $BV_{\mathrm{loc},H}(\Omega)$ the space of functions of H-bounded variation and of locally H-bounded variation, respectively.

By Riesz Representation theorem we get the existence of a nonnegative Radon measure $|D_H u|$ and of a Borel section ν of $H\Omega$ such that we have $|\nu| = 1$ $|D_H u|$-a.e. and for any $\phi \in \Gamma_c(H\Omega)$ the following integration by parts formula holds

$$\int_\Omega u(x)\, \mathrm{div}_H \phi(x)\, dx = - \int_\Omega \langle \phi, \nu \rangle\, d\,|D_H u| \,. \tag{8}$$

The vector valued measure $\nu\,|D_H u|$ is denoted by $D_H u$. A measurable set $E \subset \Omega$ is said to be of H-*finite perimeter* if its characteristic function $\mathbf{1}_E$ belongs to $BV_H(\Omega)$. We use the symbol $|\partial E|_H$ to denote the total variation $|D_H \mathbf{1}_E|$ and the Borel section ν is denoted by ν_E and it is called *generalized inward normal.*

Definition 1.11 (H-linear maps) Let $\mathbb{G}$ and $\mathbb{M}$ be two stratified groups. We say that $L : \mathbb{G} \longrightarrow \mathbb{M}$ is an H-linear map if it is a group homomorphism with the property $L(\delta_t p) = \delta'_t L(p)$ for any $p \in \mathbb{G}$ and $t > 0$, where δ_t and δ'_t are dilations of $\mathbb{G}$ and $\mathbb{M}$, respectively. The space of all H-linear maps from $\mathbb{G}$ to $\mathbb{M}$ is denoted by $\mathrm{HL}(\mathbb{G}, \mathbb{M})$.

Definition 1.12 (H-differentiability) Let $A \subset \mathbb{G}$ be a measurable set and let $p \in A$ be a density point, i.e. $|A \cap D_{p,t}|/|D_{p,t}| \to 1$ as $t \to 0^+$. We say that a function $u : A \longrightarrow \mathbb{R}$ is H-*differentiable* at $p \in A$ if there exists an H-linear map $L : \mathbb{G} \longrightarrow \mathbb{R}$ such that

$$\lim_{A \ni r \to p} \frac{|u(r) - u(p) - L(p^{-1} r)|}{d(p, r)} = 0 \,. \tag{9}$$

The unique map L which satisfies (9) is denoted by $d_H u(p)$ and it is called the H-*differential* of u at p. The family of continuously H-differentiable functions $u : \Omega \longrightarrow \mathbb{R}$ is denoted by $C^1_H(\Omega)$.

We will also write that a function is C^1_H to indicate that it belongs to $C^1_H(\Omega)$. It is worth observing that C^1 functions in the usual sense are always C^1_H functions, as one can directly check from the definition of H-differentiability. The converse is not true and in Section 4 we will show an extreme example, where a C^1_H function might not be differentiable in the Euclidean sense on a set of positive measure.

Lipschitz functions on stratified groups satisfy a Rademacher-type theorem.

Theorem 1.13 *Let $u : A \longrightarrow \mathbb{R}$ be a Lipschitz function, where $A \subset \mathbb{G}$. Then u is a.e. H-differentiable.*

This theorem has been proved by Monti and Serra Cassano in the more general framework of Carnot-Carathéodory spaces, [31]. Note that the Rademacher-type theorem proved in [31] is obtained for functions defined on all the space. Taking into account that real-valued Lipschitz maps on metric spaces can be extended from their domain to all the space, it is easily proved the a.e. H-differentiability of Lipschitz functions defined on an arbitrary subset. Let us mention that Rademacher theorem in the general case of Lipschitz maps between stratified groups is a deep result due to Pansu, [33].

When a graded metric g is fixed on the group we can see the usual identification between the gradient of a function and its differential. Let $L : \mathbb{G} \longrightarrow \mathbb{R}$ be an H-linear map. It is easy to see that there exists a unique horizontal vector $X \in V_1$ such that $L(\exp W) = \langle X, W\rangle_g$ for any $W \in \mathcal{G}$. In view of the fact that the exponential map is a diffeomorphism between $\mathcal{G}$ and $\mathbb{G}$, it will be also convenient to read the H-linear map L as a linear map $L : \mathcal{G} \longrightarrow \mathbb{R}$ simply writing $L(W) = \langle X, W\rangle_g$. We will use this convention in the following definition.

Definition 1.14 (H-gradient) Let $d_H u(p) : \mathcal{G} \longrightarrow \mathbb{R}$ be the H-differential of the function $u : A \longrightarrow \mathbb{R}$. We denote by $\nabla_H u(p)$ the unique horizontal vector of V_1 such that

$$d_H u(x)(W) = \langle \nabla_H u(p), W\rangle_g \tag{10}$$

for any $W \in \mathcal{G}$. We say that $\nabla_H u(p)$ is the *H-gradient* of u at p.

Throughout the paper we will utilize both notions of H-differential and of H-gradient.

2 Measure of level sets

In this section we establish an explicit formula for the perimeter measure of level sets of C^1_H functions. An important tool to obtain this result is the following implicit function theorem, proved by Franchi, Serapioni and Serra Cassano in [16]. In the sequel the notation ∂E will denote the topological boundary of a subset $E \subset \Omega$ in the topology induced by Ω. Note that by Theorem 2.1 the piece of boundary given by the level set of a C^1_H function with nonvanishing H-gradient is indeed a manifold boundary in the sense of topological manifolds.

Theorem 2.1 (Implicit function theorem) *Let $f \in C^1_H(\Omega)$, $p \in \Omega$ and $f(p) = 0$. Suppose that $X_1 f(p) > 0$, where X_1 is an element of the adapted orthonormal basis (4). Then there exists a connected open neighbourhood $U \subset \Omega$ of p such that the open subset $E = \{r \in U \mid f(r) < 0\}$ has H-finite perimeter in U and there exist connected open subsets $A \subset \mathbb{R}^{q-1}$, $I \subset \mathbb{R}$ with $(0,0) \in I \times A$ and a continuous embedding $\Phi : A \longrightarrow U$ such that $f(\Phi(\xi)) = 0$ for every $\xi \in A$, $\Phi(0) = p$ and $\Phi(A) = U \cap \partial E$. The generalized inward normal is continuous and for every $r \in U \cap \partial E$ it is expressed by*

$$\nu_E(r) = -\frac{\nabla_H f(r)}{|\nabla_H f(r)|}. \tag{11}$$

The following formula for the perimeter measure with respect to graded coordinates holds

$$|\partial E|_H(U) = \int_A \frac{\sqrt{\sum_{j=1}^m X_j f(\Phi(\xi))^2}}{X_1 f(\Phi(\xi))}\, d\xi\,. \tag{12}$$

Remark 2.2 Note that Theorem 2.1 also provides a rather explicit form of Φ. In fact, using graded coordinates we have $\Phi(\xi) = p\,\gamma(\varphi(\xi), \xi)$, where $\varphi : A \longrightarrow \mathbb{R}$ is a continuous function and $\gamma : I \times A \longrightarrow p^{-1}U$ is a diffeomorphism, where $\gamma(t, \xi) = F(0, \xi)\exp(tX_1)$ and $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ is the system of graded coordinates associated to the frame (4).

The next theorem is the main result of this section.

Theorem 2.3 (Blow-up) *Let $f \in C^1_H(\Omega)$, $p \in \Omega$ with $f(p) = 0$ and $\nabla_H f(p) \neq 0$ and define the open subset $E = \{r \in \Omega \mid f(r) < 0\}$. Then the following limit holds*

$$\lim_{t \to 0^+} \frac{|\partial E|_H(B_{p,t})}{t^{Q-1}} = \theta^g_{Q-1}(\nabla_H f(p))\,. \tag{13}$$

Proof. By hypothesis there exist an open neighbourhood $V \subset \Omega$ of p and $f \in C^1_H(V)$ such that $V \cap \Sigma = f^{-1}(0)$ and $d_H f(r) : \mathbb{G} \longrightarrow \mathbb{R}$ is nonvanishing for any $r \in V$. We fix an orthonormal system of the form (4), where $X_1(p) = \nabla_H f(p)/|\nabla_H f(p)|$. This particular choice implies that

$$X_j f(p) = 0 \qquad \text{for every } j = 2, \ldots, m\,. \tag{14}$$

In view of this assumption there exists an open neighbourhood $V' \subset V$ of p such that $X_1 f(r) > 0$ for every $r \in V'$. By virtue of Theorem 2.1 there exist a connected open neighbourhood $U \subset V'$ of p and a homeomorphism $\Phi : A \longrightarrow U \cap \Sigma$ defined by

$$\Phi(\xi) = p\,F(0, \xi)\exp(\varphi(\xi)X_1) = p\gamma(\varphi(\xi), \xi)$$

for every $\xi \in A$, where $A \subset \mathbb{R}^{q-1}$ is a connected open neighbourhood of zero. The function $\varphi : A \longrightarrow \mathbb{R}$ is continuous with $\varphi(0) = 0$ and $\gamma : I \times A \longrightarrow U$ is a diffeomorphism, where I is an open interval with $0 \in I$. We choose $t_0 > 0$ such that $D_{p,t_0} \subset U$ and define the compact subset $A_0 = \Phi^{-1}(D_{p,t_0}) \subset A$. Under these conditions the open set

$$E = \{\gamma(t, \xi) \mid (t, \xi) \in I \times A,\ t < \varphi(\xi)\} = \{r \in U \mid f(r) < 0\}$$

has H-finite perimeter in U and by (12) there exists $t_0 > 0$ such that for every positive $t \le t_0$ we have

$$|\partial E|_H(B_{p,t}) = \int_{\Phi^{-1}(B_{p,t})} \frac{\sqrt{\sum_{j=1}^m X_j f(\Phi(\xi))^2}}{X_1 f(\Phi(\xi))}\, d\xi\,. \tag{15}$$

We make the change of variable $\xi = \tilde{\Lambda}_t \eta$, where $\tilde{\Lambda}_t : \mathbb{R}^{q-1} \longrightarrow \mathbb{R}^{q-1}$ is the restriction of the coordinate dilation Λ_t to the hyperplane $\{0\} \times \mathbb{R}^{q-1}$. The jacobian of the restriction $\tilde{\Lambda}_t$ is t^{Q-1}, hence for every positive $t \le t_0$ we obtain that

$$|\partial E|_H(B_{p,t}) = t^{Q-1} \int_{\tilde{\Lambda}_{1/t}\Phi^{-1}(B_{p,t})} \frac{\sqrt{\sum_{j=1}^m X_j f(\Phi(\tilde{\Lambda}_t\eta))^2}}{X_1 f(\Phi(\tilde{\Lambda}_t\eta))}\, d\eta \tag{16}$$

To perform the limit of $|\partial E|_H(B_{p,t})/t^{Q-1}$ as $t \to 0^+$ we have to study how the sets

$$\tilde{\Lambda}_{1/t}\Phi^{-1}(B_{p,t}) = \{\xi \in \tilde{\Lambda}_{1/t}A_0 \mid \tilde{\Lambda}_t\xi \in \Phi^{-1}(B_{p,t})\}$$

behave as $t \to 0^+$. We observe that

$$\Phi^{-1}(B_{p,t}) = \{\xi \in A_0 \mid F(0,\xi)\exp(\varphi(\xi)X_1) \in B_t\}\,,$$

hence we have

$$\tilde{\Lambda}_{1/t}\Phi^{-1}(B_{p,t}) = \left\{\xi \in \tilde{\Lambda}_{1/t}A_0 \mid F(0,\tilde{\Lambda}_t\xi)\exp(\varphi(\tilde{\Lambda}_t\xi)X_1) \in B_t\right\} \tag{17}$$

$$= \left\{\xi \in \tilde{\Lambda}_{1/t}A_0 \mid \delta_t F(0,\xi)\delta_t \exp\left(\tfrac{\varphi(\tilde{\Lambda}_t\xi)}{t}X_1\right) \in \delta_t B_1\right\} \tag{18}$$

$$= \left\{\xi \in \tilde{\Lambda}_{1/t}A_0 \mid F(0,\xi)\exp\left(\tfrac{\varphi(\tilde{\Lambda}_t\xi)}{t}X_1\right) \in B_1\right\}. \tag{19}$$

We want to prove that $\varphi(\tilde{\Lambda}_t\xi)/t \to 0$ as $t \to 0^+$. This is a rather delicate fact in that we know only that φ is continuous. For any $\xi \in A_0$ we define the open set

$$I_1(\xi) = \left\{t \in]0,+\infty[\cap I \;\mid\; \varphi(\tilde{\Lambda}_t\xi) \neq 0\right\}.$$

In the case when the topological closure $\overline{I_1(\xi)}$ does not contain zero, then $\varphi(\tilde{\Lambda}_t\xi) = 0$ for every $t \in]0,\varepsilon'[$, where $\varepsilon' > 0$ is suitable small, hence our limit becomes trivial. Suppose now that $0 \in \overline{I_1(\xi)}$. In this case we can choose a sequence $\{t_j\}$ contained in $I_1(\xi)$ and converging to zero. We can also suppose that $\{t_j\} \subset]0,\varepsilon[$, where $\varepsilon > 0$ is chosen so that $\tilde{\Lambda}_t A_0 \subset A_0$ for every $t \in]0,\varepsilon[$. By the fact that f is C^1_H the derivative of $t \longrightarrow f(p\gamma(t,\xi))$ is $X_1 f(p\gamma(t,\xi)) > 0$ for every $t \in I$ and for every $\xi \in A_0$. Thus, the function $t \longrightarrow f(p\gamma(t,\xi))$ is strictly increasing and we have

$$f(p\gamma(\varphi(\tilde{\Lambda}_{t_j}\xi),\tilde{\Lambda}_{t_j}\xi)) - f(p\gamma(0,\tilde{\Lambda}_{t_j}\xi)) = -f(p\gamma(0,\tilde{\Lambda}_{t_j}\xi)) \neq 0 \tag{20}$$

for every $\xi \in A_0$, due to the condition $\varphi(\tilde\Lambda_{t_j}\xi) \neq 0$. By the the mean value theorem there exists $c(t_j,\xi) \in]0,t_j[$ such that

$$\frac{f(p\gamma(t_j,\xi)) - f(p\gamma(0,\xi))}{t_j} - X_1 f(p\gamma(0,\xi)) = X_1 f(p\gamma(c(t_j,\xi),\xi)) - X_1 f(p\gamma(0,\xi))\,.$$

Thus, the uniform continuity of $(X_1 f)\circ(p\gamma)$ on the compact set $[0,\varepsilon]\times A_0$ implies that

$$\max_{\xi\in A_0}\left|\frac{f(p\gamma(t,\xi)) - f(p\gamma(0,\xi))}{t} - X_1 f(p\gamma(0,\xi))\right| \longrightarrow 0^+ \quad \text{as} \quad t\to 0^+\,.$$

Utilizing (20) and the previous uniform convergence it follows that

$$0 \neq -\frac{\varphi(\tilde\Lambda_{t_j}\xi)}{f(p\gamma(0,\tilde\Lambda_{t_j}\xi))} = \frac{\varphi(\tilde\Lambda_{t_j}\xi)}{f(p\gamma(\varphi(\tilde\Lambda_{t_j}\xi),\tilde\Lambda_{t_j}\xi)) - f(p\gamma(0,\tilde\Lambda_{t_j}\xi))} \longrightarrow \frac{1}{X_1 f(p)} \tag{21}$$

as $t_j\to 0^+$. In view of the particular choice of X_1 we have $d_H f(p)(F(x)) = X_1 f(p)\, x_1$, where $x\in\mathbb{R}^q$ and $F:\mathbb{R}^q\longrightarrow\mathbb{G}$ is the system of graded coordinates associated to the basis (4). It follows that

$$d_H f(p)(F(0,\xi)) = 0 \quad \text{for any } \xi\in\mathbb{R}^{q-1}.$$

As a result, the H-differentiability of f at p and the identity $\gamma(0,\tilde\Lambda_t\xi) = F(0,\tilde\Lambda_t\xi)$ yield

$$\frac{f(p\gamma(0,\tilde\Lambda_t\xi))}{t} = \frac{f(p\,F(0,\tilde\Lambda_t\xi)) - d_H f(p)(F(0,\tilde\Lambda_t\xi))}{t} \longrightarrow 0 \tag{22}$$

uniformly in $\xi\in A_0$ as $t\to 0^+$. Joining the limits (21) and (22), for any $\xi\in A_0$, we obtain that

$$\lim_{I_1(\xi)\ni t_j\to 0^+} \frac{\varphi(\tilde\Lambda_{t_j}\xi)}{t_j} = 0\,. \tag{23}$$

Taking into account that $\varphi(\tilde\Lambda_t\xi) = 0$ whenever $t>0$ and $t\notin I_1(\xi)$, we can conclude that limit (23) holds for every infinitesimal sequence $\{t_j\}\subset]0,\varepsilon[$, then

$$\lim_{t\to 0^+} \frac{\varphi(\tilde\Lambda_t\xi)}{t} = 0\,. \tag{24}$$

The limit (24) and the expression (19) yield

$$\mathbf{1}_{\tilde\Lambda_{1/t}\Phi^{-1}(B_{p,t})} \longrightarrow \mathbf{1}_{\tilde B_1\cap(\{0\}\times\mathbb{R}^{q-1})} \quad \text{as} \quad t\to 0^+, \tag{25}$$

where $\tilde B_1 = F^{-1}(B_1)$. We also notice that $F(\{0\}\times\mathbb{R}^{q-1})\subset\mathbb{G}$ is exactly the hyperplane orthogonal to $\nabla_H f(p) = |\nabla_H f(p)|\, X_1(p)$. Hence, from (11) and the definition of metric factor we conclude that

$$\mathcal{H}^{q-1}_{|\cdot|}(\tilde B_1\cap(\{0\}\times\mathbb{R}^{q-1})) = \theta^g_{Q-1}(\nu_E(p))\,. \tag{26}$$

Now, in view of (14) and the continuity of $\nabla_H f$ we obtain

$$\frac{\sqrt{\sum_{j=1}^m X_j f(\Phi(\tilde{\Lambda}_t \eta))^2}}{X_1 f(\Phi(\tilde{\Lambda}_t \eta))} \longrightarrow 1 \quad \text{as} \quad t \to 0^+. \tag{27}$$

By virtue of (16), (25) and (27) we deduce that the limit of $|\partial E|_H(B_{p,t})/t^{Q-1}$ as $t \to 0^+$ exists and in view of (26) it is equal to $\theta^g_{Q-1}(\nu_E(p))$. This finishes the proof. □

In Theorem 2.5 we will also utilize the following general result due to Ambrosio, [1].

Theorem 2.4 (Absolute continuity) *Let $E \subset X$ be a set of finite perimeter in a k-Ahlfors regular metric space X which supports a weak (1,1)-Poincaré inequality. Then the perimeter measure $P(E,\cdot)$ is absolutely continuous with respect to $\mathcal{H}^{k-1}$.*

Stratified groups are clearly Q-Ahlfors regular, due to the property $|B_{p,r}| = |B_1| r^Q$ for any $p \in \mathbb{G}$ and $r > 0$. They also satisfy a Poincaré inequality, [20]. By a standard approximation argument it is not difficult to check that the notion of perimeter measure in metric spaces used in [1] corresponds to the one for stratified groups or more general Carnot-Carathéodory spaces, [29]. Then Theorem 2.4 can be used for stratified groups.

In the next theorem we will use the standard notation $\mu \llcorner A$ to denote the restriction of the measure μ to the measurable set A. Precisely $\mu \llcorner A(F) = \mu(A \cap F)$ for every measurable set F.

Theorem 2.5 *Let $u \in C^1_H(\Omega)$, $s \in \mathbb{R}$ and $E = \{r \in \Omega \mid u(r) < s\}$. We set*

$$\Sigma = \Big\{ r \in \Omega \mid u(r) = s \text{ and } \nabla_H u(r) = 0 \Big\}$$

and suppose that $\mathcal{H}^{Q-1}(\Sigma) = 0$. We also assume that $|\partial E|_H(\Omega) < \infty$. Then we have

$$|\partial E|_H = \theta^g_{Q-1}(\nu_E)\, \mathcal{S}^{Q-1} \llcorner \partial E. \tag{28}$$

PROOF. By definition of perimeter measure it is easily recognized that its support is contained in the topological boundary ∂E, hence we can write

$$|\partial E|_H(\Omega) = |\partial E|_H(\partial E) = |\partial E|_H(\partial E \setminus \Sigma) + |\partial E|_H(\Sigma)\,. \tag{29}$$

Note that the singular set Σ can be highly irregular. However, the contribution to the perimeter measure from this part is negligible. In fact, the set E has H-finite perimeter, hence due to Theorem 2.4 the perimeter measure is absolutely continuous with respect to $\mathcal{H}^{Q-1}$. Using our hypothesis we know that $\mathcal{H}^{Q-1}(\Sigma) = 0$, then $|\partial E|_H(\Sigma) = 0$ and the perimeter measure $|\partial E|_H$ is clearly supported on $\partial E \setminus \Sigma$.

Let $p \in \partial E \setminus \Sigma$ and define $\varepsilon_p > 0$ such that for any $t \in]0, \varepsilon_p[$ we have $B_{p,t} \subset \Omega$. In view of $|\partial E|_H(B_{p,t}) < \infty$ we see that $|\partial E|_H(\partial B_{p,t})$ might not vanish only on a countable set of numbers t, therefore $|\partial E|_H(B_{p,t}) = |\partial E|_H(D_{p,t})$ for a.e. $t > 0$. As a consequence, the family of closed balls

$$\mathcal{C} = \{D_{p,t} \mid \ 0 < t < \varepsilon_p,\ p \in \partial E \setminus \Sigma,\ |\partial E|_H(B_{p,t}) = |\partial E|_H(D_{p,t})\} \tag{30}$$

is fine at each point of $\partial E \setminus \Sigma$, i.e. defining $I_p = \{t \in]0, \varepsilon_p[\,| D_{p,t} \in \mathcal{C}\}$, for every $p \in \partial E \setminus \Sigma$ we have $\inf I_p = 0$. Now we observe that the function $u - s$ satisfies the hypotheses of Theorem 2.3 for every $p \in \partial E \setminus \Sigma$, then by (13) we obtain

$$\lim_{t \in I_p, t \to 0^+} \frac{|\partial E|_H(D_{p,t})}{t^{Q-1}} = \theta^g_{Q-1}(\nu_E(p)) \tag{31}$$

for every $p \in \partial E \setminus \Sigma$. In order to employ Theorems 2.10.17(2) and 2.10.18(1) of [12] we have to make sure $\text{diam}(B_t) = \text{diam} B_{p,t} = 2t$, for any $t > 0$ and $p \in \mathbb{G}$. We can choose $\exp at'X \in B_t$ with $X \in V_1$, $0 < t' < t$ and $d(\exp aX, e) = 1$, where e is the unit element of the group. Then $d(\exp at'X, \exp(-at'X)) = 2t' \le \text{diam}(B_t)$. Letting $t' \to t$ we obtain $2t \le \text{diam}(B_t)$. The opposite inequality is straightforward. In the notation of Section 2.6.16 of [12] we have proved that for any $p \in \partial E \setminus \Sigma$ there exists the limit

$$(\mathcal{C}) \lim_{S \to p} \frac{|\partial E|_H(S)}{\zeta(S)} = \theta^g_{Q-1}(\nu_E(p)), \tag{32}$$

where $\mathcal{C}$ is the family (30) and $\zeta(S) = \text{diam}(S)^{Q-1}/2^{Q-1}$. Finally, Theorems 2.10.17(2) and 2.10.18(1) of [12] applied to the measure $|\partial E|_H \llcorner (\partial E \setminus \Sigma)$ lead us to the conclusion. $\Box$

3 The coarea formula

This section is devoted to the proof of the coarea formula on stratified groups. We start recalling this formula for functions of bounded variation, see [14], [18] and [31].

Theorem 3.1 *Let $u \in BV_H(\Omega)$. Then for every measurable set $A \subset \Omega$ we have*

$$|D_H u|(A) = \int_{\mathbb{R}} |\partial E_s|_H(A)\, ds\,, \tag{33}$$

where $E_s = \{x \in \Omega \mid u(x) < s\}$.

Proposition 3.3 is the key step to obtain the coarea formula in the general case of Lipschitz functions. To obtain this result we need of the following theorem, proved in [25].

Theorem 3.2 (Sard-type theorem) *Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups of Hausdorff dimension Q and P, respectively, and let $A \subset \mathbb{G}$ be a measurable set. Consider a Lipschitz map $u : A \longrightarrow \mathbb{M}$ and define the set of singular points*

$$\Sigma = \{p \in A \mid d_H u(p) \textit{ exists and it is not surjective}\}.$$

Then, for $\mathcal{H}^P$-a.e. $\xi \in \mathbb{M}$ it follows $\mathcal{H}^{Q-P}\left(\Sigma \cap u^{-1}(\xi)\right) = 0$.

We will use this result for the case $\mathbb{M} = \mathbb{R}$, where $P = 1$ and $u : \Omega \longrightarrow \mathbb{R}$ is of class C^1_H. In this case the map u is everywhere H-differentiable, hence in view of (10) for every $s \in \mathbb{R}$ we have

$$u^{-1}(s) = \left\{ p \in u^{-1}(s) \mid \nabla_H u(p) = 0 \right\} \bigsqcup \left\{ p \in u^{-1}(s) \mid \nabla_H u(p) \neq 0 \right\}, \tag{34}$$

where the symbol $\bigsqcup$ denotes the disjoint union. Note also that C^1_H functions are locally Lipschitz with respect to the Carnot-Carathéodory distance, therefore Theorem 3.2 applied to $u_{|\Omega'}$, where Ω' is a relatively compact open subset of Ω, yields

$$\mathcal{H}^{Q-1}\left(\left\{ r \in \Omega' \mid u(r) = s \text{ and } \nabla_H u(r) = 0 \right\} \right) = 0 \,. \tag{35}$$

for a.e. $s \in \mathbb{R}$. This last consequence will be exploited in the next proposition.

Proposition 3.3 *Let $u : \Omega \longrightarrow \mathbb{R}$ be a C^1_H function. Then for every measurable set $A \subset \Omega$ we have*

$$\int_A |\nabla_H u(x)| \, dx = \int_{\mathbb{R}} \int_{u^{-1}(s) \cap A} \theta^g_{Q-1}(\nabla_H u(x)) \, d\mathcal{S}^{Q-1}(x) \, ds \,, \tag{36}$$

where $\mathcal{S}^{Q-1}$, θ^g_{Q-1} are considered with respect to the same homogeneous distance.

PROOF. Consider Ω' compactly contained in Ω and recall that C^1_H functions are both locally Lipschitz and locally H-BV functions. We define the set $E_s = \{r \in \Omega' \mid u(r) < s\}$ for every $s \in \mathbb{R}$. By virtue of the H-differentiability of u it is not difficult to see that $|D_H u| = |\nabla_H u| \, v_g$, where $D_H u$ is the vector distributional derivative of u regarded as a vector measure. Thus, the coarea formula (33) implies

$$\int_{\mathbb{R}} |\partial E_s|_H(\Omega') \, ds = |D_H u|(\Omega') = \int_{\Omega'} |\nabla_H u(x)| \, dx < \infty.$$

Then for a.e. $s \in \mathbb{R}$ we have $|\partial E|_H(\Omega') < \infty$ and formula (35) holds for a.e. $s \in \mathbb{R}$. As a consequence, for a.e. $s \in \mathbb{R}$ we can apply Theorem 2.5 to the set E_s, getting

$$|\partial E_s|_H = \theta^g_{Q-1}(\nu_{E_s}) \, \mathcal{S}^{Q-1} \llcorner \partial E_s \tag{37}$$

for a.e. $s \in \mathbb{R}$. Then formulae (33) and (37) give

$$\int_A |\nabla_H u(x)| \, dx = |D_H u|(A) = \int_{\mathbb{R}} \int_{u^{-1}(s) \cap A} \theta^g_{Q-1}(\nu_{E_s}) \, d\mathcal{S}^{Q-1} \, ds \tag{38}$$

for any measurable set $A \subset \Omega'$. Finally, we replace formula (11) in (38) and we use the fact that the metric factor depends only on the direction of the horizontal vector, then (36) follows. □

To recover the coarea formula in the case of Lipschitz functions we will need of the following result, taken from [17].

Theorem 3.4 (Whitney extension theorem) *Let $u : F \longrightarrow \mathbb{R}$ be a continuous function on a closed subset $F \subset \mathbb{G}$ and let $\Psi : F \longrightarrow \mathrm{HL}(\mathbb{G},\mathbb{R})$ be continuous, such that for every compact set $K \subset F$ we have*

$$\sup\left\{\frac{|u(r)-u(p)-\Psi(p)(p^{-1}r)|}{d(p,r)}\,\Big|\, p,r\in K,\ d(p,r)<t\right\}\longrightarrow 0 \quad as \quad t\to 0^+ .$$

Then there exists a C^1_H function $\tilde u : \mathbb{G} \longrightarrow \mathbb{R}$ such that $\tilde u_{|F} = u$ and $(d_H\tilde u)_{|F} = \Psi$.

In the next theorem we prove the coarea formula for real-valued Lipschitz maps on stratified groups.

Theorem 3.5 (Coarea formula) *Let $A \subset \mathbb{G}$ be a measurable set, $u : A \longrightarrow \mathbb{R}$ be a Lipschitz function and $h : A \longrightarrow [0,+\infty]$ be a measurable function. Then we have*

$$\int_A h(x)\,|\nabla_H u(x)|\,dx = \int_{\mathbb{R}}\int_{u^{-1}(s)\cap A} h(x)\,\theta^g_{Q-1}(\nabla_H u(x))\,d\mathcal{S}^{Q-1}(x)\,ds\,, \tag{39}$$

PROOF. We first assume that A is bounded. By Theorem 1.13 we know that u is a.e. H-differentiable on A, then for a.e. $p\in A$ it follows

$$\omega(p,t) = \sup_{r\in A,\,d(p,r)<t}\frac{|u(r)-u(p)-d_Hu(p)(p^{-1}r)|}{d(p,r)}\longrightarrow 0 \quad \text{as} \quad t\to 0^+.$$

Let us choose an arbitrary $\kappa>0$. By Egorov-Severini theorem there exists a compact set $F\subset A$ such that $|A\setminus F|<\kappa$ and $\sup_{p\in F}\omega(p,t)\to 0$ as $t\to 0^+$, then we get

$$\sup\left\{\frac{|u(r)-u(p)-d_Hu(p)(p^{-1}r)|}{d(p,r)}\,\Big|\, p,r\in F,\ d(p,r)<t\right\}\longrightarrow 0 \quad \text{as} \quad t\to 0^+ .$$

The last limit enables us to apply Theorem 3.4 with $\Psi = (d_Hu)_{|F}$. Thus, there exists a C^1_H function $\tilde u : \mathbb{G}\longrightarrow\mathbb{R}$ such that $\tilde u_{|F} = u$ and $(d_H\tilde u)_{|F} = (d_Hu)_{|F}$. We can apply Proposition 3.3 to $\tilde u$, with $\Omega = \mathbb{G}$ and observing that F is a measurable subset of $\mathbb{G}$. Then we have proved that

$$\int_F |\nabla_H u(x)|\,dx = \int_{\mathbb{R}}\int_{u^{-1}(s)\cap F}\theta^g_{Q-1}(\nabla_H u(x))\,d\mathcal{S}^{Q-1}(x)\,ds\,. \tag{40}$$

Taking an infinitesimal sequence of positive numbers $\{\kappa_j\}$ we have a family of closed sets $\{F_j\}$ such that $|A\setminus\bigcup_{j=1}^\infty F_j| = 0$ and formula (40) holds replacing F with F_j for any $j\geq 1$. Then Beppo Levi convergence theorem yields

$$\int_{\bigcup_{j=1}^\infty F} |\nabla_H u(x)|\,dx = \int_{\mathbb{R}}\int_{u^{-1}(s)\cap\left(\bigcup_{j=1}^\infty F_j\right)}\theta^g_{Q-1}(\nabla_H u(x))\,d\mathcal{S}^{Q-1}(x)\,ds\,. \tag{41}$$

Now we apply the coarea estimate 2.10.25 of [12] and the fact that the metric factor is a bounded map. It follows that

$$\int_{\mathbb{R}}\int_{u^{-1}(s)\cap\left(A\setminus\bigcup_{j=1}^\infty F_j\right)}\theta^g_{Q-1}(\nabla_H u(x))\,d\mathcal{S}^{Q-1}(x)\,ds \leq \|\theta^g_{Q-1}\|_\infty\left|A\setminus\bigcup_{j=1}^\infty F_j\right| = 0$$

then (41) becomes

$$\int_A |\nabla_H u(x)|\, dx = \int_{\mathbb{R}} \int_{u^{-1}(s)\cap A} \theta^g_{Q-1}(\nabla_H u(x))\, d\mathcal{S}^{Q-1}(x)\, ds\,. \tag{42}$$

We wish to extend (42) to any measurable subset A of Ω. It suffices to choose an increasing sequence of bounded measurable sets $\{A_k\}$ whose union coincides with A and use the Beppo Levi convergence theorem for (42) with A is replaced by A_k. Then (42) holds for any measurable subset of Ω. Finally, taking an increasing sequence of nonnegative step functions which pointwise converges to h and applying again the Beppo Levi convergence theorem, our claim follows. □

The coarea formula we have previously obtained takes a more familiar form in a privileged class of stratified groups, called *rotational groups.* We briefly recall their definition.

We consider a group $\mathbb{G}$ and a fixed graded metric g. A *horizontal isometry* is an H-linear map $T : \mathbb{G} \longrightarrow \mathbb{G}$ such that $dT(e) : \mathcal{G} \longrightarrow \mathcal{G}$ is an isometry with respect to g. A *rotational group* $\mathbb{G}$ has sufficiently many horizontal isometries so that for any couple of vertical hyperplanes $\mathcal{L}(\nu), \mathcal{L}(v) \subset \mathcal{G}$, orthogonal to the horizontal directions $\nu, v \in V_1$, there exists a horizontal isometry $T : \mathbb{G} \longrightarrow \mathbb{G}$ such that $dT(e)\,(\mathcal{L}(\nu)) = \mathcal{L}(v)$. It is straightforward to recognize that Euclidean spaces are rotational. It has been proved in [26] that Heisenberg groups and more general H-type groups are rotational. The main feature of rotational groups is that the metric factor $\theta^g_{Q-1}(\nu)$ with respect to the Carnot-Carathéodory distance does not depend on the horizontal direction ν, see Remark 2.14 and Proposition 2.18 of [26]. Then it is constantly equal to a geometric constant, denoted by α_{Q-1}. It is natural to use this constant in the definition of spherical Hausdorff measure, then defining

$$\mathcal{S}^{Q-1}_{\mathbb{G}} = \alpha_{Q-1}\, \mathcal{S}^{Q-1}$$

as the intrinsic spherical Hausdorff measure. Note that in the Euclidean space $\mathbb{E}^n$ we have $Q = n$ and $\alpha_{Q-1} = \omega_{n-1}$, where ω_{n-1} denotes the volume of the unit ball of $\mathbb{E}^{n-1}$. Then we have $\mathcal{S}^{n-1}_{\mathbb{E}^n} = \omega_{n-1}\mathcal{S}^{n-1}_{|\cdot|} = \omega_{n-1}\mathcal{H}^{n-1}_{|\cdot|}$, where the last equality is a special property of Euclidean spaces and its validity on stratified groups is not known. In the case of arbitrary stratified groups is still possible to obtain a constant metric factor by using the distance d_∞, as we have seen in Proposition 1.9. Then also in this case we can define α_{Q-1} and $\mathcal{S}^{Q-1}_{\mathbb{G}}$ as above.

As an immediate corollary of Theorem 3.5 we have obtained the following result.

Corollary 3.6 *Let $\mathbb{G}$ be a stratified group, $A \subset \mathbb{G}$ be a measurable set and $u : A \longrightarrow \mathbb{R}$ be a Lipschitz function. Then for any measurable function $h : A \longrightarrow [0, +\infty]$ we have*

$$\int_A h(x)\, |\nabla_H u(x)|\, dx = \int_{\mathbb{R}} \int_{u^{-1}(s)\cap A} h(x)\, d\mathcal{S}^{Q-1}_{\mathbb{G}}(x)\, ds\,, \tag{43}$$

where the constant α_{Q-1} and the spherical Hausdorff measure $\mathcal{S}^{Q-1}_{\mathbb{G}}$ are computed either with respect to d_∞ for arbitrary stratified groups or with respect to the Carnot-Carathéodory distance if the group is rotational.

4 A surprising case

In this section we apply the technique of [22] in order to obtain an example of C^1_H function on the Heisenberg group $\mathbb{H}^3$, which is not approximately differentiable on a set of positive measure, when the classical notion of approximate differentiability for Euclidean spaces is adopted, see Definition 4.1. Note that approximate differentiability is a much weaker condition then usual differentiability. In fact, Sobolev functions are always a.e. approximately differentiable, but there are well known examples of Sobolev functions which are not differentiable in the usual sense at any point, see for instance [7] and [38]. As a surprising consequence of our example, the "sub-Riemannian" coarea formula (43) holds for this function, whereas the Euclidean one fails. We also point out that the same formula (43) is always satisfied by either Euclidean or Riemannian Lipschitz functions, [26].

Recall that the Heisenberg group $\mathbb{H}^3$ is the simplest model of noncommutative stratified group. Its Lie algebra $\mathfrak{h}^3$ has the grading $V_1 \oplus V_2$, where $\mathrm{span}\{X, Y\} = V_1$, $\mathrm{span}\{Z\} = V_2$ and the only nontrivial bracket relation is $[X, Y] = -4Z$. If we represent the Heisenberg group with respect to graded coordinates associated to the basis (X, Y, Z) we can indentify it with $\mathbb{C} \times \mathbb{R}$ endowed with the group operation

$$(z, s) \cdot (w, t) = \Big(z + w, s + t + 2\mathrm{Im}\langle z, w\rangle_{\mathbb{C}}\Big), \tag{44}$$

where $(z, s), (w, t) \in \mathbb{C} \times \mathbb{R}$ and $\langle\, ,\, \rangle_{\mathbb{C}}$ denotes the complex scalar product on $\mathbb{C}$. We will also consider on $\mathbb{C}$ the structure of two-dimensional real vector space and in this case we will denote the corresponding scalar product by $\langle\, ,\, \rangle_{\mathbb{R}^2}$. Exploiting the previously mentioned graded coordinates we can introduce a homogeneous distance on $\mathbb{H}^3$ as follows

$$d_\infty\left((z, t), 0\right) = \max\{|z|, |t|^{1/2}\}$$

and $d_\infty\left((z, t), (w, \tau)\right) = d_\infty\left((z, t)^{-1}(w, \tau), 0\right)$, whenever $(z, t), (w, \tau) \in \mathbb{C} \times \mathbb{R}$. This distance is biLipschitz equivalent to the Carnot-Carathéodory distance of $\mathbb{H}^3$, as one can easily check using the scaling property with respect to dilations of the group.

Definition 4.1 Let O be an open subset of $\mathbb{R}^m$ with $x \in O$ and let $f : O \longrightarrow \mathbb{R}^n$ be a measurable map. We say that f is *approximately differentiable* at $x \in O$ if for any $\varepsilon > 0$

$$\lim_{r \to 0^+} \frac{\mathcal{L}^m\left(\{y \in B_{x,r} \mid |f(y) - f(x) - L(y - x)| < \varepsilon\, |y - x|\}\right)}{\omega_m\, r^m} = 1\,, \tag{45}$$

where we have denoted by $\mathcal{L}^m$ the m-dimensional Lebesgue measure, by ω_m the Lebesgue measure of the unit ball of $\mathbb{R}^m$ and by $B_{x,r}$ the open ball of center x and radius $r > 0$ with respect to the Euclidean distance.

Our example is built through a Hölder continuous function which is not approximately differentiable everywhere. We utilize a classical construction due to Besicovitch and

Ursell and we refer to Section 8.2 of [11] for more details and complete proofs. We consider the 4-periodic sawtooth funtion defined as

$$v(t) = \begin{cases} t & 0 \le x < 1 \\ 2-t & 1 \le x < 3 \\ t-4 & 3 \le x < 4 \end{cases}$$

and for a fixed number $1 < s < 2$ we define

$$h(t) = \sum_{k=1}^{\infty} \lambda_k^{s-2}\, v(\lambda_k t), \qquad t \in \mathbb{R}, \tag{46}$$

where we assume that

$$\lambda_{k+1}/\lambda_k \longrightarrow +\infty \qquad \text{and} \qquad \log \lambda_{k+1}/\log \lambda_k \longrightarrow 0, \tag{47}$$

(for instance $\lambda_k = k!$ satisfies these conditions). In [11] it is proved that there exist positive constants $C_0, \beta > 0$ such that

$$|h(t) - h(\tau)| \le C_0\, |t-\tau|^{2-s} \tag{48}$$

whenever $|t-\tau| \le \beta$. We also have that for any $\alpha \in]1, s[$ there exist constants $c_\alpha, \beta_1 > 0$ such that for any $0 < r < \beta_1$ and $\tau \in \mathbb{R}$ the estimate

$$\mathcal{L}^1\left(\left\{t \in \mathbb{R} \mid (t, h(t)) \in S_{(\tau,h(\tau)),r}\right\}\right) \le c_\alpha r^\alpha \tag{49}$$

holds, where $S_{(\tau,h(\tau)),r}$ is the square of center $(\tau, h(\tau))$ and side of length r. Here we have used the estimate (8.12) of [11].

Proposition 4.2 *The function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined as in (46) under the conditions (47) is not approximately differentiable at τ for any $\tau \in \mathbb{R}$.*

PROOF. We arbitrarily choose $\tau \in \mathbb{R}$ and suppose by contradiction that h is approximately differentiable at τ. We fix $\varepsilon > 1$ so that on account of (45) there exist $r_0 > 0$ and a linear map $L(t) = \gamma\, t$ such that, defining

$$E_{\tau,r} = \{t \in \mathbb{R} \mid |t-\tau| < r(\varepsilon + |\gamma|),\ |h(t) - h(\tau)| \le (\varepsilon + |\gamma|)|t-\tau|\}$$

for any $r \le r_0$ we have $\mathcal{L}^1(E_{\tau,r}) \ge r$. We note that

$$E_{\tau,r} \subset \left\{t \in \mathbb{R} \mid (t, h(t)) \in S_{(\tau,h(\tau)),2r(\varepsilon+|\gamma|)}\right\}$$

and this inclusion together with (49) yields

$$r \le c_\alpha 2^\alpha (\varepsilon + \gamma)^\alpha r^\alpha \tag{50}$$

whenever $r \le \min\{r_0, \beta_1\}$, where we recall that $]0, \beta_1[$ is the set of values for which (49) holds and $\alpha > 1$. Inequality (50) clearly yields a contradiction when $r \to 0^+$. $\square$

Another tool we will need is the following result taken from Theorem 4.1 of [5].

Theorem 4.3 *For every $\sigma > 0$ there exists $g_\sigma \in \bigcap_{0<\theta<1} C^{1,\theta}(Q)$, with $Q = [0,1]^2$, such that the Lebesgue measure of the set*

$$A_{g_\sigma} = \{(x,y) \in Q \mid \nabla g_\sigma(x,y) = (2y, -2x)\} \tag{51}$$

is greater than or equal to the number $1 - \sigma$.

Now we make precise the notion of approximate differentiability in the Euclidean sense for a map defined on a stratified group. Let $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ be a system of graded coordinates on the stratified group $\mathbb{G}$. A function $u : \Omega \longrightarrow \mathbb{R}$ is either differentiable or approximately differentiable in the Euclidean sense if $u \circ F : F^{-1}(\Omega) \longrightarrow \mathbb{R}$ is either differentiable or approximately differentiable, respectively.

We can now state the main result of this section.

Theorem 4.4 *There exist a function $\zeta \in C^1_H(\mathbb{H}^3, \mathbb{R})$ and a measurable subset $T \subset \mathbb{H}^3$ with positive measure such that ζ is not approximately differentiable at p for any $p \in T$.*

Proof. We fix $\sigma \in]0,1[$ and define $g = g_\sigma : Q \longrightarrow \mathbb{R}$ as in Theorem 4.3. We denote by F the closed set $A_g \times \mathbb{R}$, where A_g is defined in (51). We choose $h : \mathbb{R} \longrightarrow \mathbb{R}$ as in Proposition 4.2. We recall that h is locally (2-s)-Hölder continuous and satisfies (48) on intervals of length β. The number $s \in]1,2[$ can be chosen so that $2 - s > 1/2$. We fix $a = 2 - s$ and define the function $v : F \longrightarrow \mathbb{R}$ as $v(z,t) = h(t - g(z))$, where we consider on $\mathbb{R}^3$ the algebraic structure of $\mathbb{C} \times \mathbb{R}$. Now we use graded coordinates with respect to the basis (X, Y, Z) of $\mathfrak{h}^3$, where $[X, Y] = -4Z$. These are represented by a diffeomorphism $S : \mathbb{R}^3 \longrightarrow \mathbb{H}^3$, as explained in Definition 1.2. By this diffeomorphism we can read v on $\mathbb{H}^3$ defining $u = v \circ S^{-1} : S(F) \longrightarrow \mathbb{R}$, where $F_0 = S(F) \subset \mathbb{H}^3$ is a closed subset. We want to prove that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|u(p) - u(r)| \le \varepsilon \, d_\infty(p, r)\,, \tag{52}$$

whenever $p, r \in F_0$ with $d_\infty(p,r) \le \delta(\varepsilon)$. This condition will unable us to apply Theorem 3.4 to $u : F_0 \longrightarrow \mathbb{R}$, obtaining a function $\zeta \in C^1_H(\mathbb{H}^3, \mathbb{R})$ such that $\zeta_{|F_0} = u$ and $d_H \zeta_{|F_0} = 0$. We define $p = S(z,t) \in F_0$ and $r = S(w,\tau) \in F_0$, where $z = (x,y)$ and $w = (\xi, \eta)$, noting that $u(p) = v(z,t)$ and $u(r) = v(w,\tau)$. We choose $\delta < \min\{1, \beta\}$, where β is the length of intervals where the Hölder estimate (48) holds. We have

$$\begin{aligned} |u(p) - u(r)| &= |h(t - g(z)) - h(\tau - g(w))| \le C_0 \, |t - \tau + g(w) - g(z)|^a \\ &\le C_0 \left(|t - \tau - 2\operatorname{Im}\langle w, z\rangle_{\mathbb{C}}|^a + |2\operatorname{Im}\langle w, z\rangle_{\mathbb{C}} + g(w) - g(z)|^a \right), \end{aligned}$$

where we have used the condition $a < 1$. If we assume that $d_\infty(p,r) \le \delta$, then the condition $a > 1/2$ gives

$$|u(p) - u(r)| \le C_0 (d_\infty(p,r)^{2a} + |2\operatorname{Im}\langle w, z\rangle_{\mathbb{C}} + g(w) - g(z)|^a). \tag{53}$$

Now we note that

$$2\operatorname{Im}\langle w, z\rangle_{\mathbb{C}} + g(w) - g(z) = 2\operatorname{Im}\langle w, z - w\rangle_{\mathbb{C}} - (g(z) - g(w))\,, \tag{54}$$

where we can write

$$2\,\mathrm{Im}\langle w, z-w\rangle_{\mathbb{C}} = 2\left(-\xi(y-\eta)+\eta(x-\xi)\right) \tag{55}$$

$$= \left(\partial_2 g(w)(y-\eta)+\partial_1 g(w)(x-\xi)\right) = \langle \nabla g(w), z-w\rangle_{\mathbb{R}^2}\,. \tag{56}$$

Joining equations (54), (55) and (56) we see that

$$|2\,\mathrm{Im}\langle w,z\rangle_{\mathbb{C}} + g(w)-g(z)| = |\langle \nabla g(w), z-w\rangle_{\mathbb{R}^2} - (g(z)-g(w))|\,. \tag{57}$$

Exploiting the Hölder property of g, as stated in Theorem 4.3 we can choose $\theta\in]0,1[$ such that $(1+\theta)a>1$ due to the fact that $a>1/2$. Then the θ-Hölder condition on first derivatives of g ensures the existence of a constant $C_1\geq 1$ such that

$$|g(z)-g(w)-\langle \nabla g(w), z-w\rangle| \leq C_1\,|z-w|^{1+\theta} \tag{58}$$

for any $z,w\in Q$. The equation (57) and the inequality (58) give

$$|2\,\mathrm{Im}\langle w,z\rangle_{\mathbb{C}} + g(w)-g(z)| \leq C_1\,|z-w|^{1+\theta}\,. \tag{59}$$

The inequalities (53) and (59) yield

$$|u(p)-u(r)| \leq C_0\,C_1^a\left(d_\infty(p,r)^{2a}+d_\infty(p,r)^{(1+\theta)a}\right). \tag{60}$$

The fact that $2a>1$ and $(1+\theta)a>1$ allows us to choose $0<\delta(\varepsilon)<\min\{1,\beta\}$ so that

$$C_0\,C_1^a\left(\delta(\varepsilon)^{2a-1}+\delta(\varepsilon)^{(1+\theta)a-1}\right)\leq\varepsilon$$

By virtue of (60), it follows that whenever $d_\infty(p,r)\leq\delta(\varepsilon)$ the condition (52) holds. We observe that $\mathcal{L}^2(A_g)\geq 1-\sigma>0$, hence $\mathcal{L}^3(F)=+\infty$ and $F=A_g\times\mathbb{R}$. The C^1_H function $\zeta:\mathbb{H}^3\longrightarrow\mathbb{R}$ has the property $\zeta\circ S_{|F}=v$, therefore, reasoning by contradiction, if v were approximately differentiable a.e. on F, then it would be approximately differentiable with respect to each variable a.e. on F. In fact, by Stepanoff's theorem (see Theorem 3.1.8 of [12]), if v were a.e. approximately differentiable on F, then it would be countably Lipschitz on F and Rademacher's theorem applied to the restriction would give the one-dimensional differentiability of all restrictions and eventually the a.e. one-dimensional approximate differentiability of v on F. This conflicts with the expression $v(z,t)=h(t-g(z))$ and with the fact that the function $h:\mathbb{R}\longrightarrow\mathbb{R}$ is not approximately differentiable everywhere on $\mathbb{R}$, due to Proposition 4.2. As a result, there exists a measurable set $T\subset F$ with positive measure, such that u is not approximately differentiable at any of its points. □

Remark 4.5 Let $f:\mathbb{H}^3\longrightarrow\mathbb{R}$ be a C^1 function and let ζ and T as in Theorem 4.4. We have already mentioned that C^1 functions are always C^1_H functions, then $u=f+\zeta\in C^1_H(\mathbb{H}^3)\setminus C^1(\mathbb{H}^3)$. By the construction in Theorem 4.4 we know that $\nabla_H\zeta(p)=0$ for any $p\in T$, then $\nabla_H u(p)=\nabla_H f(p)$ whenever $p\in T$. If we choose f so that $\nabla_H f\neq 0$ on T we have obtained an example of function $u:\mathbb{H}^3\longrightarrow\mathbb{R}$ such that it is not approximately differentiable in the Euclidean sense on a set of positive measure and with respect to which the sub-Riemannian coarea formula holds on this set and it is not trivial.

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