# ON THE CONVERGENCE RATE OF SOME NONLOCAL ENERGIES

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Abstract. We study the rate of convergence of some nonlocal functionals recently considered by Bourgain, Brezis and Mironescu. In particular, we establish the $\Gamma$-convergence of the corresponding rate functionals, suitably rescaled, to a limit functional of second order.

Contents



## 1. Introduction

We are interested in the rate of converge, as $h \searrow 0$, of the nonlocal functionals

$$\mathcal{F}_h(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_h(z) f\left(\frac{|u(x+z)-u(x)|}{|z|}\right) dzdx,$$

to the limit functional

$$\mathcal{F}_0(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z) f(|\nabla u(x)\cdot \hat{z}|) dzdx.$$

Here $f\colon \mathbb{R}\to[0,+\infty)$ is a convex function of class $C^2$ satisfying $f(0)=f'(0)=0$, $K:\mathbb{R}^d\to[0,+\infty)$ is a kernel such that $K(z)=K(-z)$ for a.e. $z\in\mathbb{R}^d$, and

$$\int_{\mathbb{R}^d} K(z)\left(1+|z|^2\right) dz < +\infty,$$

and we set $K_h(z) := h^{-d}K(z/h)$.

It has been proved by Bourgain, Brezis and Mironescu in [3] that $\mathcal{F}_h(u)$ tends to $\mathcal{F}_0(u)$ as $h\searrow 0$ for all $u\in H^1(\mathbb{R}^d)$, and in [8, 11] (see also [2]) it is shown that such convergence also holds in the sense of $\Gamma$-convergence [4, 6], with respect to the $L^2(\mathbb{R}^d)$-topology.

Let now

$$\mathcal{E}_h(u) := \frac{\mathcal{F}_0(u)-\mathcal{F}_h(u)}{h^2} = \frac{1}{h^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[K(z)f(|\nabla u(x)\cdot\hat{z}|) - K_h(z) f\left(\frac{|u(x+z)-u(x)|}{|z|}\right)\right] dzdx \tag{1}$$

be the functional which measures the rate of convergence of $\mathcal{F}_h$ to $\mathcal{F}_0$. In this paper, under the assumption that the function $f$ is strongly convex (see condition (12) below), we prove that the family $\{ \mathcal{E}_h \}$ $\Gamma$-converges, with respect to the $H^1(\mathbb{R}^d)$-topology, to the *second order* limit functional

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) \, |z|^2 \, f''(|\nabla u(x) \cdot \hat{z}|) \left|\nabla^2 u(x) \hat{z} \cdot \hat{z}\right|^2 dz dx & \text{if } u \in H^2(\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\hat{z} = z/|z|$. The uniform convexity assumption on $f$, which is needed for the $\Gamma$-liminf inequality, excludes from our analysis some interesting cases such as $f(x) = |x|^p$ with $p \geq 1$, $p \neq 2$. In particular, when $f(x) = |x|$ and $K$ is radially symmetric, the problem is related to a geometric problem considered in [9] in the context of a physical model for liquid drops with dipolar repulsion. We also observe that the problem we study is different from a higher order $\Gamma$-limit of $\mathcal{F}_h$ (see [5]), which would rather correspond to considering the $\Gamma$-limit of the functionals

$$\frac{\mathcal{F}_h - \min \mathcal{F}_0}{h^\alpha} \qquad \text{for some } \alpha > 0\,.$$

As a consequence of our result (see Remark 4) we also get that, if the rate of convergence of $\mathcal{F}_h(u)$ to $\mathcal{F}_0(u)$ is fast enough, more precisely if $|\mathcal{E}_h(u)| \leq M$ for all $h$'s sufficiently small, then $u \in H^2(\mathbb{R}^d)$.

We notice that our result is reminiscent to the one obtained by Peletier, Planqué and Röger in [10], motivated by a model for bilayer membranes, where they consider the convolution functionals

$$\mathcal{G}_h(u) := \int_{\mathbb{R}^d} f\left(K_h * u\right) dx,$$

which converge to the functional $\mathcal{G}_0(u) = c \int_{\mathbb{R}^d} f\left(u\right) dx$ as $h \searrow 0$, where $c = c(K, d)$ is a positive constant, and show that the corresponding rate functionals

$$\frac{\mathcal{G}_0(u) - \mathcal{G}_h(u)}{h^2} = \frac{1}{h^2} \int_{\mathbb{R}^d} \big(c\, f\left(u\right) - f\left(K_h * u\right)\big) dx \tag{2}$$

converge pointwise to the limit functional

$$\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) |z|^2 f''(u(x)) |\nabla u(x) \cdot \hat{z}|^2 dz dx \qquad \text{for } u \in H^1(\mathbb{R}^d).$$

In particular, the rate functionals are uniformly bounded if and only if $u \in H^1(\mathbb{R}^d)$.

In the proof of our convergence result, we follow a strategy similar to the one in [7, 8]: we first consider a related 1-dimensional problem, and then reduce the general case to it by a *slicing* procedure. More precisely, in Section 2 we study the functionals

$$E_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} \left[ f(u(x)) - f\left( \fint_x^{x+h} u(y) dy \right) \right] dx,$$

which are a particular case of (2), and we show their convergence (see Theorem 1) to the limit energy

$$E_0(u) := \frac{1}{24} \int_{\mathbb{R}} f''(u(x)) \left|u'(x)\right|^2 dx \qquad \text{for } u \in H^1(\mathbb{R}).$$

Then, in Section 3 we consider the general functionals in (1) and we prove the $\Gamma$-convergence to $\mathcal{E}_0$ (see Theorem 2), which is the main result of this paper. We first show the convergence for $d = 1$, using the result of Section 2, and then we reduce to the 1-dimensional case by means of a delicate slicing technique.

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## 2. Finite difference functionals in the 1-dimensional case

For $u \in L^2(\mathbb{R})$ and $h > 0$, we define the energy

$$E_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} \left[ f(u(x)) - f\left(D_h U(x)\right) \right] dx, \tag{3}$$

where

$$U(x) := \int_0^x u(y)dy \quad \text{and} \quad D_h U(x) := \frac{U(x+h) - U(x)}{h} = \fint_x^{x+h} u(y)dy. \tag{4}$$

Let us fix an open interval $I := (a, b) \subset \mathbb{R}$. We shall compute the $\Gamma$-limit of $\{ E_h \}$ regarded as a family of functionals on the closed subspace $Y \subset L^2(\mathbb{R})$ defined as

$$Y := \{ u \in L^2(\mathbb{R}) : u = 0 \text{ in } \mathbb{R} \setminus I \} \tag{5}$$

endowed with the $L^2$-topology. Let us set

$$E_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}} f''(u(x)) \left| u'(x) \right|^2 dx & \text{if } u \in Y \cap H^1(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

We shall prove the following:

**Theorem 1.** *Let us assume that there exists $\gamma > 0$ such that $2f(t) - \gamma t^2$ is convex. Then, the restriction to $Y$ of the family $\{ E_h \}$ $\Gamma$-converges, as $h \searrow 0$, to $E_0$ w.r.t. the $L^2(\mathbb{R})$-topology, that is, for every $u \in Y$ the following properties hold:*

1. *For any family $\{ u_h \} \subset Y$ that converges to $u$ in $L^2(\mathbb{R})$ we have*
$$E_0(u) \leq \liminf_{h \searrow 0} E_h(u_h).$$
2. *There exists a sequence $\{ u_h \} \subset Y$ converging to $u$ in $L^2(\mathbb{R})$ such that*
$$\limsup_{h \searrow 0} E_h(u_h) \leq E_0(u).$$

The $\Gamma$-upper limit is established in Proposition 1, while Proposition 2 takes care of the lower limit. In turn, the latter is achieved by exploiting a suitable lower bound on the energy (see Lemma 1) and a compactness result (see Lemma 2).

### 2.1. Pointwise limit and upper bound.

We now compute the limit of $E_h(u)$, as $h \searrow 0$, for a function $u \in Y \cap C^2(\mathbb{R})$.

**Proposition 1.** *Let $u \in Y \cap C^2(\mathbb{R})$. Then, there exists a continuous, bounded, and increasing function $m \colon [0, +\infty) \to [0, +\infty)$ such that $m(0) = 0$ and*

$$|E_h(u) - E_0(u)| \leq c\, m(h), \tag{7}$$

*where $c := c(b - a, \|u\|_{C^2(\mathbb{R})}, \|f\|_{C^2([-\|u\|_{C^2(\mathbb{R})}, \|u\|_{C^2(\mathbb{R})}])}) > 0$ is a constant. In particular,*

$$\lim_{h \searrow 0} E_h(u) = E_0(u),$$

*moreover for every $u \in Y$ there exists a sequence $\{ u_h \} \subset Y$ that converges to $u$ in $L^2(\mathbb{R})$ and satisfies*

$$\limsup_{h \searrow 0} E_h(u_h) \leq E_0(u).$$

*Proof.* Since $u \in Y \cap C^2(\mathbb{R})$ and $f \in C^2(\mathbb{R})$, it is easy to see that $F_h(u) - F_0(u)$ and $E_0(u)$ are uniformly bounded in $h$. Thus, there exists a constant $c_\infty > 0$ such that

$$|E_h(u) - E_0(u)| \leq c_\infty \qquad \text{for } h > 1. \tag{8}$$

Next, we focus on the case $h \in (0,1]$. If $x \notin (a-h, b)$, then $D_h U(x) = 0$, and hence

$$F_0(u) - F_h(u) = \int_a^b [f(u(x)) - f(D_h U(x))]\, dx - \int_{a-h}^a f(D_h U(x)) dx.$$

Being $u$ regular, for any $x \in (a-h, b)$ we have the Taylor's expansion

$$D_h U(x) = u(x) + \frac{h}{2} u'(x) + \frac{h^2}{6} u''(x_h), \qquad \text{with } x_h \in (x, x+h),$$

which we rewrite as

$$D_h U(x) = u(x) + h v_h(x), \qquad \text{with } \ v_h(x) \coloneqq \frac{u'(x)}{2} + \frac{h}{6} u''(x_h); \tag{9}$$

note that $v_h$ converges uniformly to $u'/2$ as $h \searrow 0$.

Plugging (9) into the definition of $F_h$, we get

$$\begin{aligned} F_0(u) - F_h(u) = &- \int_a^b \big[f\big(u(x) + h v_h(x)\big) - f(u(x))\big]\, dx - \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx \\ = &- h \int_a^b f'(u(x)) v_h(x) dx - \frac{h^2}{2} \int_a^b f''(w_h(x)) v_h(x)^2 dx \\ &- \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx, \end{aligned}$$

where $w_h$ fulfils $w_h(x) \in (u(x), u(x) + h v_h(x_h))$ for all $x \in (a,b)$.

It easy to see that

$$\left| \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx \right| \leq c_1 h^5,$$

for a constant $c_1 > 0$ that depends only on $N \coloneqq \|u\|_{C^2(\mathbb{R})}$ and on $\|f''\|_{L^\infty([-N,N])}$. Moreover, recalling the definition of $v_h$, we have

$$\int_a^b f'(u(x)) v_h(x) dx = \frac{h}{6} \int_a^b f'(u(x)) u''(x_h) dx,$$

and therefore

$$\begin{aligned} |E_h(u) - E_0(u)| \leq &\frac{1}{6} \left| - \int_a^b f'(u(x)) u''(x_h) dx - \int_a^b f''(u(x)) u'(x)^2 dx \right| \\ &+ \frac{1}{2} \left| \frac{1}{4} \int_a^b f''(u(x)) u'(x)^2 dx - \int_a^b f''(w_h(x)) v_h(x)^2 dx \right| + c_1 h^5. \end{aligned} \tag{10}$$

Since $u \in Y \cap C^2(\mathbb{R})$, $u''$ admits a uniform modulus of continuity $m_{u''} \colon [0, +\infty) \to [0, \infty)$. An integration by parts gives that

$$\begin{aligned} \left| - \int_a^b f'(u(x)) u''(x_h) dx - \int_a^b f''(u(x)) u'(x)^2 dx \right| &\leq \int_a^b |f'(u(x))|\, |u''(x) - u''(x_h)|\, dx \\ &\leq c_2 m_{u''}(h), \end{aligned}$$

where $c_2 \coloneqq (b-a)\|f'\|_{L^\infty([-N,N])}$.

In a similar manner, denoting by $m_{f''}$ the modulus of continuity of the restriction of $f''$ to the interval $[-N,N]$, we also find

$$\left|\frac{1}{4}\int_a^b f''(u(x))u'(x)^2dx - \int_a^b f''(w_h(x))v_h(x)^2dx\right|$$
$$\le \int_a^b |f''(u(x))| \left|\frac{1}{4}u'(x)^2 - v_h(x)^2\right| dx + \int_a^b |f''(u(x)) - f''(w_h(x))|\, v_h(x)^2dx$$
$$\le c_3(h + m_{f''}(h)),$$

with $c_3$ depending on $b-a$, $N$, and $\|f''\|_{L^\infty([-N,N])}$.

By combining (10) with the inequalities above, we obtain

$$|E_h(u) - E_0(u)| \le c_0\big(m_{u''}(h) + m_{f''}(h) + h + h^5\big) \qquad \text{for } h \in (0,1], \tag{11}$$

for a suitable constant $c_0 > 0$.

The conclusion now follows by (8) and (11). □

Notice that, by standard density arguments, the second statement of Theorem 1 follows directly by Proposition 1.

**Remark 1.** Notice that, as a consequence of Proposition 1, the Γ-limit of the rate functionals

$$hE_h(u) = \frac{1}{h}\int_{\mathbb{R}} [f(u(x)) - f(D_hU(x))]\, dx$$

is equal to zero.

2.2. **Lower bound in the strongly convex case.** In view of Proposition 1, to complete the proof of the Theorem 1, it only remains to establish statement 1, that is, for any $u \in Y$ and for any family $\{u_h\} \subset Y$ converging to $u$ in $L^2(\mathbb{R})$ it holds

$$E_0(u) \le \liminf_{h\searrow 0} E_h(u_h).$$

We prove the inequality under the hypothesis that the function $f$ is strongly convex, i.e., we assume that

$$\text{there exists } \gamma > 0 \text{ such that } 2f(t) - \gamma t^2 \text{ is convex.} \tag{12}$$

Thanks to this additional assumption on $f$, we are able to provide a lower bound on the energy $E_h$, and we use it to prove that sequences with equibounded energy are relatively compact w.r.t. the $L^2$-topology.

**Lemma 1** (Lower bound on the energy)**.** *Assume that $f$ fulfils* (12)*. Then, for any $u \in Y$, it holds*

$$E_h(u) \ge \sup_{\varphi\in C_c^\infty(\mathbb{R}^2)} \left\{\int_{\mathbb{R}} \fint_x^{x+h} \left(\frac{u(y) - D_hU(x)}{h}\varphi(x,y) - \frac{\varphi(x,y)^2}{4\lambda_h(x,y)}\right) dydx\right\}, \tag{13}$$

*with*

$$\lambda_h(x,y) := \int_0^1 (1-\vartheta)f''\big((1-\vartheta)D_hU(x) + \vartheta u(y)\big)d\vartheta. \tag{14}$$

*Moreover,*

$$E_h(u) \ge \frac{\gamma}{4}\int_{\mathbb{R}}\int_{-h}^h J_h(r)\left(\frac{u(y+r) - u(y)}{h}\right)^2 drdy, \tag{15}$$

*where*

$$J(r) := (1-|r|)^+ \quad \textit{and} \quad J_h(r) := \frac{1}{h}J\left(\frac{r}{h}\right).$$

*Proof.* For a given $h > 0$, let us consider $u \in Y$ such that $E_h(u)$ is finite. We write

$$E_h(u) = \frac{1}{h^2} \int_{\mathbb{R}} e_h(x) dx, \quad \text{where } e_h(x) := \fint_x^{x+h} [f(u(y)) - f(D_h U(x))] dy.$$

Thanks to the identity

$$f(s) - f(t) = f'(t)(s-t) + (s-t)^2 \int_0^1 (1-\vartheta) f''((1-\vartheta)t + \vartheta s)) d\vartheta,$$

we find

$$e_h(x) = \fint_x^{x+h} \lambda_h(x,y) \left(u(y) - D_h U(x)\right)^2 dy, \tag{16}$$

where $\lambda_h(x,y)$ is as in (14). Observe that, by the strong convexity of $f$, $\lambda_h(x,y) \geq \gamma/2$ for all $(x,y) \in \mathbb{R}^2$ and $h > 0$, so that (13) holds.

By the same bound on $\lambda_h$, we also deduce that

$$e_h(x) \geq \frac{\gamma}{2} \fint_x^{x+h} \left(u(y) - D_h U(x)\right)^2.$$

Hence, we get

$$\begin{aligned} E_h(u) \geq & \frac{\gamma}{2} \int_{\mathbb{R}} \fint_x^{x+h} \left(\frac{u(y) - D_h U(x)}{h}\right)^2 dy dx \\ \geq & \frac{\gamma}{4} \int_{\mathbb{R}} \fint_x^{x+h} \fint_x^{x+h} \left(\frac{u(z) - u(y)}{h}\right)^2 dz dy dx, \end{aligned}$$

where the last inequality follows from the identity

$$\int |\varphi(y)|^2 d\mu(y) = \left| \int \varphi(y) d\mu(y) \right|^2 + \frac{1}{2} \int \int |\varphi(z) - \varphi(y)|^2 d\mu(z) d\mu(y),$$

which holds whenever $\mu$ is a probability measure and $\varphi \in L^2(\mu)$. By Fubini's Theorem and neglecting contributions near the boundary, we find the lower bound on the energy:

$$\begin{aligned} E_h(u) \geq & \frac{\gamma}{4} \int_{\mathbb{R}} \fint_{y-h}^{y} \fint_x^{x+h} \left(\frac{u(z) - u(y)}{h}\right)^2 dz dx dy \\ = & \frac{\gamma}{4h} \int_{\mathbb{R}} \int_{y-h}^{y+h} \left(1 - \frac{|z-y|}{h}\right) \left(\frac{u(z) - u(y)}{h}\right)^2 dz dy. \end{aligned}$$

The conclusion (13) is now achieved by the change of variables $r = z - y$. □

**Lemma 2** (Compactness)**.** *Assume that $f$ fulfils (12). Let $\{ u_h \} \subset Y$ be a sequence of functions such that $E_h(u_h) \leq M$ for some $M \geq 0$. Then, there exist a subsequence $\{ u_{h_\ell} \}$ and a function $u \in Y \cap H^1(\mathbb{R})$ such that $u_{h_\ell} \to u$ in $L^2(\mathbb{R})$.*

*Proof.* We adapt the strategy of [1, Theorem 3.1].

By Lemma 1, we infer that

$$\frac{\gamma}{4} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r) \left(\frac{u_h(y+r) - u_h(y)}{h}\right)^2 dr dy \leq M. \tag{17}$$

Observe that $J_h(r)dr$ is a probability measure on $[-h, h]$.

We now introduce the mollified functions $v_h := \rho_h * u_h$, where $\{ \rho_h \}$ is the family

$$\rho_h(r) := \frac{1}{ch} \rho\left(\frac{r}{h}\right), \qquad \text{with } c := \int_{\mathbb{R}} \rho(r) dr.$$

Here, $\rho \in C_c^\infty(\mathbb{R})$ is an even kernel, and it is chosen in such a way that its support is contained in $[-1,1]$,

$$0 \le \rho \le J, \quad \text{and} \quad |\rho'| \le J.$$

Note that, for all $h>0$, $v_h\colon \mathbb{R}\to\mathbb{R}$ is a smooth function whose support is a subset of $(a-h, b+h)$. Moreover, the family of derivatives $\{ v_h' \}_{h\in(0,1)}$ is uniformly bounded in $L^2(\mathbb{R})$; indeed, since $\int_{\mathbb{R}} \rho'(r)dr = 0$, it holds

$$\begin{aligned}
\int_{\mathbb{R}} |v_h'(y)|^2\, dy &= \int_{\mathbb{R}} \left| \int_{-h}^{h} \rho_h'(r)[u_h(y+r)-u_h(y)]dr \right|^2 dy \\
&\le \int_{\mathbb{R}} \left( \int_{-h}^{h} |\rho_h'(r)|\, |u_h(y+r)-u_h(y)|\, dr \right)^2 dy \\
&\le \frac{1}{c^2} \int_{\mathbb{R}} \left( \int_{-h}^{h} J_h(r) \left| \frac{u_h(y+r)-u_h(y)}{h} \right| dr \right)^2 dy \\
&\le \frac{1}{c^2} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r) \left| \frac{u_h(y+r)-u_h(y)}{h} \right|^2 drdy,
\end{aligned}$$

and thus

$$\int_{\mathbb{R}} |v_h'(y)|^2\, dy \le \frac{4M}{c^2\gamma}. \tag{18}$$

For all $h\in(0,1)$, let $\tilde v_h$ be the restriction of $v_h$ to the interval $(a-1, b+1)$. By Poincaré inequality, (18) entails boundedness in $H_0^1((a-1,b+1))$ of the family $\{ \tilde v_h \}_{h\in(0,1)}$, and, in view of Sobolev's Embedding Theorem, this grants in turn that there exists a subsequence $\{ \tilde v_{h_\ell} \}$ uniformly converging to some $\tilde u \in H_0^1([a-1,b+1])$. Since each $\tilde v_{h_\ell}$ is supported in $(a-h_\ell, b+h_\ell)$, we see that $\tilde u \in H_0^1(\bar I)$; therefore, if we set

$$u(x) := \begin{cases} \tilde u(x) & \text{if } x\in \bar I, \\ 0 & \text{otherwise}, \end{cases}$$

we deduce that $\{ v_{h_\ell} \}$ converges uniformly to $u \in Y\cap H^1(\mathbb{R})$.

Lastly, to achieve the conclusion, we provide a bound on the $L^2$-distance between $u_h$ and $v_h$. Similarly to the previous computations, we have

$$\begin{aligned}
\int_{\mathbb{R}} |v_h(y)-u_h(y)|^2\, dy &= \int_{\mathbb{R}} \left| \int_{-h}^{h} \rho_h(r)[u_h(y+r)-u_h(y)]dr \right|^2 dy \\
&\le \int_{\mathbb{R}} \int_{-h}^{h} \rho_h(r)\, |u_h(y+r)-u_h(y)|^2\, drdy \\
&\le \frac{1}{c} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r)\, |u_h(y+r)-u_h(y)|^2\, drdy,
\end{aligned}$$

and, by (17), we get

$$\int_{\mathbb{R}} |v_h(y)-u_h(y)|^2 dy \le \frac{4M}{c\gamma} h^2. \tag{19}$$

Since there exists a subsequence $\{ v_{h_\ell} \}$ uniformly converging to a function $u\in Y\cap H^1(\mathbb{R})$, (19) gives the conclusion. □

Now we can prove statement (1) of Theorem 1.

**Proposition 2.** *Let $f$ satisfy (12). Then, for any $u \in Y$ and for any family $\{ u_h \} \subset Y$ that converges to $u$ in $L^2(\mathbb{R})$, it holds*

$$E_0(u) \le \liminf_{h\searrow 0} E_h(u_h). \tag{20}$$

*Proof.* Fix $u, u_h \in Y$ in such a way that $u_h \to u$ in $L^2(\mathbb{R})$. We can suppose that the inferior limit in (20) is finite, otherwise the conclusion holds trivially. Consequently, up to extracting a subsequence, which we do not relabel, there exists $\lim_{h\searrow 0} E_h(u_h)$ and it is finite. In particular, there exists $M \ge 0$ such that $E_h(u_h) \le M$ for all $h > 0$, and, by Lemma 2, this yields that $u \in Y \cap H^1(\mathbb{R})$.

We use formula (13) for each $u_h$, choosing, for $(x,y) \in \mathbb{R}^2$,

$$\varphi(x,y) = \psi\left(x, \frac{y-x}{h}\right), \qquad \text{with } \psi \in C_c^\infty(\mathbb{R}^2).$$

We get

$$\begin{aligned} E_h(u) \ge & \int_{\mathbb{R}} \fint_x^{x+h} \frac{u_h(y) - \fint_x^{x+h} u_h}{h} \psi\left(x, \frac{y-x}{h}\right) dydx \\ & - \frac{1}{4} \int_{\mathbb{R}} \fint_x^{x+h} \frac{\psi\left(x, \frac{y-x}{h}\right)^2}{\lambda_h(x,y)} dydx, \end{aligned} \tag{21}$$

where, coherently with (14),

$$\lambda_h(x,y) \coloneqq \int_0^1 (1-\vartheta) f''\left( (1-\vartheta) \fint_x^{x+h} u_h(z)dz + \vartheta u_h(y) \right) d\vartheta \ge \frac{\gamma}{2}.$$

Let us focus on the first quantity on the right-hand side of (21). We have

$$\begin{aligned} & \frac{1}{h} \int_{\mathbb{R}} \fint_x^{x+h} \left( \fint_x^{x+h} u_h(z)dx \right) \psi\left(x, \frac{y-x}{h}\right) dydx \\ & = \frac{1}{h^3} \int_{\mathbb{R}} \int_x^{x+h} \int_x^{x+h} u_h(z) \psi\left(x, \frac{y-x}{h}\right) dydzdx \\ & = \frac{1}{h^3} \int_{\mathbb{R}} \int_{z-h}^{z} \int_x^{x+h} u_h(z) \psi\left(x, \frac{y-x}{h}\right) dydxdz, \end{aligned}$$

and, by similar computations, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \fint_x^{x+h} \frac{u(y) - \fint_x^{x+h} u_h}{h} \psi\left(x, \frac{y-x}{h}\right) dydx \\ & = \frac{1}{h} \int_{\mathbb{R}} \fint_{y-h}^{y} \fint_x^{x+h} u_h(y) \left[ \psi\left(x, \frac{y-x}{h}\right) - \psi\left(x, \frac{z-x}{h}\right) \right] dzdxdy. \end{aligned} \tag{22}$$

By a simple change of variable, we get

$$\begin{aligned} \fint_{y-h}^{y} \fint_x^{x+h} \psi\left(x, \frac{y-x}{h}\right) dzdx &= \int_0^1 \int_0^1 \psi(y-hr, r) dqdr, \\ \fint_{y-h}^{y} \fint_x^{x+h} \psi\left(x, \frac{z-x}{h}\right) dzdx &= \fint_{y-h}^{y} \int_0^1 \psi(x,r) drdx \\ &= \int_0^1 \int_0^1 \psi(y-hq, r) dqdr, \end{aligned}$$

hence

$$\begin{aligned}\frac{1}{h}\fint_{y-h}^{y}\fint_{x}^{x+h}&\left[\psi\left(x,\frac{y-x}{h}\right)-\psi\left(x,\frac{z-x}{h}\right)\right]dzdx\\&=\int_0^1\int_0^1\frac{\psi(y-hr,r)-\psi(y-hq,r)}{h}dqdr\\&=-\int_0^1\int_0^1\int_q^r\partial_1\psi(y-hs,r)dsdqdr\\&=-\int_0^1\int_0^1(r-q)\fint_q^r\partial_1\psi(y-hs,r)dsdqdr.\end{aligned}$$

Being $\psi$ smooth, we have that $\partial_1\psi(y-hs,r)=\partial_1\psi(y,r)+O(h)$ as $h\searrow 0$, uniformly for $s\in[0,1]$. Consequently,

$$\frac{1}{h}\fint_{y-h}^{y}\fint_{x}^{x+h}\left[\psi\left(x,\frac{y-x}{h}\right)-\psi\left(x,\frac{z-x}{h}\right)\right]dzdx=-\int_0^1\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)dr+O(h).$$

Plugging this equality in (22) yields

$$\int_{\mathbb{R}}\fint_x^{x+h}\frac{u(y)-\fint_x^{x+h}u_h}{h}\psi\left(x,\frac{y-x}{h}\right)dydx=-\int_{\mathbb{R}}u_h(y)\int_0^1\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)dr+O(h).$$

It is possible to take the limit $h\searrow 0$ in the previous formula, since $u_h\to u$ in $L^2(\mathbb{R})$. We then get

$$\lim_{h\searrow 0}\int_{\mathbb{R}}\fint_x^{x+h}\frac{u(y)-\fint_x^{x+h}u_h}{h}\psi\left(x,\frac{y-x}{h}\right)dydx=-\int_{\mathbb{R}}\int_0^1u(y)(r-\tfrac{1}{2})\partial_1\psi(y,r)drdy. \tag{23}$$

Now, we turn to the second addendum on the right-hand side of (21). By Fubini's Theorem and a change of variables, we have

$$\int_{\mathbb{R}}\fint_x^{x+h}\frac{\psi(x,\frac{y-x}{h})^2}{\lambda_h(x,y)}dydx=\int_{\mathbb{R}}\int_0^1\frac{\psi\left(y-hr,r\right)^2}{\lambda_h(y-hr,y)}drdy.$$

The function $\psi$ has compact support and $\lambda_h\geq\gamma/2$ for all $h>0$, therefore we can apply Lebesgue's Convergence Theorem to let $h\searrow 0$ in the previous expression, and we get

$$\begin{aligned}\lim_{h\searrow 0}\int_{\mathbb{R}}\int_0^1&\frac{\psi\left(y-hr,r\right)^2}{\int_0^1(1-\vartheta)f''\left((1-\vartheta)\fint_{y-hr}^{y+(1-r)h}u_h(z)dz+\vartheta u_h(y)\right)d\vartheta}drdy\\&=\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{\int_0^1(1-\vartheta)f''(u(y))d\vartheta}drdy,\end{aligned}$$

thus

$$\lim_{h\searrow 0}\int_{\mathbb{R}}\fint_x^{x+h}\frac{\psi(x,\frac{y-x}{h})^2}{\lambda_h(x,y)}dydx=2\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{f''(u(y))}drdy. \tag{24}$$

Summing up, by (23) and (24), we deduce

$$\liminf_{h\searrow 0}E_h(u_h)\geq-\int_{\mathbb{R}}\int_0^1\left[u(y)\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)drdy+\frac{1}{2}\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{f''(u(y))}\right]drdy, \tag{25}$$

for all $\psi\in C_c^\infty(\mathbb{R}^2)$.

We can reach the conclusion from the last inequality by a suitable choice of the test function $\psi$. To see this, we let $\eta \in C_c^\infty(\mathbb{R})$ and we choose a standard sequence of mollifiers $\{ \rho_k \}$. We then set

$$\psi(x,y) = \psi_k(x,y) \coloneqq \eta(x)\left(\zeta_k(y) - \frac{1}{2}\right), \quad \text{with } \zeta_k(y) \coloneqq \int_{\mathbb{R}} \rho_k(z-y) z dz,$$

so that (25) reads

$$\begin{aligned}\liminf_{h\searrow 0} E_h(u_h) \geq & -\int_0^1 \left(r - \frac{1}{2}\right)\left(\zeta_k(r) - \frac{1}{2}\right) dr \int_{\mathbb{R}} u(y)\eta'(y) dy \\ & - \frac{1}{2}\int_0^1 \left(\zeta_k(r) - \frac{1}{2}\right)^2 dr \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy.\end{aligned}$$

Because of the identity $\int_0^1 (r-1/2)^2 dr = 1/12$, letting $k \to +\infty$ yields

$$\begin{aligned}\liminf_{h\searrow 0} E_h(u_h) \geq & -\frac{1}{12}\left[\int_{\mathbb{R}} u(y)\eta'(y) dy + \frac{1}{2}\int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy\right] \\ = & \frac{1}{12}\left[\int_{\mathbb{R}} u'(y)\eta(y) dy - \frac{1}{2}\int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy\right],\end{aligned}$$

where $u' \in L^2(\mathbb{R})$ is the distributional derivative of $u$, which exists since $u \in H^1(\mathbb{R})$. Recall that, in the previous formula, the test function $\eta$ is arbitrary, thus, to recover (20), it suffices to take the supremum w.r.t. $\eta \in C_c^\infty(\mathbb{R})$. □

## 3. Γ-limit in arbitrary dimension

Let us fix an open, bounded set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and a function $K\colon \mathbb{R}^d \to [0,+\infty)$ such that

$$\int_{\mathbb{R}^d} K(z)\left(1 + |z|^2\right) dz < +\infty. \tag{26}$$

We require that $K(z) = K(-z)$ for a.e. $z \in \mathbb{R}^d$ and that the support of $K$ contains a sufficiently large annulus centered at the origin. More precisely, let us set

$$\sigma_d \coloneqq \begin{cases} 1 & \text{when } d = 2, \\ \dfrac{d-2}{d-1} & \text{when } d > 2; \end{cases} \tag{27}$$

we suppose that there exist $r_0 \geq 0$ and $r_1 > 0$ such that $r_0 < \sigma_d r_1$ and

$$\operatorname{ess\,inf}\{ K(z) : z \in B(0,r_1) \setminus B(0,r_0) \} > 0. \tag{28}$$

The simplest case for which (28) holds is when there exists $k > 0$ such that $K(z) \geq k$ for all $z \in B(0,r_1)$.

Let $f\colon [0,+\infty) \to [0,+\infty)$ be a $C^2$ function such that $f(0) = f'(0) = 0$, and (12) is satisfied. For $u \in H^1(\mathbb{R}^d)$, we define the functionals

$$\mathcal{F}_0(u) \coloneqq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z) f(|\nabla u(x) \cdot \hat{z}|) dz dx,$$

$$\mathcal{F}_h(u) \coloneqq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_h(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|}\right) dy dx,$$

where $K_h(z) \coloneqq h^{-d} K(z/h)$.

By results in [11], we know that $\mathcal{F}_h(u)$ tends to $\mathcal{F}_0$ as $h \searrow 0$ when $u \in H^1(\mathbb{R}^d)$, and also that $\mathcal{F}_0$ is the Γ-limit of the family $\{ \mathcal{F}_h \}$. As before, we are interested in the asymptotics of

$$\frac{\mathcal{F}_0 - \mathcal{F}_h}{h} \quad \text{and} \quad \frac{\mathcal{F}_0 - \mathcal{F}_h}{h^2}$$

Analogously to the 1-dimensional case, we define

$$\mathcal{E}_h(u) := \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h^2}.$$

Let us set

$$X := \{ u \in H^1(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega \} \tag{29}$$

and

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) \, |z|^2 \, f''(|\nabla u(x) \cdot \hat{z}|) \left| \nabla^2 u(x) \hat{z} \cdot \hat{z} \right|^2 dz dx & \text{if } u \in X \cap H^2(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{30}$$

**Remark 2** (Radial case)**.** When $K$ is radial, that is $K(z) = \bar{K}(|z|)$ for some $\bar{K} \colon [0, +\infty) \to [0, +\infty)$, we have

$$\mathcal{F}_0(u) = \|K\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f(|\nabla u(x) \cdot e|) d\mathcal{H}^{d-1}(e) dx,$$

$$\mathcal{E}_0(u) = \frac{1}{24} \left( \int_{\mathbb{R}^d} K(z) \, |z|^2 \, dz \right) \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f''(|\nabla u(x) \cdot e|) \left| \nabla^2 u(x) e \cdot e \right|^2 d\mathcal{H}^{d-1}(e) dx.$$

This Section is devoted to the proof of the following Γ-convergence result:

**Theorem 2.** *Under the previous assumptions on $\Omega$, $K$, and $f$, there holds:*

1. *For any family $\{ u_h \} \subset X$ such that $\mathcal{E}_h(u_h) \le M$ for some $M > 0$, there exists a subsequence $\{ u_{h_\ell} \}$ and a function $u \in X \cap H^2(\mathbb{R}^d)$ such that $\nabla u_{h_\ell} \to \nabla u$ in $L^2(\mathbb{R}^d)$.*
2. *For any family $\{ u_h \} \subset X$ that converges to $u \in X$ in $H^1(\mathbb{R}^d)$*
$$\mathcal{E}_0(u) \le \liminf_{h \searrow 0} \mathcal{E}_h(u_h).$$
3. *For any $u \in X$, there exists a family $\{ u_h \} \subset X$ that converges to $u$ in $H^1(\mathbb{R}^d)$ with the property that*
$$\limsup_{h \searrow 0} \mathcal{E}_h(u_h) \le \mathcal{E}_0(u).$$

3.1. **Slicing and upper bound.** When the dimension is 1, in virtue of the analysis in Section 2, it is not difficult to derive the Γ-convergence of the functionals $\mathcal{E}_h$.

**Corollary 1.** *Let $K \colon \mathbb{R} \to [0, +\infty)$ be an even function such that* (26) *holds. For $h > 0$ and $u \in H^1(\mathbb{R})$, let us define the family*

$$\mathcal{E}_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} \int_{\mathbb{R}} K_h(z) \left[ f(|u'(x)|) - f\left( \left| \frac{u(x+z) - u(x)}{z} \right| \right) \right] dz dx.$$

*Let also $\Omega = (a, b)$ be an open interval, and let $X \subset\in H^1(\mathbb{R})$ be defined as in* (29)*. Then, the restrictions of the functionals $\mathcal{E}_h$ to $X$ Γ-converge w.r.t. the $H^1(\mathbb{R})$-topology to*

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{24} \left( \displaystyle\int_{\mathbb{R}} K(z) z^2 dz \right) \int_{\mathbb{R}} f''(u'(x)) \, |u''(x)|^2 \, dx & \text{if } u \in X \cap H^2(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof.* A change of variables gives

$$\mathcal{E}_h(u) = \int_{\mathbb{R}} K(z) z^2 \left[ \frac{1}{(hz)^2} \int_{\mathbb{R}} f(|u'(x)|) - f\left( \left| \frac{u(x+hz) - u(x)}{hz} \right| \right) dx \right] dz.$$

Recalling (3), we notice that the quantity between square brackets is equal to $E_{hz}(u')$, therefore the conclusion follows by a straightforward adaptation of the proof of Theorem 1 (see also the proof of Proposition 3). □

Corollary 1 concludes the analysis when $d = 1$, so we may henceforth assume that $d \geq 2$. Our aim is proving that the restrictions to $X$ of the functionals $\mathcal{E}_h$ $\Gamma$-converge w.r.t. the $H^1(\mathbb{R}^d)$-topology to $\mathcal{E}_0$. The gist of our proof is a slicing procedure, which amounts to express the $d$-dimensional energies $\mathcal{E}_h$ as superpositions of the 1-dimensional energies $E_h$, regarded as functionals on each line of $\mathbb{R}^d$.

**Lemma 3** (Slicing)**.** *For $u \in X$, $z \in \mathbb{R}^d \setminus \{ 0 \}$, and $\xi \in \hat{z}^\perp$, we define $w_{\hat{z},\xi} \colon \mathbb{R} \to \mathbb{R}$ as $w_{\hat{z},\xi}(t) \coloneqq u(\xi + t\hat{z})$. Then, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z}$ and*

$$\mathcal{E}_h(u) = \int_{\mathbb{R}^d} \int_{z^\perp} K(z) \left|z\right|^2 E_{h|z|}(w'_{\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz, \tag{31}$$

*where $E_{h|z|}$ is as in* (3) *(note that the function $f$ in* (3) *must be replaced here by $f(|t|)$).*

*Proof.* The formula (31) is an easy consequence of Fubini's Theorem. Indeed, once the direction $\hat{z} \in \mathbb{S}^{d-1}$ is fixed, we can write $x \in \mathbb{R}^d$ as $x = \xi + t\hat{z}$ for some $\xi \in \mathbb{R}^d$ such that $\xi \cdot z = 0$ and $t \in \mathbb{R}$. Using this decomposition, we have

$$\begin{aligned}\mathcal{F}_h(u) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f\left(\frac{|u(x+hz) - u(x)|}{h\,|z|}\right) dz dx \\ &= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} K(z) f\left(\frac{|w_{\hat{z},\xi}(t + h\,|z|) - w_{\hat{z},\xi}(t)|}{h\,|z|}\right) dt d\mathcal{H}^{d-1}(\xi) dz,\end{aligned}$$

whence

$$\mathcal{E}_h(u) = \frac{1}{h^2} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} K(z) \left[ f\left(\left|w'_{\hat{z},\xi}(t)\right|\right) - f\left(\frac{|w_{\hat{z},\xi}(t+h|z|) - w_{\hat{z},\xi}(t)|}{h\,|z|}\right)\right] dt d\mathcal{H}^{d-1}(\xi) dz.$$

To obtain (31), it now suffices to multiply and divide the integrands by $\left|z\right|^2$. □

The connection with the 1-dimensional case provided by Lemma 3 suggests that the $\Gamma$-convergence of the functionals $E_h$ might be exploited to prove Theorem 2. Though, to be able to apply the results of Section 2, we need the functions $w_{\hat{z},\xi}$ in (31) to admit a second order weak derivative for a.e. $z$ and $\xi$. This poses no real problem for the proof of the upper limit inequality, because we may reason on regular functions; as for the lower limit one, we shall tackle the difficulty in the next subsection by means of a compactness criterion, see Lemma 6 below.

We now establish the following:

**Proposition 3.** *Let $u \in X \cap H^2(\mathbb{R}^d)$. Then:*

1. *For any family $\{ u_h \} \subset X$ that converges to $u$ in $H^1(\mathbb{R}^d)$, there holds*
$$\mathcal{E}_0(u) \leq \liminf_{h \searrow 0} \mathcal{E}_h(u).$$
2. *If $u \in X \cap C^3(\mathbb{R}^d)$, then*
$$\mathcal{E}_0(u) = \lim_{h \searrow 0} \mathcal{E}_h(u).$$

*Proof.* We prove both the assertions by using the slicing formula (31).

1. For all $h > 0$, $z \in \mathbb{R}^d \setminus \{ 0 \}$, and $\xi \in \hat{z}^\perp$, we let $w_{h;\hat{z},\xi} \colon \mathbb{R} \to \mathbb{R}$ be defined as $w_{h;\hat{z},\xi}(t) \coloneqq u_h(\xi + t\hat{z})$. Then,
$$\mathcal{E}_h(u_h) = \int_{\mathbb{R}^d} \int_{z^\perp} K(z) \left|z\right|^2 E_{h|z|}(w'_{h;\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz,$$
and, by Fatou's Lemma,
$$\liminf_{h \searrow 0} \mathcal{E}_h(u_h) \geq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) \left|z\right|^2 \left[\liminf_{h \searrow 0} E_{h|z|}(w'_{h;\hat{z},\xi})\right] d\mathcal{H}^{d-1}(\xi) dz. \tag{32}$$

Let us set $w_{\hat z,\xi}(t) := u(\xi+t\hat z)$ and note that if $\rho\colon \mathbb{R}^d \to [0,+\infty)$ is any kernel such that $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$, we may write

$$\int_{\mathbb{R}^d} |\nabla u_h - \nabla u|^2 = \int_{\mathbb{R}^d} \rho(z) \int_{\hat z^\perp} \int_{\mathbb{R}} \left| w'_{h;\hat z,\xi}(t) - w'_{\hat z,\xi}(t) \right|^2 dt d\mathcal{H}^{d-1}(\xi) dz.$$

Since the left-hand side vanishes as $h \searrow 0$, it follows that there exists a subsequence of $\{ w'_{h;\hat z,\xi} \}$, which we do not relabel, that converges in $L^2(\mathbb{R})$ to $w'_{\hat z,\xi}$ for $\mathcal{L}^d$-a.e. $z \in \mathbb{R}^d$ and $\mathcal{H}^{d-1}$-a.e. $\xi \in \hat z^\perp$. In particular, by assumption, $w'_{\hat z,\xi} \in H^1(\mathbb{R})$ for a.e. $(z,\xi)$ and it equals 0 on the complement of some open interval $I_{\hat z,\xi}$.

From the previous considerations, we see that Proposition 2 can be applied on the right-hand side of (32), yielding

$$\liminf_{h\searrow 0} \mathcal{E}_h(u_h) \geq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 E_0(w'_{\hat z,\xi}) d\mathcal{H}^{d-1}(\xi) dz = \mathcal{E}_0(u).$$

(2) As above, for any fixed $z \in \mathbb{R}^d \setminus \{ 0 \}$ and $\xi \in \hat z^\perp$, we define the function $w_{\hat z,\xi} \in C^3(\mathbb{R})$ setting $w(t) := u(\xi + t\hat z)$. Since $\Omega$ is bounded, there exists $r > 0$ such that, for any choice of $z$, $w'_{\hat z,\xi}(t) = \nabla u(\xi + t\hat z) \cdot \hat z = 0$ whenever $\xi \in z^\perp$ satisfies $|\xi| \geq r$, while $w'_{\hat z,\xi}(t)$ is supported in an open interval $I_{\hat z,\xi}$ if $|\xi| < r$.

By virtue of the slicing formula (31), we obtain

$$\begin{aligned} |\mathcal{E}_h(u) - \mathcal{E}_0(u)| &\leq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 \left| E_{h|z|}(w'_{\hat z,\xi}) - E_0(w'_{\hat z,\xi}) \right| d\mathcal{H}^{d-1}(\xi) dz \\ &\leq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 \left| E_{h|z|}(w'_{\hat z,\xi}) - E_0(w'_{\hat z,\xi}) \right| d\mathcal{H}^{d-1}(\xi) dz \end{aligned}$$

Proposition 1 gets the existence of a constant $c > 0$ and of a continuous, bounded, and increasing function $m\colon [0,+\infty) \to [0,+\infty)$ such that $m(0) = 0$ and

$$|\mathcal{E}_h(u) - \mathcal{E}_0(u)| \leq c \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 m(h|z|) d\mathcal{H}^{d-1}(\xi) dz.$$

Note here that $m$ can be chosen depending only on $\nabla u$, and not on $\hat z$ and $\xi$.

Recalling (26), to achieve the conclusion it now suffices to appeal to Lebesgue's Convergence Theorem.

□

**Remark 3.** As observed in Remark 1, from Proposition 3 it follows that the Γ-limit of the rate functionals

$$h\mathcal{E}_h(u) = \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h}$$

is equal to zero (even with respect to the $L^2(\mathbb{R}^d)$-topology on $X$).

3.2. **Lower bound and compactness.** Similarly to the 1-dimensional case, we shall prove the compactness of functions with equibounded energy by first establishing a lower bound on the functionals $\mathcal{E}_h$. More precisely, Lemma 4 below shows that, when $f$ is strongly convex, $\mathcal{E}_h(u)$ is greater than a double integral which takes into account, for each $z \in \mathbb{R}^d \setminus \{ 0 \}$, the squared projection of the difference quotients of $\nabla u$ in the direction of $z$. Thanks to the slicing formula, the inequality follows with no effort by applying Lemma 1 on each line of $\mathbb{R}^d$.

We point out that our approach results in the appearance of an effective kernel $\tilde K$ in front of the difference quotients. This function stands as a multidimensional counterpart of the kernel $J$ in Lemma 1; actually, $\tilde K$ depends both on $K$ and on $J$ (see (33) for the precise definition). In Lemma 5, we shall collect some properties of the effective kernel that will be useful in the proof of Lemma 6.

**Lemma 4** (Lower bound on the energy)**.** *Let* $\Omega$*,* $K$*, and* $f$ *be as above, and suppose that*

$$\tilde{K}(z) \coloneqq \int_{-1}^{1} J(r) K_{|r|}(z) dr \qquad \text{for a.e. } z \in \mathbb{R}^d, \tag{33}$$

*with* $J$ *as in Lemma 1. Then, it holds*

$$\mathcal{E}_h(u) \geq \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[ \frac{\left(\nabla u(x+hz) - \nabla u(x)\right) \cdot \hat{z}}{h} \right]^2 dxdz. \tag{34}$$

*Proof.* Thanks to Lemma 3, we can reduce to the 1-dimensional case, and we take advantage of the lower bound provided by Lemma 1. Keeping the notation of Lemma 3, we find

$$\begin{aligned} \mathcal{E}_h(u) \geq & \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \, |z|^2 \left( \frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h\,|z|} \right)^2 drdtd\mathcal{H}^{d-1}(\xi)dz \\ = & \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \left( \frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 drdtd\mathcal{H}^{d-1}(\xi)dz. \end{aligned}$$

To cast this bound in the form of (34), we change variables and use Fubini's Theorem:

$$\begin{aligned} I \coloneqq & \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \left( \frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 drdtd\mathcal{H}^{d-1}(\xi)dz \\ & = \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-1}^{1} J(r) K(z) \left( \frac{w'_{\hat{z},\xi}(t+h\,|z|\,r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 drdtd\mathcal{H}^{d-1}(\xi)dz \\ & = \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left( \frac{w'_{\hat{z},\xi}(t-h\,|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dtd\mathcal{H}^{d-1}(\xi)dzdr \\ & \qquad + \int_{0}^{1} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{r}(z) \left( \frac{w'_{\hat{z},\xi}(t+h\,|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dtd\mathcal{H}^{d-1}(\xi)dzdr \end{aligned}$$

Note that

$$\begin{aligned} & \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left( \frac{w'_{\hat{z},\xi}\big(t+h\,|z|\,\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dtd\mathcal{H}^{d-1}(\xi)dzdr \\ = & \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left( \frac{w'_{-\hat{z},\xi}\big(-(t+h\,|z|)\big) - w'_{-\hat{z},\xi}(-t)}{h} \right)^2 dtd\mathcal{H}^{d-1}(\xi)dzdr \\ & = \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left( \frac{w'_{\hat{z},\xi}\big(t+h\,|z|\,\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dtd\mathcal{H}^{d-1}(\xi)dzdr, \end{aligned}$$

because $w'_{-\hat{z},\xi}(-s) = -w'_{\hat{z},\xi}(s)$ for all $s \in \mathbb{R}$. Thus, we conclude that

$$\begin{aligned} I = & \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \left( \int_{-1}^{1} J(r) K_{|r|}(z) dr \right) \left( \frac{w'_{\hat{z},\xi}\big(t+h\,|z|\,\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dtd\mathcal{H}^{d-1}(\xi)dz \\ & = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[ \frac{\big(\nabla u(x+hz) - \nabla u(x)\big) \cdot \hat{z}}{h} \right]^2 dxdz, \end{aligned}$$

which concludes the proof. □

Let us remind that, by assumption, the kernel $K$ is bounded away from 0 in a suitable annulus. The next lemma shows that the effective kernel appearing $\tilde{K}$ in (34) inherits a similar property.

**Lemma 5.** *Let $\tilde{K}\colon \mathbb{R}^d \to [0,+\infty)$ be as in* (33)*. Then,*

$$\int_{\mathbb{R}^d} \tilde{K}(z)\left(1+|z|^2\right)dz < +\infty. \tag{35}$$

*Moreover, if $\sigma_d$ and $r_1$ are the constants in* (27) *and* (28)*, then,*

$$\operatorname{ess\,inf}\left\{\tilde{K}(z) : z \in B(0,\sigma_d r_1)\right\} > 0. \tag{36}$$

*Proof.* The convergence of the integral in (35) follows easily from (26). Indeed, by the definition of $\tilde{K}$, we see that

$$\int_{\mathbb{R}^d} \tilde{K}(z)dz = \int_{-1}^{1}\int_{\mathbb{R}^d} J(r)K_{|r|}(z)dzdr = \int_{\mathbb{R}^d} K(z)dz;$$

analogously, one finds that

$$\int_{\mathbb{R}^d} \tilde{K}(z)\,|z|^2\,dz = c\int_{\mathbb{R}^d} K(z)\,|z|^2\,dz,$$

for some $c>0$.

For what concerns (36), let us set $k \coloneqq \operatorname{ess\,inf}\{\, K(z) : z \in B(0,r_1)\setminus B(0,r_0)\,\}$. In view of (28), $k>0$.

We distinguish between the case $z \in B(0,r_0)$ and the case $z \in B(0,r_1)\setminus B(0,r_0)$. In the first situation, for a.e. $z \in \mathbb{R}^d$,

$$\begin{aligned}\tilde{K}(z) \geq& 2\int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} J(r)K_r(z)dr \geq 2k\int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} \frac{1}{r^d}J(r)dr\\ =&\frac{2k}{|z|^{d-1}}\int_{r_0}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds.\end{aligned}$$

When $z \in B(0,r_1)\setminus B(0,r_0)$, instead, similar computations get

$$\tilde{K}(z) \geq 2\int_{\frac{|z|}{r_1}}^{1} J(r)K_r(z)dr = \frac{2k}{|z|^{d-1}}\int_{|z|}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds \qquad \text{for a.e. } z\in\mathbb{R}^d,$$

so that we obtain

$$\tilde{K}(z) \geq \frac{2k}{|z|^{d-1}}\int_{\max(r_0,|z|)}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds \qquad \text{for a.e. } z\in\mathbb{R}^d. \tag{37}$$

When $d=2$, the estimate above becomes

$$\tilde{K}(z) \geq 2k\left[\frac{r_1-\max(r_0,|z|)}{|z|} - \log\left(\frac{r_1}{\max(r_0,|z|)}\right)\right] \qquad \text{for a.e. } z\in\mathbb{R}^d.$$

Exploiting the concavity of the logarithm, we see that the lower bound that we have obtained is strictly positive if $|z| < r_1 = \sigma_2 r_1$.

On the other hand, putting $M \coloneqq \max(r_0,|z|)$ for shortness, if $d\geq 3$, the right-hand side in (37) equals

$$\frac{2k}{(d-1)(d-2)\,|z|^{d-1}}\left[(d-2)\left(r_1^{d-1}-M^{d-1}\right)-(d-1)\,|z|\left(r_1^{d-2}-M^{d-2}\right)\right],$$

and therefore

$$\begin{aligned}\tilde{K}(z) \geq& \frac{2kM^{d-2}}{(d-1)(d-2)\,|z|^{d-1}}\\ &\cdot\left\{\left(\frac{r_1}{M}\right)^{d-2}\left[(d-2)r_1-(d-1)\,|z|\right]-\left[(d-2)M-(d-1)\,|z|\right]\right\}\end{aligned}$$

for a.e. $z \in \mathbb{R}^d$. When $|z| < \frac{d-2}{d-1} r_1 = \sigma_d r_1$, the quantity between braces is strictly positive if

$$\frac{(M-|z|)d-(2M-|z|)}{(r_1-|z|)d-(2r_1-|z|)} < \Big(\frac{r_1}{M}\Big)^{d-2}.$$

Observe that both the left-hand side and the right-hand one are strictly increasing in $d$; also, the left-hand side is bounded above by $(M-|z|)/(r_1-|z|)$, so the last inequality holds if

$$\frac{M-|z|}{r_1-|z|} < \frac{r_1}{M},$$

which, in turn, is true for all $z \in B(0, r_1)$. □

We are now in the position to prove that families with equibounded energy are compact in $H^1(\mathbb{R}^d)$, and that their accumulation points admit second order weak derivatives.

**Lemma 6** (Compactness)**.** *Assume that $\Omega$, $K$, and $f$ are as above. Then, if $\{u_h\} \subset X$ satisfies $\mathcal{E}_h(u_h) \leq M$ for some $M \geq 0$, there exist a subsequence $\{u_{h_\ell}\}$ and a function $u \in X \cap H^2(\mathbb{R}^d)$ such that $u_{h_\ell} \to u$ in $H^1(\mathbb{R}^d)$.*

*Proof.* Let $\tilde{k} := \operatorname{ess\,inf}\{\tilde{K}(z) : z \in B(0, \sigma_d r_1)\}$; Lemma 5 ensures that $\tilde{k} > 0$. We consider a function $\rho \in C_c^\infty([0,+\infty))$ such that

$$\rho(r) = 0 \qquad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty\right),$$

and we further require that

$$0 \leq \rho(r) \leq \tilde{k} \quad \text{and} \quad |\rho'(r)| \leq \tilde{k}.$$

For $h > 0$ and $y \in \mathbb{R}^d$, we set

$$\rho_h(y) := \frac{1}{ch^d}\rho\left(\frac{|y|}{h}\right), \qquad \text{with } c := \int_{\mathbb{R}^d} \rho(|y|)dy,$$

and we introduce the functions $v_h := \rho_h * u_h$, as above.

Each function $v_h$ is a smooth function and, for all $\tilde{h} \in (0,1)$, its support is contained in

$$\Omega_{\tilde{h}} := \{x : \operatorname{dist}(x,\Omega) \leq 2^{-1/2}\tilde{h}\sigma_d r_1\}$$

if $h \in (0,\tilde{h})$. In particular, we can choose $\tilde{h}$ so small that $\partial\Omega_{\tilde{h}}$ is still Lipschitz. For such an $\tilde{h}$, we assert that the family $\{v_h\}_{h\in(0,\tilde{h})}$ is relatively compact in $H^1_0(\Omega_{\tilde{h}})$. In order to prove this, we first remark that

$$\int_{\Omega_{\tilde{h}}} \left|\nabla^2 v_h\right|^2 = \int_{\Omega_{\tilde{h}}} |\Delta v_h|^2, \tag{38}$$

and next we show that the right-hand side is uniformly bounded.

We observe that $\int_{\mathbb{R}^d} \nabla\rho_h(y)dy = 0$ for all $h > 0$, because $\rho$ is compactly supported. Hence,

$$\begin{aligned}
\|\Delta v_h\|^2_{L^2(\Omega_{\tilde{h}})} &= \int_{\mathbb{R}^d} |\Delta v_h|^2 \\
&= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} \nabla\rho_h(y)\cdot\big(\nabla u_h(x+y)-\nabla u_h(x)\big)dy\right|^2 dx \\
&\leq \int_{\mathbb{R}^d}\left[\frac{1}{ch^{d+1}}\int_{\mathbb{R}^d}\left|\rho'\left(\frac{|y|}{h}\right)\right|\left|\big(\nabla u_h(x+y)-\nabla u_h(x)\big)\cdot\hat{y}\right|dy\right]^2 dx.
\end{aligned}$$

By our choice of $\rho$ and (36), we find

$$\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} \leq \int_{\mathbb{R}^d} \left[\frac{1}{ch}\int_{\mathbb{R}^d} \tilde K_h(y)\left|\big(\nabla u_h(x+y)-\nabla u_h(x)\big)\cdot \hat y\right| dy\right]^2 dx$$
$$\leq \int_{\mathbb{R}^d} \left[\frac{1}{ch}\int_{\mathbb{R}^d} \tilde K(z)\left|\big(\nabla u_h(x+hz)-\nabla u_h(x)\big)\cdot \hat z\right| dz\right]^2 dx$$

Further, since $\tilde K \in L^1(\mathbb{R}^d)$, Jensen's inequality and Fubini's Theorem yield

$$\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} \leq \frac{\|\tilde K\|_{L^1(\mathbb{R}^d)}}{c^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde K(z)\left[\frac{\big(\nabla u_h(x+hz)-\nabla u_h(x)\big)\cdot\hat z}{h}\right]^2 dxdz.$$

The lower bound (34) entails

$$\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} \leq \frac{4}{c^2\gamma}\|\tilde K\|_{L^1(\mathbb{R}^d)}\mathcal{E}_h(u_h),$$

so that, in view of the assumption $\mathcal{E}_h(u_h)\leq M$ and of (38), we get

$$\|\nabla^2 v_h\|^2_{L^2(\Omega_{\tilde h})} \leq \frac{4M}{c^2\gamma}\|\tilde K\|_{L^1(\mathbb{R}^d)}. \tag{39}$$

We argue as in the proof of Lemma 2. We recall that, for $h\in(0,\tilde h)$, each $v_h$ vanishes on the complement of $\Omega_{\tilde h}$, and thus, by Poincaré inequality, (39) implies a uniform bound on the norms $\|v_h\|_{H^2_0(\Omega_{\tilde h})}$. As a consequence, by Rellich-Kondrachov Theorem, the family $\{\tilde v_h\}_{h\in(0,\tilde h)}$ of the restrictions of the functions $v_h$ to $\Omega_{\tilde h}$ admits a subsequence $\{\tilde v_{h_\ell}\}$ that converges in $H^1_0(\Omega_{\tilde h})$ to a function $\tilde u\in H^2_0(\Omega_{\tilde h})$. Actually, the support of $\tilde u$ is contained in $\bar\Omega$, and, if we put,

$$u(x) := \begin{cases} \tilde u(x) & \text{if } x\in\bar\Omega,\\ 0 & \text{otherwise,}\end{cases}$$

we infer that $\{v_{h_\ell}\}$ converges in $H^1(\mathbb{R}^d)$ to $u\in X\cap H^2(\mathbb{R}^d)$.

To accomplish the proof, it suffices to show that the $L^2$ distance between $\nabla u_h$ and $\nabla v_h$ vanishes when $h\searrow 0$. Since $\rho_h$ has unit $L^1(\mathbb{R}^d)$-norm and it is radial, we have

$$\int_{\mathbb{R}^d}|\nabla v_h(x)-\nabla u_h(x)|^2dx = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\rho_h(y)\big(\nabla u_h(x+y)-\nabla u_h(x)\big)dy\right|^2 dx$$
$$\leq \frac14\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\rho_h(y)\big(\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x)\big)dy\right|^2 dx$$
$$\leq \frac14\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_h(y)\left|\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x)\right|^2 dydx.$$

We remark that for any fixed $y\in\mathbb{R}^d\setminus\{0\}$ and for all $p\in\mathbb{R}^d$, the identity $|p|^2=|p\cdot y|^2+|(\mathrm{Id}-y\otimes y)p|^2$ can be reformulated as

$$\begin{aligned}|p|^2 &= |p\cdot y|^2 + \int_{\hat y^\perp}\pi(|\eta|)\,|p\cdot\eta|^2\,d\mathcal{H}^{d-1}(\eta)\\ &= |p\cdot y|^2+\frac{1}{h^2}\int_{\hat y^\perp}\pi_h(\eta)\,|p\cdot\eta|^2\,d\mathcal{H}^{d-1}(\eta),\end{aligned} \tag{40}$$

where $\pi\colon[0,+\infty)\to[0,+\infty)$ is a continuous function such that

$$\int_{e_d^\perp}\pi(|\eta|)\,|\eta|^2\,d\mathcal{H}^{d-1}(\eta)=1,$$

and $\pi_h(\eta) := h^{-d+1}\pi(|\eta|/h)$. We further prescribe that

$$\pi(r) = 0 \qquad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty\right)$$

and that $\lim_{r\searrow 0} \eta(r)/r \in \mathbb{R}$.

We apply the formula (40) to $p_h(x,y) := \nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x)$ and we find that

$$\int_{\mathbb{R}^d} |\nabla v_h(x) - \nabla u_h(x)|^2 \, dx \leq \frac{1}{4}\left(I_1 + I_2\right), \tag{41}$$

where

$$I_1 := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_h(y)\,|y|^2\,|p_h(x,y)\cdot\hat{y}|^2\,dydx,$$
$$I_2 := \frac{1}{h^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho_h(y)\pi_h(\eta)\,|p_h(x,y)\cdot\eta|^2\,d\mathcal{H}^{d-1}(\eta)dydx.$$

We first consider $I_1$. Keeping in mind that $\rho$ is compactly supported and $\rho(|y|) \leq \tilde{k} \leq \tilde{K}(y)$ for a.e. $y \in B(0, 2^{-1/2}\sigma_d r_1)$, we get

$$I_1 \leq \frac{(\sigma_d r_1)^2}{c}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{K}_h(y)\left|\big(\nabla u_h(x+y) - \nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx \right.$$
$$\left. + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{K}_h(y)\left|\big(\nabla u_h(x-y) - \nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx\right],$$

and, by (34),

$$I_1 \leq \frac{8(\sigma_d r_1)^2 M}{c\gamma} h^2. \tag{42}$$

As for $I_2$, we assert that there exist a constant $L > 0$, depending on $d$, $\sigma_d$, $r_1$, $\tilde{k}$, and $c$, such that

$$I_1 \leq \frac{LM}{\gamma} h^2. \tag{43}$$

To prove the claim, we write the integrand appearing in $I_2$ as follows:

$$\begin{aligned}
p_h(x,y)\cdot\eta =& \big(\nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x)\big)\cdot\eta \\
=& \big(\nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x-\eta)\big)\cdot\eta \\
&+ 2\big(\nabla u_h(x-\eta) - \nabla u_h(x)\big)\cdot\eta \\
=& \big(\nabla u_h(x+y) - \nabla u_h(x-\eta)\big)\cdot(\eta+y) \\
&+ \big(\nabla u_h(x-y) - \nabla u_h(x-\eta)\big)\cdot(\eta-y) \\
&- \big(\nabla u_h(x+y) - \nabla u_h(x-y)\big)\cdot y + 2\big(\nabla u_h(x-\eta) - \nabla u_h(x)\big)\cdot\eta.
\end{aligned}$$

We plug this identity in the definition of $I_2$ and we find that

$$\begin{aligned}
I_2 \leq& \frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy) - \nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\
&+ \frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy) - \nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\
&+ \frac{8}{c}\|\pi\|_{L^1(e_d^\perp)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(|y|)\,|y|^2\left|\big(\nabla u_h(x+hy) - \nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx \\
&+ \frac{16}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta) - \nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx.
\end{aligned}$$

We estimate separately each of the contributions on the right-hand side.

Let us set $\mathbb{S}^{d-1}_+ \coloneqq \{\, e \in \mathbb{S}^{d-1} : e \cdot e_d > 0 \,\}$ and $\mathbb{S}^{d-1}_- \coloneqq \{\, e \in \mathbb{S}^{d-1} : e \cdot e_d < 0 \,\}$. Hereafter, we denote by $L$ any strictly positive constant depending only on $d$, $\sigma_d$, $r_1$, and on the norms of $\rho$ and $\pi$.

Taking advantage of the Coarea Formula, we rewrite the first addendum as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&\qquad\qquad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h(\eta+y))-\nabla u_h(x)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dxdy\\
&= \int_{\mathbb{S}^{d-1}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{e^\perp} r^{d-1}\rho(r)\pi(|\eta|)\left|\big(\nabla u_h(x+h(\eta+re))-\nabla u_h(x)\big)\cdot(\eta+re)\right|^2 d\mathcal{H}^{d-1}(\eta)drdxd\mathcal{H}^{d-1}(e)\\
&= \int_{\mathbb{S}^{d-1}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^2\,|y\cdot e|^{d-1}\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&= \int_{\mathbb{S}^{d-1}_-}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^2\,|y\cdot e|^{d-1}\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e),
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\
&= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y\cdot e|^{d-1}\,|y|^2\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e).
\end{aligned}$$

Let us recall that $\rho(r)=\eta(r)=0$ if $r\notin[0,2^{-1/2}\sigma_d r_1)$, whence, for any $e\in\mathbb{S}^{d-1}$, the product $\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)$ vanishes outside the cylinder

$$C_e \coloneqq \{\, y\in\mathbb{R}^d : |y\cdot e|\,, |(\mathrm{Id}-e\otimes e)y| \in [0,2^{-1/2}\sigma_d r_1)\,\} \subset B(0,\sigma_d r_1).$$

We therefore see that the last multiple integral equals

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{C_e} |y\cdot e|^{d-1}\,|y|^2\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e)\\
&\qquad\qquad \le L\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{C_e} \tilde{K}(y)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le \frac{LM}{\gamma}h^2.
\end{aligned}$$

We then obtain

$$\begin{aligned}(44)\quad &\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\ &+\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\ &\leq \frac{LM}{\gamma}h^2.\end{aligned}$$

Next, we have

$$(45)\qquad \frac{8}{c}\|\pi\|_{L^1(e_d^\perp)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(|y|)\,|y|^2\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx\leq\frac{LM}{\gamma}h^2,$$

$$(46)\qquad \frac{16}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\leq\frac{LM}{\gamma}h^2.$$

The bound in (45) may be deduced as the one in (42), so, to establish (43), we are only left to prove (46). To this aim, let $\psi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ be a test function. By a standard argument and Fubini's Theorem we have that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\psi(y,\eta)d\mathcal{H}^{d-1}(\eta)dy\\ &\quad=\lim_{\varepsilon\searrow0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|y|}{2\varepsilon}\chi_{\{\,t<\varepsilon\,\}}(|\eta\cdot y|)\rho(|y|)\pi(|\eta|)\psi(y,\eta)d\eta dy\\ &\quad=\lim_{\varepsilon\searrow0}\int_{\mathbb{R}^d}\frac{\pi(|\eta|)}{|\eta|}\left(\int_{\mathbb{R}^d}\frac{|\eta|}{2\varepsilon}\chi_{\{\,t<\varepsilon\,\}}(|\eta\cdot y|)\rho(|y|)\,|y|\,\psi(y,\eta)dy\right)d\eta\\ &\quad=\int_{\mathbb{R}^d}\int_{\hat{\eta}^\perp}\frac{\pi(|\eta|)}{|\eta|}\rho(|y|)\,|y|\,\psi(y,\eta)d\mathcal{H}^{d-1}(y)d\eta.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\ &\quad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{\eta}^\perp}\frac{\pi(|\eta|)}{|\eta|}\rho(|y|)\,|y|\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(y)d\eta dx\\ &\quad\leq L\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{K}(\eta)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\eta dx\end{aligned}$$

(recall that we assume $\lim_{r\searrow0}\eta(r)/r$ to be finite). In view of the bound on the energy, we retrieve (46).

In conclusion, from (41), (42), and (43), we obtain

$$\int_{\mathbb{R}^d}|\nabla v_h(x)-\nabla u_h(x)|^2\,dx\leq\frac{LM}{\gamma}h^2,$$

as desired. This concludes the proof. □

**Remark 4.** By choosing $u_h=u$ in Lemma 6 we obtain a criterion for an $H^1$-function to belong to $H^2(\mathbb{R}^d)$, namely a function $u\in X$ is in $H^2(\mathbb{R}^d)$ if and only if $\mathcal{E}_h(u)\leq M$ for some $M>0$ and for all $h$ small enough.

We can now conclude the proof of Theorem 2.

*Proof of Theorem 2.* The compactness result in statement (1) of Theorem 2 is contained in Lemma 6.

The upper limit inequality (3) follows by Proposition 3 and a standard density argument, as in the 1-dimensional case.

As for the lower limit inequality (2), for any $u \in X$ and for any family $\{ u_h \} \subset X$ that converges to $u$ in $H^1(\mathbb{R}^d)$, we may focus on the situation when there exists $M \geq 0$ such that $|\mathcal{E}_h(u_h)| \leq M$ for all $h > 0$. By Lemma 6, we have that $u \in H^2(\mathbb{R}^d)$. Then, (2) follows by Proposition 3. □

**Remark 5.** We conclude by noticing that the Γ-convergence result in Theorem 2, statements (2) and (3), still holds if we replace $X$ with $H^1(\mathbb{R}^d)$, with essentially the same proof. On the other hand, being $\mathbb{R}^d$ non-compact, the compactness result in statement (1) of Theorem 2 does not hold in this case.

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