Abstract. We investigate the term structure of zero coupon bonds, in the case where the forward rate evolves as a Wiener sheet. We introduce a definition of stochastic integral with respect to a continuous semimartingale with values in the set of continuous functions and characterize the dynamics of the zero coupon bonds. We also define a notion of generalized strategy, in order to admit the (theoretical) possibility of investing in a continuum of bonds. Finally we study the problem of utility maximization from terminal wealth in this setting and deduce a “mutual fund” theorem.

1. Introduction

The description of the term structure of interest rates is an open problem. Various models have been proposed in order to fit the yield curve and to find a consistent pricing rule for interest rates contingent claims, in a similar way to what happens in the stock market. One of the most successful approaches is due to Heath, Jarrow and Morton [14], who specified the dynamics of all possible forward rates, allowing them to depend on multiple stochastic factors, and derived from it the dynamics of the short rate and of the bond prices. This model is consistent with the initial term structure but it does not permit consistency with term structure innovation. In the spirit of this approach, Kennedy [17, 18] proposed to model the term structure of interest rates, and in particular, the forward rates, as a Gaussian random field (deterministic volatility structure). He derived a drift condition for the instantaneous forward rate, in order for the zero-coupon bond discounted prices to be martingales and obtained a pricing formula for some interest rates derivatives. More recently, Goldstein [13] generalized Kennedy’s results to non-Gaussian random fields and illustrated the advantages of this type of modelization, also from a practitioners point of view: in particular, he showed that recalibration of parameters is unnecessary, besides the fact that random fields offers a more parsimonious description of term structure dynamics than the multi-factor models. However, both Kennedy and Goldstein left open the question of how to define a strategy in such a market. The problem is that, since zero coupon bonds form a continuum, a concept of "infinite-dimensional" portfolio is needed, in the sense that a consistent definition of strategy must take into account the possibility of investing in a continuum of securities. An answer to this issue was proposed first by Björk et alii. [2], who introduced the notion of measure-valued strategies and suggested two constructions of a stochastic integral with respect to processes taking values in a space of
continuous functions. A different approach was used in [8], by making use of a theory on stochastic integration with respect to cylindrical locally square integrable martingales, developed by Mikulevicius and Rozovskii [23, 24]: it was shown that measure-valued processes are not sufficient to describe all financial portfolios. In the same paper, a first analysis of Kennedy’s model was carried out. However, this type of approach is limited to the martingale case, which means that it is necessary to work under an equivalent martingale measure. Unfortunately, there are some questions, such as completeness or utility maximization, which need to be posed under the original measure.

The purpose of this paper is to overcome this difficulty, by defining a stochastic integral with respect to a continuous semimartingale with values in the space of continuous functions. Following the approach used in [9] for the case of a sequence of semimartingales, we introduce the concept of a generalized integrand as limit of a sequence of simple integrands, such that the corresponding sequence of stochastic integral converges in the topology of semimartingales. This allows us to analyze the class of interest rates models based on continuous random fields by adapting more standard techniques. We derive the dynamics of the zero-coupon bond prices and the drift condition under the equivalent-martingale measure, adapting a technique from Heath-Jarrow-Morton [14]. We give a definition of generalized portfolio and analyze the question of completeness. Since we assume the random field to be a Wiener sheet, our model is a particular case of Kennedy’s and Goldstein’s model. However, it is not difficult to recognize that most of our results can be adapted to the general case by replacing the correlation structure of the Wiener sheet, with a more general correlation structure and the deterministic volatility with a stochastic one. Finally we study the problem of utility maximization in the bond market: adapting the technique of Kramkov and Schachermayer [19], we prove the existence of the optimal portfolio. Furthermore, we state an “infinite-dimensional” version of a mutual fund-theorem: we show that the optimal portfolio is the limit of a sequence of portfolios, obtained by allocating the wealth between the riskless bond and a sequence of bond portfolios which do not depend on the particular utility function.

2. STOCHASTIC INTEGRATION WITH RESPECT TO A CONTINUOUS SEMIMARTINGALE WITH VALUES IN THE SET OF CONTINUOUS FUNCTIONS

We assume as given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual assumptions. Let $\mathcal{S}(P)$ be the space of real semimartingales endowed with the semimartingale topology introduced by Emery [12]: $\mathcal{S}(P)$ is a complete metric space. Furthermore, we denote by $\mathcal{M}^2(P)$ the space of square integrable martingales and by $\mathcal{A}^1(P)$ the space of predictable processes whose paths are of finite variation, such that the variation is integrable. Finally, $\mathcal{M}^2 \oplus \mathcal{A}^1(P)$ denotes the space of semimartingales introduced by Mémin [21] and is defined as follows: a special semimartingale $S$ belongs to $\mathcal{M}^2 \oplus \mathcal{A}^1(P)$ if and only if the canonical decomposition $S = M + B$ is such that $M$ is in $\mathcal{M}^2(P)$ and $B$ in $\mathcal{A}^1(P)$.

We assume that for every $x \in [0,1]$, a continuous semimartingale $(S^x_t)_{t\leq T}$ is defined: every $S^x$ admits a unique decomposition:

\begin{equation}
S^x = M^x + B^x
\end{equation}
where $M^x$ is a continuous (locally square integrable) martingale and $B^x$ is a predictable process with finite variation.

We make the following assumptions:

**Assumption 2.1.**

1. For all $(t, \omega)$, the mapping $x \rightarrow S^x_{t, \omega}$ is continuous;
2. for all $x$, we have that
   \[ d\langle M^x, M^x \rangle_t \ll dt \quad dB^x_t \ll dt \]
   where $M^x$ and $B^x$ are uniquely determined by (1);
3. there exist a predictable process $Q$ with values in $C([0,1] \times [0,1]; \mathbb{R})$ and a predictable process $b$ with values in $C[0,1]$ such that $\int_0^T Q_t(x, y)dt$ is still continuous, and for all $x, y \in [0,1],$
   \[ d\langle M^y, M^x \rangle_t = Q_t(x, y)dt \quad dB^x_t = b_t(x)dt \]

Our aim is to define a stochastic integral with respect to the infinite-dimensional process $S$. Since $S$ is a process with values in $C[0,1]$, it seems natural to take, as value set of the integrands, the dual set of $C[0,1]$, that is the set of Radon measures on $[0,1]$, which we denote by $\mathcal{M}$ and which is a separable space with respect to the $\sigma(\mathcal{M},C)$ topology. Björk et alii. [2] suggested the construction of a stochastic integral for measure-valued processes. Métivier [22] observed, in the case of square integrable martingales with values in a Hilbert space $H$, that the value space of the integrands may contain also non-continuous operators on $H$. This was also shown in [8], for the case of a locally square integrable martingale in a locally convex vector space. Here, we follow the approach used in [9] for the case of a sequence of semimartingales.

We start by taking as simple integrands the set of linear combination of Dirac measures:

**Definition 2.1.** A simple integrand is a process $h$ of the form:

\[ h = \sum_{i \leq n} \alpha^i \delta_{x_i} \]

where $\alpha^i$ are predictable bounded processes, whereas the $\delta_{x_i}$ are the Dirac deltas at points $x_i \in [0,1]$.

The stochastic integral of a simple integrand is naturally defined by:

\[ (h \cdot S)_t = \int_0^t h_s dS_s = \int_0^t \sum_{i \leq n} h_s^i dS_s^i. \]

Consider now a measure-valued process $\mu$. We first need to give a definition of measurability:

**Definition 2.2.** A measure-valued process $\mu$ is weakly predictable if for all $f \in C[0,1]$, the process $\mu(f)$ is predictable.
We remark that this notion is in fact equivalent to that of strong predictability, which is defined as follows: a measure-valued process is strongly predictable if there exists a sequence of simple integrands \( h^n \) which converges a.s. to \( \mu \) in the \( \sigma(M, C) \)-topology ([11], Proposition I.22). We will use the latter characterization to give a notion of measurability for non-measure valued processes.

We denote by \( U \) the set of not necessarily bounded (continuous) operators on \( C[0, 1] \). Given \( k \in U \), we denote by \( D(k) \) the domain of \( k \). Clearly, if \( \mu \) is a Radon measure, then \( \mu \) belongs to \( U \) and \( D(\mu) = C[0, 1] \). We give the following definitions, analogously to [9]:

**Definition 2.3.** Let \( k \in U \): we say that a sequence \( \mu^n \in M \) converges to \( k \) if and only if \( \lim_n \mu^n(f) = k(f) \) for all \( f \in D(k) \).

**Definition 2.4.** We say that a \( U \)-valued process \( h \) is predictable if there exists a sequence of predictable measure-valued processes \( (\mu^n) \), such that \( (\mu^n) \) converges to \( h \) almost surely in the sense of Definition 2.3.

**Definition 2.5.** We say that a \( U \)-valued process \( h \) is \( S \)-integrable if there exists a sequence of simple integrands \( (h^n) \) such that

(i) the sequence \( h^n \) converges a.s. to \( h \), in the sense of Definition 2.3;
(ii) the sequence of semimartingales \( (h^n \cdot S) \) is a Cauchy sequence in \( S(P) \) and converges to a semimartingale \( Y \).

We then define \( h \cdot S = \int h dS = Y \). The process \( h \) is called generalized integrand.

In the next subsection, we show that the above definitions make sense (namely, are independent on the approximating sequence) in the case of a continuous local martingale with values in the space of continuous functions. Then, we prove that the integral is well-defined even in the case of a semimartingale.

### 2.1. The martingale case

Assume that for all \( x \), \( S^x = M^x \). The theory on stochastic integration with respect to the process \( M \) can be seen as a particular case of the theory of cylindrical stochastic integration developed by Mikulevicius and Rozovskii [23, 24]: it is a generalization of some results of Kunita [20] concerning the integrals with respect to stochastic flows. This special case has also been analyzed in [8], with application to the bond market. We will recall the main facts. We begin by observing that the family \( M = (M^x)_{0 < x < 1} \) can be seen as a cylindrical martingale on \( C[0, 1] \), namely as a linear mapping from \( M \) to \( M_{loc}^2(P) \) (see [8] for details). The function \( Q \), defined in (2), can be identified with a linear, weakly continuous, function from \( M \) to \( C[0, 1] \), defined by

\[
Q\mu(y) = \int Q(x, y) \mu(dx);
\]

it can be proved that \( Q \) is symmetric and non-negative definite with respect to the duality \( (\cdot, \cdot)_{M, C} \) (see [8], [23], [24], for details).

For fixed \( (t, \omega) \) (which will be omitted for simplicity), we define a scalar product on the set \( Q(M) \) by the formula

\[
(Q\mu, Q\nu) = (\mu, Q\nu)_{M, C}.
\]

Then
the set $Q(M)$ admits a unique completion $H$ in $C[0, 1]$, which is a separable Hilbert space and can be continuously embedded in $C[0, 1]$;
• the dual set $H'$ of $H$ (which is clearly a Hilbert space) is the completion of $M/\ker Q$ with respect to the norm induced by the scalar product
  $$(\mu, \nu) = (\mu, Q\nu)_{M, C};$$
• the function $Q$ can be extended to the canonical isomorphism from $H'$ to $H$.

So, a family $(H'_t, \omega)_{t, \omega} \in [0, T] \times \Omega$ of Hilbert spaces is defined. A process $h$ such that $h_{t, \omega} \in H'_t, \omega$ for all $(t, \omega)$ is said to be weakly predictable if the process $(\mu, h_{t, \omega})_{H'_t, \omega}$ is predictable for all $\mu \in M$. The following results can be proved ([23, 24]):

1. the set $L^2(M, P)$ of all weakly predictable $H'$-valued processes $h$ such that
   $$\mathbb{E} \left[ \int_0^T |h_t|^2_{H'_t} dt \right] < \infty;$$
   is a Hilbert space; the set $L^2_{loc}(M, P)$ is defined in the natural way;
2. an isometric stochastic integral can be defined on $L^2(M, P)$ (and by localization on $L^2_{loc}(M, P)$) such that
   $$(\mathcal{I}(h), \mathcal{I}(h))_T = \int_0^T |h_t|^2_{H'_t} dt,$$
   where we set $\mathcal{I}(h) = \int h_t dM = h \cdot M$
3. every element $h \in L^2_{loc}(M, P)$ can be approximated by a sequence of measure-valued processes;
4. the stable subspace generated by $M$ in the set of square integrable martingales coincides with the set of stochastic integrals $\{\mathcal{I}(h) : h \in L^2(M, P)\}$.

We can show that Definition 2.5 is coherent with this theory:

**Proposition 2.1.** The following are equivalent:
(i) $h \in L^2_{loc}(M, P)$;
(ii) there exists a sequence of simple integrands $(h^n)$ such that $h^n$ converges to $h$ in the sense of Definition 2.3 and $(h^n \cdot M)$ is a Cauchy sequence in the semimartingale topology.

**Proof.** Assume that (i) holds and consider first the case when $h$ has the form

$$h = \sum_{i \leq N} \alpha^i m_i$$

where $\alpha^i$ are predictable processes and $m_i$ are Radon measures. By Lemma 41 in [23], for all $i$, there exists a sequence $\mu^n_i$ of linear combination of Dirac measures, such that $\mu^n_i$ converges weakly to $m_i$ and

$$\int_0^T (\mu^n_i - m^i, Q(\mu^n_i - m^i))_{M, C} dt = \int_0^T |\mu^n_i - m^i|^2_{H'_t} dt \to 0$$
in probability as $n$ tends to infinity. Then, one can choose a subsequence $k^n$ in such a way that the sequence of simple integrands $(h^n)$ defined by

$$h^n = \sum_{i \leq N} a^i 1_{\{a^i \leq n\}} u^k_{i,n},$$

is such that $\int_0^T |h^n_t - h_t|^2_{H^t} dt$ goes to 0 in probability, that is, the sequence $(\int_0^T h^n_t dM_t)$ converges in $L^2_{loc}(P)$, hence in $S(P)$ (see Theorem IV.5 in [21]).

Let now $h$ be any element of $L^2_{loc}(M, P)$. As it was proved by Mikulevicius and Rozovskii [23, 24], there exists a sequence $\tilde{e}^n$ of predictable measure-valued processes, such that $\{\tilde{e}^n_{t,\omega}\}$ is an orthonormal basis for $H^t_{t,\omega}$; then, the process $h$ can be written as $h = \lim_n \nu^n$, where

$$\nu^n_t = \sum_{i \leq n} (h_t, \tilde{e}^i_t) H^t_{t,\omega} \tilde{e}^i_t,$$

and $\int_0^T |\nu^n_t - h_t|^2_{H^t} dt$ goes to 0 in probability (as $n \to \infty$). By construction, $\tilde{e}^n$ has the form (3), so, in fact $h$ is the limit of processes of the form (3). By a diagonalization procedure, one can then extract a sequence of simple integrands satisfying (ii) and the claim is proved.

Assume now that (ii) holds and let $(h^n)$, $h$ be defined as in (ii). It is easy to verify that $h$ is weakly predictable. Furthermore, every $(h^n \cdot M)$ is a continuous local martingale. Since the set of continuous local martingales is closed in $S(P)$ ([21], Theorem IV.5), then the sequence $(h^n \cdot M)$ converges in $L^2_{loc}(P)$ to a continuous martingale, which proves that $h \in L^2_{loc}(M, P)$.

2.2. Generalized integrands. We have just seen that every process which is integrable with respect to a $C[0,1]$-valued continuous local martingale is a generalized integrand. We want now to show that Definition 2.5 is a good definition even for an integral with respect to a semimartingale. We first make the following observation:

**Remark 2.1.** Consider the case when $S^x = B^x$ for all $x$. Then, it is not difficult to prove that a $\mathcal{U}$-valued process $h$ is a generalized integrand for $S$ if and only if $b_{t,\omega}$ belongs to $D(h_{t,\omega})$ for all $(t, \omega)$ and

$$\int_0^T |h_t(b_t)| dt < \infty \quad P\text{-a.s.}$$

Let us go back to the general case, when $S$ is a family of semimartingales, satisfying Assumption 2.1. Then, we can prove the following:

**Proposition 2.2.** Let $h$ be as in Definition 2.5. Then:

(i) the process $h \cdot S$ is well-defined;

(ii) if $S = M + B$ is the canonical decomposition, then $h$ is integrable with respect to both $M$ and $B$ and

$$h \cdot S = h \cdot M + h \cdot B.$$
Proof. (i) Let be given two sequences of simple integrands \((h^n)\) and \((k^n)\), both converging to \(h\) and such that \(Y^n = h^n \cdot S\) and \(Z^n = k^n \cdot S\) are Cauchy in \(\mathcal{S}(P)\). Then, as in the proof of Proposition 5.1 in [9], it can be shown that \(Z^n\) and \(Y^n\) converge to the same limit.

(ii) Let \(h^n\) be an approximating sequence of simple integrands for \(h\), such that \(h^n \cdot S\) is Cauchy in \(\mathcal{S}(P)\). Thanks to the uniqueness of canonical decomposition for continuous semimartingales, the canonical decomposition of \(h^n \cdot S\) will be \(h^n \cdot \mathcal{M} + h^n \cdot \mathcal{B}\). Furthermore, if a sequence of continuous semimartingales converges in \(\mathcal{S}(P)\), then the sequence of martingale parts converges in \(\mathcal{M}^2_{loc}(P)\) while the sequence of parts with finite variation converges in \(\mathcal{A}^1_{loc}(P)\) (see for instance Remark IV.3 in [21]). Hence by Proposition 2.1 and Remark 2.1, we can assert that \(h\) is both \(\mathcal{M}\)-integrable and \(\mathcal{B}\)-integrable.

\(\square\)

Remark 2.2. The converse of Proposition 2.2 (ii) may not hold: it may be the case that \(h\) is integrable with respect to \(\mathcal{M}\) and \(\mathcal{B}\) in the sense of Definition 2.5, but the approximating sequences are different. However, if \(b\) belongs to \(H\) almost surely (where \(H\) is defined as in section 2.1), then the integral \(h \cdot S\) does exist.

2.3. The Wiener sheet. A typical example of a continuous local martingale, with values in the set of continuous functions, is the Brownian sheet (Wiener noise in space and time) \(W = (W^x_t)_{t,x \in [0,T] \times [0,1]}\), namely, the Gaussian process with covariance:

\[
\text{Cov}(W^x_t, W^y_s) = \min(x, y) \min(s, t).
\]

This is a well-known process and several attempts to define a stochastic integral with respect to it have been done (see [5] and references therein). The Mikulevicius-Rozovskii integral for this example has been analyzed in [8] (Example 3.1), where the set of integrands was fully characterized. We refer to it for all details. Here, we will simply give some results which will be needed below. From (4), it follows that \(Q(x, y) = \min(x, y)\) and does not depend on \((t, \omega)\), so also \(H'\) will not. Various results on the finite-dimensional Wiener process can be extended to the infinite-dimensional version. We recall some of them, which will be needed in the following.

We start by recalling a version of the Girsanov theorem for the Wiener sheet (see for instance [24], Proposition 3.1)

**Theorem 2.1.** Let \(k \in L^2(W, P)\) and define the process \(Z\) by

\[
Z_t = \exp \left( - \int_0^t k_s dW_s - {1 \over 2} \int_0^t \|k_s\|^2_{H'} ds \right),
\]

and assume that \(\mathbb{E}[Z_t] = 1\) for all \(t\). Then \(Z_T\) is the density of a probability measure \(\hat{P}\) \((d\hat{P}/dP = Z_T)\) such that the process \(\hat{W} = (\hat{W}^x)_{0 \leq x \leq 1}\) defined by

\[
\hat{W}^x_t = W^x_t + \int_0^t Qk_s(x) ds
\]

is a \(\hat{P}\)-Wiener sheet.
Finally, we state a representation theorem, which is an extension of a well-known result in the finite-dimensional setting:

**Proposition 2.3.** Let $M$ be a uniformly integrable martingale with respect to the filtration generated by the Wiener sheet $W$. Then, there exists $h \in L^2_{\text{loc}}(\mathcal{W}, P)$ such that

$$M_t = \mathbb{E}[M_0] + \int_0^t h_s dw_s.$$

3. The financial model

We consider a financial market model on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t)_{t \in [0,T^*]}, P)$, fulfilling the usual assumptions. For simplicity, we assume that $T^* = 1$. The main objects of the market are the zero coupon bonds (ZCB) with different maturities. We denote by $p(t, T)$ the price at time $t$ of a ZCB maturing at time $T \geq t$.

**Assumption 3.1.**

1. There exists a (frictionless) market for the zero coupon bonds for all maturity $T \leq 1$.
2. For each fixed $t$, the bond price $p(t, T)$ is differentiable with respect to $T$.
3. $p(t, t) = 1$ for all $t \leq 1$.

For the basic definitions, notations and results of the theory of bond markets, we will refer to [1]. We recall that the instantaneous forward rate at $T$, contracted at time $t$, is given by

$$f(t, T) = -\frac{\partial p(t, T)}{\partial T},$$

which implies that the zero coupon bond price $p(t, T)$ can be written as

$$p(t, T) = \exp \left(-\int_t^T f(t, u)du \right).$$

The short rate is defined by $r(t) = f(t, t)$.

We model the term structure of interest rates as a random field. In particular we make the following assumption on the dynamics of the forward rate:

**Assumption 3.2.** We assume that for all fixed $T \leq 1$, the dynamics of the forward rate $f(\cdot, T)$ evolves according to the following equation

$$(7) \quad df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW^T_t,$$

where $W = (W^x)_{x \leq 1}$ is a Wiener sheet, whereas $\alpha(t, x)$ and $\sigma(t, x)$ are deterministic functions which are assumed to be continuously differentiable in the $x$-variable and $\sigma(t, x) > 0$ for all $(t, x)$ such that $t \leq x$ (we set $\alpha(t, x) = \sigma(t, x) = 0$ for $t > x$).
We thus assume that the forward rate evolves as a continuous random field, whose covariance structure is specified by the formula:
\[
d\langle W_{T_1}^t, W_{T_2}^t \rangle = Q_{Wt}^W(T_1, T_2)dt,
\]
where \( Q_{Wt}^W(T_1, T_2) = T_1 \wedge T_2 \). This model is a particular case of the more general class of models considered by Goldstein [13], which in turn extended the model introduced by Kennedy [17, 18]. Both Kennedy and Goldstein assumed the forward rate to evolve according to (7) but they assumed a more general correlation structure, namely \( Q_{Wt}^W(T_1, T_2) = c(t, T_1, T_2) \) for some given function \( c \). However, we want to remark that most of our results can be proved in the more general setting of Goldstein, replacing \( Q_{Wt}^W \) with the generic function \( c \).

We assume that all the information in the market is given by the forward rate curve:

**Assumption 3.3.** The filtration \((F_t)_{t \leq 1}\) is generated by \((f(t, T))_{t \leq T \leq 1}\).

### 3.1. Dynamics of the zero coupon bonds \( p \)

Our aim is to derive the dynamics of \( p \) from equation (7). We work as in [3], adapting a technique from Heath-Jarrow-Morton [14].

We assume that all processes are regular enough to allow us to interchange the order of integration.

We set \( Y(t, T) = -\int_t^T f(t, u)du \), so that we can write

\[
(8) \quad p(t, T) = \exp\left[Y(t, T)\right]
\]

Writing (7) in integrated form and inserting it the expression of \( Y \), we have

\[
(9) \quad Y(t, T) = -\int_t^T f(0, u)du - \int_t^T \int_0^t \alpha(s, u)dsdu - \int_t^T \int_0^t \sigma(s, u)dW_s^u du.
\]

**Lemma 3.1.** Under the previous assumptions, the following equality holds a.s. for all \( t_1 \leq t_2, u_1 \leq u_2 \):

\[
\int_{u_1}^{u_2} \int_{t_1}^{t_2} \sigma(s, u)dW_s^u du = \int_{t_1}^{t_2} \Sigma(s, u_1, u_2)dW_s
\]

where \( \Sigma(\cdot, u_1, u_2) \) is the process with values in \( \mathcal{U} \) defined as follows: for \( f \in C[0, 1] \), we have

\[
(10) \quad \Sigma(s, u_1, u_2) = \int_{u_1}^{u_2} \sigma(s, u)f(u)du.
\]

(Observe that \( \Sigma \) can also be represented as the measure-valued process \( \mu_s(du) = \sigma(s, u)1_{[u_1, u_2]}(u)du \).}
Proof. For simplicity, we prove the lemma in the case where \( t_1 = 0, t_2 < u_1 \) and \( \sigma(s, u) \equiv \sigma \in \mathbb{R} \) for \( s \leq u \). We have that

\[
\int_{u_1}^{u_2} \int_{t_1}^{t_2} \sigma dW_s^u du = \sigma \int_{u_1}^{u_2} W_s^u du.
\]

The term on the right-hand side is a.s. limit of processes of the form

\[
\sum_{v_i \in \pi_n} \sigma_{v_i} W_{t_2}^{v_i-1} (v_i - v_{i-1})
\]

where \((\pi_n)\) is a sequence of finite partitions of \([u_1, u_2]\), such that \(\text{mesh}(\pi_n)\) tends to 0. Observe now that (11) is equal to

\[
\sum_{v_i \in \pi_n} \int_0^{t_2} \sigma(v_i - v_{i-1}) dW_s^{v_i-1},
\]

which can also be written as

\[
\int_0^{t_2} \mu_s dW_s,
\]

where \(\mu_s = \sum_{v_i \in \pi_n} \sigma(v_i - v_{i-1}) \delta_{v_i-1}^s\). It is not difficult to recognize that \(\mu_s\) converges to \(\Sigma(s, u_1, u_2)\) in the sense of Definition 2.3 and that the sequence of integrals (12) is Cauchy in \(M^2_{loc}(P)\).

We are now allowed to interchange the integrals in (9): let \(\Sigma\) be defined as (10). Observing that \(\Sigma(s, u_1, u_2) + \Sigma(s, u_2, u_3) = \Sigma(s, u_1, u_3)\), for all \(u_1 \leq u_2 \leq u_3\), we obtain

\[
Y(t, T) = -\int_t^T f(0, u) du - \int_t^T \int_s^T \alpha(s, u) duds - \int_0^t \Sigma(s, t, T) dW_s
\]

\[
= -\int_0^T f(0, u) du - \int_t^T \int_s^T \alpha(s, u) duds - \int_0^t \Sigma(s, t, T) dW_s
+ \int_t^T f(0, u) du + \int_0^t \int_s^T \alpha(s, u) duds + \int_0^t \Sigma(s, s, T) dW_s
= Y(0, T) - \int_0^t \int_s^T \alpha(s, u) duds - \int_0^t \Sigma(s, s, T) dW_s
+ \int_0^t f(0, u) du + \int_0^t \int_s^u \alpha(s, u) dsdu + \int_0^t \int_s^u \sigma(s, u) dW_s^u du,
\]

where in the last term above, we can again interchange the integrals, similar to the proof of Lemma 3.1. Now, by definition of the short rate, we have that the last line above equals to \(\int_0^T r_u du\). If we set

\[
A(t, T) = -\int_t^T \alpha(t, u) du \quad \text{and} \quad S(t, T) = -\Sigma(t, t, T)
\]

we finally obtain

\[
A(t, T) = -\int_t^T \alpha(t, u) du \quad \text{and} \quad S(t, T) = -\Sigma(t, t, T)
\]
\[ Y(t, T) = Y(0, T) + \int_0^t (r_s + A(s, T)) ds + \int_0^t S(s, T) dW_s. \]

Thus, if we apply the Ito formula to (8) and recall that
\[ d\langle Y(\cdot, T), Y(\cdot, T) \rangle_t = |S(t, T)|^2_{H'_W} dt \]
where \( H'_W \) is defined as in section 2.1 for the cylindrical martingale \( W \), we prove the following:

**Proposition 3.1.** If \( f(t, T) \) satisfies (7), then the bond dynamics is given by:

\[ dp(t, T) = p(t, T) \left[ \left( r(t) + A(t, T) + \frac{1}{2} |S(t, T)|^2_{H'_W} \right) dt + S(t, T) dW_t \right] \]

where \( A \) and \( S \) are defined by (13). In particular, \( S(t, T) \) is the \( U \)-valued process such that, for all \( f \in C[0, 1] \),
\[ S(t, T)(f) = -\int_t^T \sigma(t, u)f(u)du. \]

**Remark 3.1.** Equation (14) is well-defined for \( t \leq T \). When \( t > T \), then \( A(t, T) = 0, S(t, T) \equiv 0 \), so that \( p(t, T) = \exp(\int_t^T \sigma(s)ds) \). This is equivalent, in financial terms, to assume that the owner of a zero-coupon bond maturing at time \( T \), puts the money received at maturity time in a bank account with interest rate equal to \( r \). This assumption was done also in [2] and [8].

Proposition 3.1 allows to easily extend the HJM drift condition to this setting. We will now state the result, but will discuss on the relation with no arbitrage in section 3.3 (see Proposition 3.3).

Define the **money account** process
\[ B(t) = \exp \left( \int_0^t r_s ds \right); \]
we will sometimes refer to it as the **locally risk-free asset**. Denote by \( \bar{p}(t, T) \) the discounted price of the zero coupon bond

\[ \bar{p}(t, T) = \frac{p(t, T)}{B(t)} = p(t, T) \exp \left( -\int_0^t r_s ds \right). \]

**Proposition 3.2.** The following conditions are equivalent:

(i) the process \( \bar{p}(\cdot, T) \) is a local martingale for all \( T \).

(ii) \( \alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) u du \) for all \( t, T \).

(iii) \( \alpha(t, T) = -\sigma(t, T)S(t, T)(Id) \) for all \( t, T \) (where \( Id(x) = x \)
Proof. The equivalence between (ii) and (iii) is trivial. Let us prove that (i) is equivalent to (ii). If we apply the Ito formula to (15) and insert (14), we obtain that the process \( \overline{\rho}(t,T) \) is a martingale for all \( T \) if and only if

\[
A(t,T) + \frac{1}{2} |S(t,T)|_{\mathcal{H}_W}^2 = 0
\]

for all \( (t,T) \). We recall that \( |S(t,T)|_{\mathcal{H}_W}^2 = (S(t,T),Q^WS(t,T)) \) and \( Q^WS(t,T)(x) = S(t,T)(Q^W(x,\cdot)) \), where \( Q^W(x,y) = \min(x,y) \). So, we have

\[
|S(t,T)|_{\mathcal{H}_W}^2 = \int_t^T \sigma(t,u) \int_t^T \sigma(t,v) \min(u,v) dvdu = 2 \int_t^T \sigma(t,u) \int_u^T \sigma(t,v) v dvdu.
\]

Then, it must be

\[
\int_t^T \left( -\alpha_t(u) + \sigma_t(u) \int_u^T \sigma_t(v) v dv \right) du = 0,
\]

and since this must hold for all \( t \leq T \leq 1 \), the claim follows.

Remark 3.2. This result can also be obtained as an application of Theorem 1.1 in [17] or Proposition 1 in [13]. In fact, it is not difficult to show that our proof can be adapted to prove Proposition 1 in [13], by replacing \( Q^W(x,y) \) with the more general (possibly stochastic) correlation structure \( c(t,x,y) \). In this case condition (ii) can be written as \( \alpha_t(T) = \sigma_t(T) \int_t^T \sigma_t(u) c(t,u,T) du \) for all \( (t,T) \).

3.2. Portfolio strategies. We want to define what is a strategy in the bond market, allowing the possibility to invest in a continuum of bonds. Some answers were given in [2] and [8], involving measure-valued strategies. In [8], in particular, a definition of generalized strategy as a process taking values in a larger space than the space of Radon measure was used, but it was limited to the case of a martingale. We will extend that definition, by using the generalized integrands defined in Section 2. We start by the simple case of elementary strategies.

Definition 3.1. (1) A elementary (simple) portfolio strategy is a pair \( \pi = (\alpha^0, h) \) such that \( \alpha^0 \) is a predictable process, \( h \) is a process of the form \( h = \sum_{i \leq N} \alpha^i \delta_{T_i} \) with \( \alpha^i \) predictable processes, \( T_i \in [0,1], i = 1, \ldots N \).

(2) The value process corresponding to this portfolio is defined by

\[
V^\pi(t) = \alpha_t^0 B(t) + \sum_{i \leq N} \alpha_t^i p(t,T_i)
\]

(16)

(3) The portfolio \( \pi \) is called self-financing if

\[
V^\pi(t) = V(0) + \int_0^t \alpha_s^0 dB(s) + \int_0^t \sum_{i \leq N} \alpha_s^i dp(s,T_i)
\]

(17)
Definition 3.1 can be reformulated in terms of the discounted prices of the zero coupon bonds. In particular, equations (16) and (17) become respectively
\[ \tilde{V}^\pi_t = \frac{V^\pi_t}{B(t)} = \alpha_0^t + \sum_{i \leq N} \alpha_i^t \tilde{p}(t, T_i) \]
and
\[ \tilde{V}^\pi(t) = V(0) + \int_0^t \sum_{i \leq N} \alpha_i^s \tilde{d}\tilde{p}(s, T_i). \]

It follows that a self-financing elementary portfolio is completely determined by the pair \((V(0), h)\).

Before giving the definition of generalized portfolio, let us first observe that the family of processes \(p = (p(\cdot, T)_{T \in [0,1]})\) satisfies Assumption 2.1. In particular, for all \(T\), we have the decomposition
\[ p(t, T) = M^p(t, T) + B^p(t, T), \]
where
\[ M^p(t, T) = p(t, T)S(t, T) \cdot W, \]
\[ B^p(t, T) = \int_0^t b^p_s(T) ds, \]
and
\[ b^p_s(T) = p(t, T) \left( r(t) + A(t, T) + \frac{1}{2} |S(t, T)|^2 \mu_t \right). \]
It follows that \(Q^p(x, y)\) is given by the formula:
\[ Q^p_t(x, y) = p(t, x)p(t, y)(S(t, x), S(t, y)) \mu_t. \]

Furthermore, an analogous result holds for the process \(\tilde{p}\). So, it makes sense to define an integral with respect to the infinite-dimensional processes \(p\) and \(\tilde{p}\), by using Definition 2.5. We give the following definition:

**Definition 3.2.** A **generalized self-financing portfolio strategy** is a pair \(\pi = (V_0, h)\), where \(V_0\) is a real number and \(h\) is a generalized integrand for \(\tilde{p}\). The discounted portfolio value process is given by the formula
\[ V_t^\pi = V_0 + h \cdot \tilde{p}. \]

**Remark 3.3.** Let \((h^n)\) be an approximating sequence of simple integrands for the generalized strategy \(h\). The process \(V\) is the limit in \(S(P)\) of the sequence \(V^n = V_0 + h^n \cdot \tilde{p}\). Then, the sequence \(V^n = B\tilde{V}^n\) converges to the non-discounted portfolio value process \(V = BV\). However, it may be not possible to specify which portion of the limit portfolio has been invested in the riskless bond. Indeed, for all \(n\), we can define the amount invested in the money market account as \(\alpha^{0,n} = V_0 + h^n \cdot \tilde{p} - h^n(\tilde{p})\), but the sequence \(\alpha^{0,n}\) may not converge (see [6], Definition 3.4 for an analogous discussion in the setting of a large financial market, with countably many securities). However, if \(\tilde{p}_{t,\omega} \in D(h_{t,\omega})\) for almost all \((t, \omega)\), then, \(\alpha^0 = \lim_n \alpha^{0,n}\) is well-defined a.s. (possibly up to a subsequence). This is, for instance, the case when \(h\) is a measure-valued process. Furthermore, from (17), it
follows that a necessary and sufficient condition is that \( h \) is integrable with respect to both \( p \) and \( \tilde{p} \).

### 3.3. No Arbitrage and completeness.

A usual requirement in the study of a financial market is that there are No Arbitrage opportunities, in the sense that it is not possible to make money from nothing. It is well-known that a sufficient condition for No Arbitrage to hold is the existence of an equivalent martingale measure, that is a probability measure equivalent to \( P \), under which every \( \tilde{p}(\cdot, T) \) is a local martingale (see, for instance, [10]). In the present setting, the following holds:

**Proposition 3.3.** There exists an equivalent martingale measure for the market if and only if the \( C[0,1] \)-valued process \( g \) defined as

\[
g_t(x) = \frac{\alpha(t,x)}{\sigma(t,x)} - \int_t^x u \sigma(t,u)du,
\]

for \( t \leq x \) (\( g_t(x) = 0 \) for \( t > x \)) belongs to \( H_W \) (where \( H_W \) is the dual set of \( H'_W \) defined in section 2.1) and the process \( k \) such that \( Q^Wk = g \) satisfies the assumptions of Theorem 2.1.

**Proof.** Let \( \hat{P} \) be a probability measure equivalent to \( P \). By Assumption 3.3, we can assume that \( d\hat{P}/dP \) has the form (5). If we apply Theorem 2.1 to find the dynamics of the forward rate under \( \hat{P} \), and then apply Proposition 3.2, we deduce that \( \tilde{p}(\cdot, T) \) is a \( \hat{P} \)-martingale for all \( T \) if and only if

\[
\alpha(t,T) - \sigma(t,T)g_t(T) = \sigma(t,T)\int_t^T u \sigma(t,u)du
\]

where \( g = Q^Wk \), and \( k \) is the process of Theorem 2.1. \( \square \)

**Remark 3.4.** The set \( H_W \) was characterized in [8], Example 2.2, as the set of continuous functions \( g \), such that \( g(0) = 0 \) and \( g' \in L^2(0,1) \), and \( |g|_{H_W} = |g'|_{L^2} \). So the condition \( k \in L^2(\mathcal{W}, P) \), with \( k \) and \( g \) defined as in Proposition 3.3, can be written as

\[
\int_0^1 \int_t^1 |g'_t(x)|^2 dx dt < \infty.
\]

From now on, we work under the following assumption:

**Assumption 3.4.** There exists an equivalent martingale measure \( \hat{P} \) for the market.

It has already been proved in [8] (Theorem 5.3) that the financial market defined by Assumptions 3.1, 3.2, 3.3 is complete. Furthermore, we recall that in this case, since all processes are continuous, completeness of the market is equivalent to the uniqueness of the martingale measure (see, for instance, [15]).

In the present setting, we also know that, because of Assumption 3.3, every integrable random variable admits a representation as a stochastic integral with
respect to the Wiener sheet. Starting from this result, we will show how to find the replicating strategy.

For this purpose, we define, for all $T$, the process $D(t, T)$ by the formula

$$D(t, T) = \int_0^t \tilde{p}(s, T)^{-1}d\tilde{p}(s, T).$$

The family of processes $D = (D(\cdot, T))_{T \leq 1}$ satisfies Assumption 2.1, with $Q^D_t(x, y) = Q^\tilde{p}_t(x, y)/\tilde{p}(t, x)\tilde{p}(t, y)$. Furthermore, it is a $C[0, 1]$-valued martingale under the probability measure $\tilde{P}$, so we can apply techniques and results from section 2.1.

**Lemma 3.2.** Let $h$ be $D$-integrable process. Then, there exists a $\tilde{p}$-integrable process $k$, such that $h \cdot D = k \cdot \tilde{p}$.

**Proof.** The claim is trivial when $h$ is a simple integrand and, in this case, $k$ is also a simple strategy. Let $h$ be a generalized integrand and let $h^n$ be an approximating sequence of simple integrands. Moreover, let $k^n$ a sequence of simple integrands such that $h^n \cdot D = k^n \cdot \tilde{p}$. Since $(h^n, Q^D h^n) = (k^n, Q^\tilde{p} k^n)$, and $h^n$ is Cauchy in $L^2_{loc}(D, \tilde{P})$, it follows that $k^n$ is a Cauchy sequence in $L^2_{loc}(\tilde{p}, \tilde{P})$, hence there exists a process $k = \lim_n k^n$ which satisfies the requirements. $\square$

**Lemma 3.3.** Let $\tilde{W}$ be defined as (6). For all $t \leq T$, there exists a $\tilde{p}$-integrable process $\tau(t, T)$ such that

$$\tilde{W}_t^T = \int_0^t \tau(s, T)d\tilde{p}_s.$$

**Proof.** Thanks to Lemma 3.2, it is sufficient to find a $\mathcal{U}$-valued process $\tilde{\tau}$ such that $\tilde{W}^T = \int \tilde{\tau}(s, T)d\mathcal{D}_s$. Let $T > 0$ be fixed and let $\varepsilon_n$ be a sequence of positive numbers decreasing to 0 as $n$ goes to infinity: define the sequence of simple integrands $\tau^n(t, T)$ as follows:

$$\tau^n(t, T) = -\frac{1}{\sigma(t, T)} \frac{\delta_{T+\varepsilon_n} - \delta_T}{\varepsilon_n}$$

Then, recalling that $D(t, T) = (S(\cdot, T) \cdot \tilde{W})_t$, we have that

$$\int_0^t \tau^n(s, T)d\mathcal{D}_s = -\int_0^t S(s, T + \varepsilon_n) - S(s, T) \frac{\sigma(s, T)\varepsilon_n}{\sigma(s, T)\varepsilon_n}d\tilde{W}_s = \int_0^t \Sigma(s, T, T + \varepsilon_n) \frac{\sigma(s, T)\varepsilon_n}{\sigma(s, T)\varepsilon_n}d\tilde{W}_s$$

It is not difficult to check that $\Sigma(s, T, T + \varepsilon_n)/\sigma(s, T)\varepsilon_n$ converges weakly and in $L^2_{loc}(\tilde{W}, \tilde{P})$ to $\delta_T$, hence $\int_0^t \tau^n(s, T)d\mathcal{D}_s$ converges in $\mathcal{S}(\tilde{P})$ to $\tilde{W}^T$. So the process $\tilde{\tau}(\cdot, T) = \lim_n \tau^n(\cdot, T)$ satisfies the claim. $\square$
Remark 3.5. The process $\tilde{\tau}(t, T)$ defined in the proof of Lemma 3.3 is the $U$-valued process defined as follows: for $f \in C[0, 1]$, $f$ differentiable in $T$, we have that $\tilde{\tau}(t, T)(f) = -f'(T)\sigma(t, T)$.

So, we have built a process with values in the set of not necessarily bounded operator on $C[0, 1]$, which does not take values in the set of Radon measures.

Denote by $\hat{W}^*$ the family of martingales $\hat{W}^x$ such that $\hat{W}^x_t = \hat{W}^x_T$ for $t \leq x \leq 1$, and $\hat{W}^x_t = \hat{W}^x_x$ for $x < t < 1$. We can finally prove the following result:

Proposition 3.4. Let $h$ be a $\hat{W}^*$-integrable process. Then, there exists a $\hat{p}$-integrable process $k$ such that

\[ h \cdot \hat{W}^* = k \cdot \hat{p} \tag{18} \]

Proof. It is not difficult to check that $\hat{W}^*$ is a cylindrical martingale with respect to the filtration $(F_t)_{t \leq T}$. Let $h$ be a simple integrand of the form $\sum_{i \leq N} \alpha_i \delta_{T_i}$ and let $\tau$ be defined as in Lemma 3.3. Then, $k_t = h_t(\tau(t, \cdot)) = \sum_{i \leq N} \alpha_i \tau(t, T_i)$ is $\hat{p}$-integrable and satisfies (18). Let now $h$ be a generalized integrand, $h^n$ the approximating sequence of simple integrands and $k^n$ the relative $\hat{p}$-integrable processes. Since $(h^n, Q^{W^n} h^n) = (k^n, Q^{W^n} k^n)$ the claim follows analogously to Lemma 3.2. \qed

4. Utility maximization

In this section we consider an economic agent who has a utility function $U : (0, +\infty) \to \mathbb{R}$: for a given capital $x > 0$, its aim is to maximize the expected value from terminal wealth $\mathbb{E}[U(X_T)]$. We do not consider in this setting the case when negative wealth is allowed. We make the following assumption (see, for instance, [19]):

Assumption 4.1. (1) The function $U$ is strictly increasing, strictly concave, continuously differentiable and satisfies the so-called Inada conditions, namely

\[ U'(0) = \lim_{x \to 0} U'(x) = \infty; \quad U'(\infty) = \lim_{x \to +\infty} U'(x) = 0. \]

(2) The function $U$ has reasonable asymptotic elasticity, namely:

\[ AE(U) = \limsup_{x \to +\infty} \frac{xU'(x)}{U(x)} < 1. \]

Our purpose is now to define the problem of maximizing expected utility on a properly chosen class of strategies. We have seen in the case of a market containing countably many securities that several notions of admissible strategies can be defined in order to extend the notion of admissibility from elementary to generalized
strategies. In general, these notions do not coincide (see [7] for a discussion on this topic); in particular, the discontinuity of processes enforces to give admissibility conditions on the approximating sequences. However, this is not the case in this setting, as we will prove below (Proposition 4.1), thanks to the continuity of all processes. So, we give the following definition which is a mere extension of the concepts of admissibility of the finite-dimensional case:

**Definition 4.1.** Given \(x > 0\), we say that a generalized strategy \(h\) is \(x\)-admissible \((h \in \mathcal{A}_x)\) if \(x + (h \cdot \tilde{p})_t \geq 0 \) a.s. for all \(t\).

**Proposition 4.1.** A generalized strategy \(h\) is \(x\)-admissible if and only if there exists a sequence \(\{h^n\}_{n=1}^\infty\) of elementary strategies, such that \(x + h^n \cdot \tilde{p} \geq 0\) and \((h^n \cdot \tilde{p}) \to (h \cdot \tilde{p})\) in the semimartingale topology.

**Proof.** Necessity is trivial. Conversely, let \(h\) be in \(\mathcal{A}_x\). Thanks to the closedness of the set of continuous local martingales in the semimartingale topology ([21], Theorem IV.5), the process \((x + h \cdot \tilde{p})\) is a \(\hat{P}\)-local martingale bounded by below, hence it is a \(\hat{P}\)-supermartingale. Let \((h^n)\) be an approximating sequence of simple integrands. We define the stopping times

\[
\tau = \inf\{t : (h \cdot \tilde{p})_t < -x\} \land 1 \quad \tau_n = \inf\{t : (h^n \cdot \tilde{p})_t < -x\} \land 1
\]

Since \(x + h \cdot \tilde{p}\) is a continuous non-negative supermartingale, by Proposition II.3.4 in [25], it vanishes on \([\tau, 1]\), hence \(h1_{[0,\tau]} \cdot \tilde{p} = h \cdot \tilde{p}\). Furthermore, for the same reason, \(x + h^n \cdot \tilde{p}\) vanishes on \([\tau_n, 1]\), hence \(\lim_n P(\tau_n < \tau) = 0\); then, we can find a subsequence (still denoted by \(\tau_n\)) such that \(\sum_n P(\tau_n < \tau) < \infty\). We set \(S_n = \inf_{m \geq n} \tau_m\), so \(S_n\) is an increasing sequence of stopping times converging to \(\tau\) a.s. We denote by \(\tilde{h}^n = h_n 1_{[0,S_n]}\), \(\tilde{h} = h1_{[0,\tau]}\). Then, we have that \(x + \tilde{h}^n \cdot \tilde{p} \geq 0\), \(\tilde{h}^n \cdot \tilde{p} = h^n \cdot \tilde{p}\) on the stochastic interval \([0, S_n]\) and \(\tilde{h}^n \cdot \tilde{p}\) converges to \(h \cdot \tilde{p}\). \(\square\)

We can now define the problem of utility maximization:

\[
(19) \quad u(x) = \sup_{h \in \mathcal{A}_x} \mathbb{E}[U(x + (h \cdot \tilde{p})_T)]
\]

To exclude the trivial case, we work under the following assumption:

**Assumption 4.2.** \(u(x) < \infty\) for some \(x > 0\).

As a first step, we extend to this setting the basic superreplication result (see, for instance [10]):

**Lemma 4.1.** Let \(X \geq 0\). Then the following are equivalent:

(i) \(\mathbb{E}_{\hat{P}}[X] \leq x\)

(ii) There exists \(h \in \mathcal{A}_x\) such that \(X \leq x + h \cdot \tilde{p}\).

**Proof.** It is trivial that (ii) implies (i). Conversely, let \(X \geq 0\) be such that \(\mathbb{E}_{\hat{P}}[X] \leq x\). Since \(X\) is in \(L^1(\hat{P})\), we can define \(X_t = \mathbb{E}_{\hat{P}}[X|\mathcal{F}_t]\) and take the continuous version. The process \((X_t)_{t \leq T}\) is a uniformly integrable martingale, hence, by Propositions 2.3 and 3.4, we have that
\[ X_t = \mathbb{E}_{\tilde{\mathbb{P}}} [X] + (h \cdot \tilde{\mathbb{P}})_t \]

for some \( h \in \mathcal{A}_x \). Then the claim follows. \( \square \)

We recall that the polar set of a subset \( A \subset L^0_+ \) is defined as
\[ A^\circ = \{ Y \in L^0_+ : \mathbb{E} [XY] \leq 1 \text{ for all } X \in A \} \]

We also recall that a subset \( A \subset L^0_+ \) is called solid if \( 0 \leq Y \leq X \) and \( X \in A \) implies that \( Y \in A \).

Define the sets:
\[ \mathcal{C} = \{ X \in L^0_+ : X \leq 1 + (h \cdot \tilde{\mathbb{P}})_T \text{ for some } h \in \mathcal{A}_1 \} \]
\[ \mathcal{D} = \left\{ Y \in L^0_+ : Y \leq \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right\} \]

**Proposition 4.2.** The sets \( \mathcal{C} \) and \( \mathcal{D} \) have the following properties:

(i) \( \mathcal{C} \) and \( \mathcal{D} \) are convex, solid and closed in the topology of convergence in measure;

(ii) \( \mathcal{C}^\circ = \mathcal{D} \) and \( \mathcal{D}^\circ = \mathcal{C} \);

(iii) the constant function \( 1 \) is in \( \mathcal{C} \).

**Proof.** Condition (iii) is trivial. It is also easy to prove that \( \mathcal{D} \) is convex, solid and closed in probability. Denote by \( \mathcal{D} = \{ \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \} \); by Lemma 4.1, it follows that \( \mathcal{C} = \mathcal{D}^\circ \), which by definition, is the set \( \mathcal{D}^\circ = \{ X \in L^0_+ : \mathbb{E}_{\tilde{\mathbb{P}}} [X] \leq 1 \} \). Then, it is clearly convex and solid. Furthermore, let \( X_n \) be a sequence in \( \mathcal{C} = \mathcal{D}^\circ \), converging in probability to a random variable \( X \); up to a subsequence, we can assume that it converges almost surely. Then, as a consequence of Fatou’s lemma, \( \mathbb{E}_{\tilde{\mathbb{P}}} [X] \leq 1 \), that is \( X \in \mathcal{D}^\circ = \mathcal{C} \). So (i) is proved.

Finally, let us prove (ii). From the definition of \( \mathcal{D} \), it follows that \( \mathcal{D} \subset \mathcal{C}^\circ = \mathcal{D}^\circ \). On the other hand, since \( \mathcal{D} \supset \mathcal{D} \) and it is convex, solid and closed in probability, by the bipolar theorem (see [4] for a version of this theorem in \( L^0_+ \)), we can conclude that \( \mathcal{D} \supset \mathcal{D}^\circ \). Hence, \( \mathcal{D} = \mathcal{C}^\circ \). Furthermore, from the bipolar theorem applied to \( \mathcal{C} \), it follows that \( \mathcal{C} = \mathcal{C}^\circ \). \( \square \)

Define the sets \( \mathcal{C}_x = x\mathcal{C} \), \( \mathcal{D}_y = y\mathcal{D} \). Observe that
\[ \mathcal{C}_x = \{ X \in L^0_+ : X \leq x + (h \cdot \tilde{\mathbb{P}})_T \text{ for some } h \in \mathcal{A}_x \}, \]

that is \( \mathcal{C}_x \) is the set of random variables which are dominated by the final outcome of a portfolio with initial wealth \( x \). Then, problem (19) can be reformulated as:

\[ (20) \quad u(x) = \sup_{X \in \mathcal{C}_x} \mathbb{E} [U(X)] . \]
Let now \( V \) be the convex conjugate function of \( U \), that is

\[
V(y) = \sup_{x > 0} (U(x) - xy) \quad y > 0.
\]

The function \( V \) is continuously differentiable, decreasing, strictly convex and satisfies \( V'(0) = -\infty, V'(\infty) = 0, V(0) = U(\infty) \) and \( V(\infty) = U(0) \). Furthermore,

\[
U(x) = \inf_{y > 0} (V(y) + xy) \quad x > 0.
\]

Consider the following optimization problem:

\[
(21) \quad v(y) = \inf_{Y \in \mathcal{D}_o} \mathbb{E}[V(Y)].
\]

The assumptions of Theorem 3.1 and 3.2 in [19] are satisfied, so we can assert the following:

(i) \( u(x) < \infty \) for all \( x > 0 \) and \( v(y) < \infty \) for all \( y > 0 \);
(ii) the functions \( u \) and \( v \) are continuously differentiable on \((0, \infty)\) and they are conjugate;
(iii) the optimal solutions \( \hat{X}(x) \) to (20) and \( \hat{Y}(y) \) to (21) do exist and are unique;

furthermore the following dual relation holds:

\[
(22) \quad \hat{X}(x) = I(\hat{Y}(y))
\]

where \( I = (U')^{-1} = -V' \).

Moreover, the market is complete, so an analogous result as Lemma 4.3 in [19] can be proved: for the function \( v(y) \) we have that

\[
(23) \quad v(y) = \mathbb{E} \left[ V \left( y \frac{dP}{dP} \right) \right].
\]

4.1. A mutual fund theorem. The previous results allow to give a version of the “mutual fund theorem” for the bond market. The classical version of the mutual fund theorem in continuous time was originally presented by Merton (one can see, for instance, [1]): it concerns the problem of maximizing expected utility from terminal wealth in a market with a riskless asset and a finite number of risky assets, and essentially states that the optimal portfolio consists of an allocation between the riskless asset and a portfolio of risky assets, which is identical for all utility functions. In the present setting, we extend this result to the bond market, where a continuum of securities is available and prove that the optimal portfolio is the limit of a sequence of portfolios, obtained by allocating the wealth again between the riskless bond and a sequence of bond portfolios which do not depend on the utility function.

**Theorem 4.1.** The optimal portfolio is the limit of a sequence of portfolios, each of which consists of an allocation between the riskless bond and a fund \( w^n \) such that:
(i) the fund \( w_f^n \) consists of a finite number of bonds;
(ii) \( w_f^n \) does not depend on \( U \).

Proof. Without loss of generality, we can assume \( x = 1 \). Let \( \hat{X} \) be the optimal portfolio and denote by \( X_t \) the continuous version of the martingale \( \left( \mathbb{E}_P \left[ \hat{X} \mid \mathcal{F}_t \right] \right)_{t \leq 1} \).

By (22) and (23), \( \hat{X} = I(\hat{Y}) = I(d\hat{P}/dP) \). We know by Theorem 2.1 that \( \hat{Y} \) has the form

\[
\hat{Y} = \exp \left( -\int_0^1 k_s d\hat{W}_s + \frac{1}{2} \int_0^1 |k_s|^2_{H^2_0} ds \right),
\]

where \( \hat{W} \) is the \( \hat{P} \)-Wiener sheet defined by (6) and \( k \) is the deterministic process defined in Proposition 3.3. The process \( M_t = \int_0^t k_s d\hat{W}_s \) is a continuous local martingale with deterministic quadratic variation, so, by Theorem II.4.4 in [16], it is a \( \hat{P} \)-Wiener process. Because of (24), the process \( X_t \) is adapted and is a continuous local martingale with respect to the filtration generated by \( (M_t)_{t \leq 1} \), hence, it admits a representation \( X_t = X_0 + (\varphi \cdot M)_t \). Furthermore, it is not difficult to check that in fact \( M = k \cdot \hat{W}^* \), so, by Proposition 3.4, we have that \( M = h_f \cdot \tilde{p} \) for some generalized strategy \( h_f \), which can be approximated by the sequence of simple strategies \( h^n_f \). If we set \( w_f^n = h^n_f(\tilde{p}) \), then the sequence \( X^n = X_0 + \varphi h^n_f \cdot \tilde{p} = \alpha^n + \varphi w^n_f \), for some appropriate \( \alpha^n \), converges to the optimal portfolio, where only \( \varphi \) and \( X_0 \) depend on the utility function. \( \square \)

References