# Approximation of the anisotropic mean curvature flow

**A. Chambolle** *, **M. Novaga** †

**Abstract**

In this note, we provide simple proofs of consistency for two well known algorithms for mean curvature motion, Almgren-Taylor-Wang's [1] variational approach, and Merriman-Bence-Osher's algorithm [24]. Our techniques, based on the same notion of strict sub- and superflows, also work in the (smooth) anisotropic case.

## 1 Introduction

The Mean curvature flow refers to the motion of a hypersurface $\Gamma(t) \subset \mathbb{R}^N$ whose normal velocity, at each point, is equal to (minus) its mean curvature. We will consider only compact hypersurfaces $\Gamma(t)$, that are the boundary of some evolving set $E(t)$ (bounded or unbounded). In this case, the motion is also known as the "area-diminishing" flow, and is in some sense the gradient flow of the perimeter of $E(t)$. It is well-known that this motion can be characterized in terms of the distance function to $\Gamma = \partial E$ [18, 2]. More precisely, if we define $d(x,t)$ as

$$d(x,t) \ := \ \mathrm{dist}(x, E(t)) - \mathrm{dist}(x, \mathbb{R}^N \setminus E(t))$$

(the signed distance function to $\partial E(t)$), then the exterior normal to $E$ is given by $\nabla d$ whereas the curvature is $\Delta d$. On the other hand, the normal velocity of a point of the boundary is given, at each time, by $-\partial d/\partial t$, so that the evolution is characterized by

$$\frac{\partial d}{\partial t}(x,t) \ = \ \Delta d(x,t) \qquad (1)$$

at any $x \in \partial E(t)$ (*i.e.*, $(x,t)$ such that $d(x,t) = 0$).

It is well known that the Mean curvature flow enjoys a comparison principle: if $E$, $F$ are two (smooth) evolutions such that $E(t) \subseteq F(t)$ at some time $t$, then $E(s) \subseteq F(s)$ at any subsequent time $s > t$ as long as the flows are defined. This key property allows to define a generalized flow for nonsmooth surfaces, by comparison with smooth flows: basically, a generalized flow will be a flow such that any smooth

*CMAP, CNRS UMR 7641, Ecole Polytechnique, 91128, Palaiseau Cedex, France,
e-mail: antonin.chambolle@polytechnique.fr

†Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy,
e-mail: novaga@dm.unipi.it

flow starting inside remains inside while any smooth flow starting outside remains outside. The formal theory that provides such a generalization is known as the barrier theory and is initially due to De Giorgi [16, 8, 5]. The theory of viscosity solutions (which is also based on the comparison principle) defines the generalized flow as the zero sub- or superlevel set of a function $u$ that solves an appropriate degenerate parabolic equation, and yields the same generalized flows as the barrier theory [6]. The generalized flow starting from a set $E$ is usually unique, except when the "fattening" phenomenon occurs, which corresponds to the fattening of the level line $\{u = 0\}$ of the corresponding viscosity solution.

It is shown in [5] that a barrier solution can be characterized by comparison with appropriate sub- and superflow: in this case, a generalized flow will be characterized by the property that any smooth flow starting inside and evolving (strictly) *faster* than the Mean curvature flow remains inside, while a smooth flow starting outside and evolving (strictly) *slower* than the Mean curvature flow remains outside. The definition of a strict super- or subflow of (1) is the following: $E(t)$ will be a strict superflow (on a small time interval $[t_0, t_1]$) iff its signed distance function satisfies

$$\frac{\partial d}{\partial t}(x,t) \ > \ \Delta d(x,t) \tag{2}$$

in a neighborhood of $\{d = 0\}$, while a strict subflow is defined with the reverse inequality.

We show in this note that such a definition (which will be slightly adapted to cover non-isotropic cases) makes very easy the proof of convergence for two well-known approximation schemes for the Mean curvature flow, namely, the Almgren-Taylor-Wang [1] approach and the Merriman-Bence-Osher [24] approach. In both schemes, a time step $h > 0$ is fixed and a discrete-in-time evolution is defined, by providing a simple evolution operator $E \mapsto T_h E$ that approximates the evolution of a initial set $E$ over a time interval of duration $h$. Given $E_0$, the discrete evolution $E_h(t)$ is simply $T_h^{[t/h]}(E_0)$ where $[\cdot]$ denotes the integer part. One then wants to know whether $E_h(t) \to E(t)$ as $h \to 0$, where $E(t)$ is the generalized evolution starting from $E_0$. The key to prove this convergence are the two properties of *monotonicity* and *consistency*. The operator $T_h$ will be monotone if given any $E$, $F$ with $E \subseteq F$, one has $T_h E \subseteq T_h F$. The notion of consistency we will use is based on our notion of strict super- and subflow: $T_h$ will be consistent if, given any superflow $E$ on $[t_0, t_1]$ and given $h > 0$ small enough, one has $E(t + h) \subseteq T_h E(t)$ for any $t \in [t_0, t_1 - h]$, while given any subflow, the same holds with the reverse inclusion. It follows from the theory of barriers that if $T_h$ is monotone and consistent in the above-defined sense, then $\partial E_h(t)$ converges to $\partial E(t)$ as $h \to 0$ (in the Hausdorff sense), at any time, as long as the generalized flow $\partial E(t)$ is uniquely defined (*i.e.*, no fattening occurs).

In our cases, the set $T_h E(t)$ will be defined as a level set of some function $u$ (depending on $h$ and $E(t)$), satisfying some elliptic or parabolic equation, and it will be quite easy to build from a function $d$ satisfying (2) a sub- or supersolution $v$ of the same equation that will be compared to $u$, yielding a comparison of the level sets.

This note is organized as follows: in Section 2 we introduce the anisotropic curvature flow and we give a rigorous definition of the corresponding super and subflows. Then, in Section 3 we introduce the Merriman-Bence-Osher's scheme and we prove its consistency. In Section 4 we do the same for the Almgren-Taylor-Wang's algorithm. We observe that in this case, a result of consistency with smooth flows is already found in [1], however, its proof is by far more complicated than ours.

## 2 Anisotropic curvature flow

We follow the definitions and notation in [7, 9]. Let us consider $(\phi, \phi^\circ)$ a pair of mutually polar, convex, one-homogeneous functions in $\mathbb{R}^N$ (i.e., $\phi^\circ(\xi) = \sup_{\phi(\eta)\le 1} \xi\cdot\eta$, $\phi(\eta) = \sup_{\phi^\circ(\xi)\le 1} \xi\cdot\eta$, see [25]). These are assumed to be locally finite, and, to simplify, even. The pair $(\phi, \phi^\circ)$ is referred as *the anisotropy* (the isotropic case corresponds to $\phi = \phi^\circ = |\cdot|$). The local finiteness implies that there is a constant $c > 1$ such that

$$c^{-1}|\eta| \le \phi(\eta) \le c|\eta| \ \text{ and } \ c^{-1}|\xi| \le \phi^\circ(\xi) \le c|\xi|$$

for any $\eta$ and $\xi$ in $\mathbb{R}^N$. We refer to [7, 9] for the main properties of $\phi$ and $\phi^\circ$.

Being convex and 1-homogeneous, $\phi^\circ$ (and $\phi$) is also subadditive, so that the function $(x, y) \mapsto \phi(x - y)$ defines a distance, the "$\phi$-distance". For $E \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, we denote by $\mathrm{dist}^\phi(x, E) := \inf_{y\in E} \phi(x - y)$ the $\phi$-distance of $x$ to the set $E$, and by

$$d^\phi_E(x) \ := \ \mathrm{dist}^\phi(x, E) \ - \ \mathrm{dist}^\phi(x, \mathbb{R}^N \setminus E)$$

the signed $\phi$-distance to $\partial E$, negative in the interior of $E$ and positive outside its closure. One easily checks that

$$|d^\phi_E(x) - d^\phi_E(y)| \ \le \ \phi(x - y) \ \le \ c|x - y|$$

for any $x, y \in \mathbb{R}^N$, so that (by Rademacher's theorem) $d^\phi_E$ is differentiable a.e. in $\mathbb{R}^N$. The former inequality shows moreover that $\nabla d^\phi_E(x)\cdot h \le \phi(h)$ for any $h \in \mathbb{R}^N$, if $x$ is a point of differentiability: hence $\phi^\circ(\nabla d^\phi_E(x)) \le 1$. In this note we will always assume that $\phi$ and $\phi^\circ$ are at least in $C^2(\mathbb{R}^N \setminus \{0\})$. In this case, one shows quite easily that $d^\phi_E$ is differentiable at each point $x$ which has a unique $\phi$-projection $y \in \partial E$ (solving $\min_{y\in\partial E} \phi(x - y)$). Then, $\nabla d^\phi_E(x)$ is given by $\nabla\phi((x - y)/d^\phi_E(x))$, so that $\phi^\circ(\nabla d^\phi_E(x)) = 1$. See [7, 9] for details.

The *Cahn-Hoffman* vector field $n_\phi$ is a vector field on $\partial E$ given by $n_\phi(x) = \nabla\phi^\circ(\nu_E(x)) = \nabla\phi^\circ(\nabla d^\phi_E(x))$ a.e. on $\partial E$. Here, $\nu_E$ is the (Euclidean) exterior normal to $\partial E$. If $E$ is smooth enough, then $\nabla d^\phi_E$ does not vanish near $\partial E$ so that one can define $n_\phi(x) = \nabla\phi^\circ(\nabla d^\phi_E(x))$ in a neighborhood of $\partial E$.

Then, we define the *$\phi$-curvature* of $\partial E$ by $\kappa_\phi = \operatorname{div} n_\phi$. The $\phi$-curvature flow is an evolution $E(t)$ such that at each time, the velocity of $\partial E(t)$ is given by

$$V = -\kappa_\phi n_\phi, \tag{3}$$

where $n_\phi$ is the Cahn-Hoffman vector field and $\kappa_\phi$ is the $\phi$-curvature. It is shown that, in some sense, it is the fastest way to diminish the anisotropic perimeter $\int_{\partial E} \phi^\circ(\nu_E)\, d\mathcal{H}^{N-1}$. If $\phi$, $\phi^\circ$ are merely Lipschitz (when, for instance, the *Wulff shape* $\{\phi \le 1\}$ is a convex polytope), then $n_\phi$ can be nonunique and the anisotropy is called *crystalline* [28, 7]. We refer to [14] for a proof of convergence of Merriman-Bence-Osher's scheme in the crystalline case.

The anisotropic variant of (1) is the following characterization of the anisotropic mean curvature flow: letting $d(x,t) = d^\phi_{E(t)}(x)$, the smooth set $E(t)$ evolves by anisotropic curvature if

$$\frac{\partial d}{\partial t}(x,t) = \operatorname{div} \nabla\phi^\circ(\nabla d(x,t)), \tag{4}$$

for any $(x,t)$ with $d(x,t) = 0$. One therefore introduces the following definition of (strict) super- and subflows, which is simplified from [15]:

**Definition 2.1** *Let $E(t) \subset \mathbb{R}^N$, $t \in [t_0, t_1]$. We say that $E(t)$ is a superflow of (4), if there exists a bounded open set $A \subset \mathbb{R}^N$, with $\bigcup_{t_0 \le t \le t_1} \partial E(t) \times \{t\} \subset A \times [t_0, t_1]$, and $\delta > 0$, such that $d(x,t) = d_{E(t)}(x) \in C^1([t_0,t_1]; C^2(A))$, and*

$$\frac{\partial d}{\partial t}(x,t) \ge \operatorname{div} \nabla\phi^\circ(\nabla d)(x,t) + \delta, \tag{5}$$

*for any $x \in A$ and $t \in [t_0, t_1]$. We say that $E(t)$ is a subflow whenever $\delta < 0$ and the reverse inequality holds in (5).*

Considering now a time discrete evolution scheme $E \mapsto T_h E$ ($T_h E$ needs not be defined for all sets $E$, in our applications, it will be sufficient to define it for closed sets with compact boundary), parametrized by the time step $h > 0$, we introduce the following definition of consistency:

**Definition 2.2** *The scheme $T_h$ is consistent if and only if for any superflow $E(t)$, $t_0 \le t \le t_1$, in the sense of Definition 2.1, there exists $h_0$ such that if $h \le h_0$, then $T_h E(t) \supseteq E(t+h)$ for any $t \in [t_0, t_1 - h]$, while for any subflow, the same holds with the reverse inclusion.*

This definition means that given a superflow, it will also go faster than the discretized evolutions, as soon as $h$ is small enough. The following results follows from the theory of barriers, see [5, 6, 8, 15].

**Proposition 2.3** *Assume $T_h$ is a consistent scheme, in the sense of Definition 2.2 above, which is also monotone: for any $E, F \subset \mathbb{R}^N$, $E \subseteq F \Rightarrow T_h E \subseteq T_h F$. Let $E_0 \subset \mathbb{R}^N$ be a closed set with compact boundary such that the generalized anisotropic* *curvature flow $E(t)$ starting from $E_0$ is uniquely defined (no fattening). For any $t \geq 0$ let $E_h(t) := T^{[t/h]}E_0$. Then, for any $t$ as long as $E(t)$ is not empty, $\partial E_h(t) \to \partial E(t)$ in the Hausdorff sense.*

In the next sections, we prove consistency (and monotonicity), first for the (anisotropic) Merriman-Bence-Osher scheme, then for the Almgren-Taylor-Wang scheme, yielding, by Proposition 2.3, convergence to the generalized solution, when unique.

## 3 The Merriman-Bence-Osher algorithm

More than ten years ago, Merriman, Bence and Osher [24] proposed the following algorithm for the computation of the motion by mean curvature of a surface. Given a closed set $E \subset \mathbb{R}^N$, they let $T_h E = \{u(\cdot, h) \geq 1/2\}$, where $u$ solves the heat equation with initial data $u(\cdot, 0) = \chi_E$, the characteristic function of $E$. They then conjectured that $E_h(t) := T_h^{[t/h]}E$ would converge to $E(t)$, where $E(t)$ is the (generalized) evolution by mean curvature starting from $E$.

The proof of convergence of this scheme was established by Evans [17], Barles and Georgelin [3]. Other proofs were given by H. Ishii [19] and Cao [11], where the heat equation was replaced by the convolution of $\chi_E$ with a more general symmetric kernel. Extensions and variants are found in [20, 27, 26, 29, 22].

As easily shown by formal asymptotic expansion, the natural anisotropic generalization of the Merriman-Bence-Osher algorithm is as follows. Given $E$ a closed set with compact boundary in $\mathbb{R}^N$, we let $T_h(E) = \{x : u(x, h) \geq 1/2\}$ where $u(x,t)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) \; \in \; \operatorname{div}\Big(\phi^\circ(\nabla u)\partial\phi^\circ(\nabla u)\Big)(x,t) & t > 0, \; x \in \mathbb{R}^N , \\ u(\cdot, 0) \; = \; \chi_E & (t = 0) . \end{cases} \tag{6}$$

The funtion $u(x,t)$ is well defined and unique by classical results on contraction semigroups [10]: if $E$ is compact, it corresponds to the flow in $L^2(\mathbb{R}^N)$ of the subdifferential of the functional $u \mapsto \int_{\mathbb{R}^N} \phi^\circ(\nabla u)^2/2\, dx$ if $u \in H^1(\mathbb{R}^N)$, and $+\infty$ otherwise. On the other hand, if $\mathbb{R}^N \setminus E$ is compact, one defines $u$ by simply letting $u = 1 + v$ where $v$ solves the same equation with initial data $\chi_E - 1$.

We first observe that the monotonicity of this scheme is obvious. Indeed, it follows from the comparision principle for equation (6)). Let us now prove the following:

**Proposition 3.1** *$T_h$, defined as above, is consistent in the sense of Definition 2.2.*

*Proof.* Let $E$ be a superflow on $[t_0, t_1]$, in the sense of Definition 2.1, and let $A$ be the associated neighborhood of $\partial E(t)$, $t \in [t_0, t_1]$.

We introduce the function $\gamma : \mathbb{R}\times[0,+\infty) \to [0,1]$ that solves the following (1D) heat equation

$$\begin{cases} \dfrac{\partial\gamma}{\partial\tau}(\xi,\tau) \;=\; \dfrac{\partial^2\gamma}{\partial\xi^2}(\xi,\tau)\,, & \xi\in\mathbb{R},\ \tau>0\,,\\ \gamma(\xi,0) = Y(\xi)\,, & \xi\in\mathbb{R},\ (\tau=0)\,. \end{cases} \tag{7}$$

where $Y = \chi_{[0,+\infty)}$ is the Heavyside function. It is well known that $\gamma$ is given by

$$\gamma(\xi,\tau) \;=\; \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\xi} e^{-\frac{s^2}{4\tau}}\,ds\,.$$

In particular, one readily sees that it is self-similar: indeed, the change of variables $s' = s/\sqrt{\tau}$ yields

$$\gamma(\xi,\tau) \;=\; \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\frac{\xi}{\sqrt{\tau}}} e^{-\frac{s'^2}{4}}\,ds' \;=\; \gamma\left(\frac{\xi}{\sqrt{\tau}},1\right) \;=:\; \gamma_1\left(\frac{\xi}{\sqrt{\tau}}\right)\,.$$

Fix $t < t_0$. The simplest idea would be to introduce the function $v(x,\tau) := \gamma(-d(x,t+\tau),\tau)$, defined in $A$ for small $\tau$. It satisfies $\{v(\cdot,\tau)\ge 1/2\} = E(t+\tau)$ and one has (using (5))

$$\frac{\partial v}{\partial\tau} \;=\; -\frac{\partial\gamma}{\partial\xi}\frac{\partial d}{\partial t} \;-\; \frac{\partial\gamma}{\partial\tau} \le -\frac{\partial\gamma}{\partial\xi}\left(\operatorname{div}\nabla\phi^\circ(\nabla d)+\delta\right) \;-\; \frac{\partial\gamma}{\partial\tau}\,.$$

Also: $\nabla v = -(\partial\gamma/\partial\xi)\nabla d$, so that $\phi^\circ(\nabla v) = (\partial\gamma/\partial\xi)$ and $\nabla\phi^\circ(\nabla v) = -\nabla\phi^\circ(\nabla d)$, hence

$$\operatorname{div}\phi^\circ(\nabla v)\nabla\phi^\circ(\nabla v) \;=\; -\operatorname{div}\frac{\partial\gamma}{\partial\xi}\nabla\phi^\circ(\nabla d) \;=\; -\frac{\partial\gamma}{\partial\xi}\operatorname{div}\nabla\phi^\circ(\nabla d) - \frac{\partial^2\gamma}{\partial\xi^2}\,.$$

Here, we have used the fact that $\phi^\circ$ is even and one-homogeneous, $\nabla\phi^\circ$ is odd and zero-homogeneous, $\phi^\circ(\nabla d) = 1$, and $\nabla d\cdot\nabla\phi^\circ(\nabla d) = \phi^\circ(\nabla d) = 1$ (by Euler's identity). Using $\partial\gamma/\partial\tau = \partial^2\gamma/\partial\xi^2$, we find:

$$\frac{\partial v}{\partial\tau} \;\le\; \operatorname{div}\phi^\circ(\nabla v)\nabla\phi^\circ(\nabla v) - \delta\frac{\partial\gamma}{\partial\xi}\,.$$

Hence, $v$ is a good candidate to be a subsolution of (6), with initial data $v(x,0) = \chi_{E(t)}(x)$. If this were the case, we would get that $v\le u$ (where $u$ solves (6) with initial data $\chi_{E(t)}$), so that $\{v(\cdot,h)\ge 1/2\}\subseteq\{u(\cdot,h)\ge 1/2\}$, in other words, $E(t+h)\subseteq T_hE(t)$, which is our consistency. However, we cannot show that this $v$ is less than $u$ at the boundary of $A$ (for instance), for $t\le t+\tau\le t+h$. This is why we define $v$ in a slightly more complicated way: we let $v(x,\tau) := \gamma(-d(x,t+\tau)+\delta\tau,\tau)-\eta h$, where $\eta < \delta/\sqrt{2\pi}$ is fixed. Since now $\partial v/\partial\tau$ differs from the previous time derivative by $\delta\partial\gamma/\partial\xi$, one still has

$$\frac{\partial v}{\partial\tau} \;\le\; \operatorname{div}\phi^\circ(\nabla v)\nabla\phi^\circ(\nabla v)\,. \tag{8}$$

at any $(x,\tau)\in A\times[0,h]$, hence $v$ is a subsolution of (6). At $\tau = 0$, $v(x,0) = \chi_{E(t)}(x)-\eta h < \chi_{E(t)}(x)$.

Let $u$ solve (6) with initial data $\chi_{E(t)}$. First of all, we observe that since $d\in C^1([t_0,t_1];C^2(A))$, $\partial E(t)$ is a $C^2$ compact hypersurface, continuous in time. Hence there exists $\rho > 0$, independent of $t$, such that each point $x \in \partial E(t)$, $E(t)$ satisfies an interior and exterior Wulff shape condition of radius $\rho$: there exist $z \in E(t)$ and $z' \notin E(t)$ with $\{\phi(\cdot - z) \le \rho\} \subset E(t)$ and $\{\phi(\cdot - z') < \rho\} \cap E(t) = \emptyset$, while $\phi(x - z) = \phi(x - z') = \rho$. One may always assume that $\{|d(\cdot, s)| \le \rho\} \subset A$ for all $s \in [t_0, t_1]$. Let $B = \{|d(\cdot, t)| < \rho\}$. If $h$ is small enough (independently of $t$), one also may assume that $|d(x, t + \tau) - d(x, t)| \le \rho/2$ in $B$ for any $\tau \in [0, h]$, so that $\mathrm{dist}^\phi(\partial E(t + \tau), \partial B) \ge \rho/2$. We assume $h \le \rho/(4\delta)$. Let $x \in \partial B$ with $d(x, t) = \rho$: then $d(x, t+\tau) \ge \rho/2$ for any $\tau \in [0, h]$, so that $-d(x, t+\tau)+\delta\tau \le \delta h - \rho/2 \le -\rho/4$, and $v(x, \tau) \le \gamma(-\rho/4, \tau) - \eta h$ for any $\tau \in [0, h]$. Hence $v(x, \tau) \le \gamma_1(-\rho/(4\sqrt{\tau})) - \eta h \le \gamma_1(-\rho/(4\sqrt{h})) - \eta h$ which is negative if $h$ is small enough. This shows that if $h$ is small enough, $v(x, \tau) < 0 \le u(x, \tau)$ for any $\tau \le h$ and $x \in \partial B \cap \{d(\cdot, t) = \rho\}$.

If now $x \in \partial B$ with $d(x, t) = -\rho$, we use the fact that $u \ge w$, where $w$ solves (6) with initial data $w_0 = \chi_{\{\phi(\cdot - x) \le \rho\}}$. One shows that $w(y, \tau) = U(\phi(y - x)/\rho, \tau/\rho^2)$ where $U(|x|, \tau) = \tilde{U}(x, \tau)$ and $\tilde{U}$ is the (radial) solution of the heat equation $\partial\tilde{U}/\partial t = \Delta\tilde{U}$ with initial datum $\chi_{B_1}$, the characteristic function of the unit ball. It is well-known that

$$\tilde{U}(y, \tau) = \frac{1}{\sqrt{4\pi\tau}^N} \int_{\{|z| \le 1\}} \exp\left(-\frac{|y - z|^2}{4\tau}\right) dz$$

so that

$$U(0, \tau) \;=\; 1 - \frac{1}{\sqrt{4\pi}^N} \int_{\{|z| \ge 1/\sqrt{\tau}\}} \exp\left(-\frac{z^2}{4}\right) dz\,.$$

Hence, $u(x, \tau) \ge 1 - (1/\sqrt{4\pi}^N) \int_{\{|z| \ge \rho/\sqrt{\tau}\}} \exp(-z^2/4)\, dz \ge 1 - c\exp(-\rho/(4\sqrt{h}))$ for some constant $c > 0$, and any $\tau \in [0, h]$. Hence, for $\tau \in [0, h]$, $v(x, \tau) - u(x, \tau) \le c\exp(-\rho/(4\sqrt{h})) - \eta h$: clearly, this is negative if $h$ is small enough (depending only on $\rho$). We have shown that $v$ is below $u$ on $\partial B \times [0, h]$, if $h$ is small enough (uniformly in $t$).

By standard results on parabolic equations, we find that $v \le u$ on $B \times [0, h]$ and in particular $v(\cdot, h) \le u(\cdot, h)$ in $B$. Hence, $\{v(\cdot, h) \ge 1/2\} \subseteq \{u(\cdot, h) \ge 1/2\}$. Observe that $v(x, h) \ge 1/2$ iff $-d(x, t + h) + \delta h \ge (\gamma(\cdot, h))^{-1}(1/2 + \eta h) = \sqrt{2\pi}\eta h + o(h)$, that is, $d(x, t+h) \le (\sqrt{2\pi}\eta - \delta)h + o(h) =: \sigma_h$. If $h$ is small enough, $\sigma_h > 0$, so that $x \in E(t + h) \Rightarrow d(x, t + h) \le \sigma_h \Leftrightarrow v(x, h) \ge 1/2$: we deduce $E(t + h) \subseteq T_h E(t)$, which was our claim. The proof of consistency with subflows is identical. □

See [14] for a proof of consistency and convergence which works in more general situations (namely, the crystalline case). See also K. Ishii [21]'s recent paper on an optimal estimate on the rate of convergence of Merriman-Bence-Osher's algorithm, in the isotropic case, where the proof of convergence is very close to ours.

# 4 The Almgren-Taylor-Wang algorithm

In Almgren, Taylor and Wang's paper [1], the transformation $T_h E$ is defined as *a* solution of

$$\min_{F\subseteq\mathbb{R}^N} P_\phi(F) \;+\; \frac{1}{h}\int_{F\triangle E} |d^\phi_E|(x)\,dx\,, \tag{9}$$

where now, $F\triangle E$ is the symmetric difference of the two sets $F$ and $E$ and $P_\phi(F)$ is the anisotropic perimeter. This is rigorously defined by $\int_{\mathbb{R}^N}\phi^\circ(D\chi_F)$, where the anisotropic total variation is given by

$$\int_{\mathbb{R}^N}\phi^\circ(Dv) \;:=\; \sup\left\{\int_{\mathbb{R}^N} v(x)\mathrm{div}\,\psi(x)\,dx \;:\; \psi\in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N),\, \phi(\psi(x))\le 1\;\forall x\in\mathbb{R}^N\right\}.$$

The same approach to curvature motion has also been proposed by Luckhaus and Sturzenhecker [23], in the isotropic case.

It is shown in [13, 12, 4] that a *monotone* selection of $T_h E$ can be built in the following way: one fixes a bounded open set $\Omega\supset\supset E$, and one lets $w$ be the (unique) minimizer of

$$\int_\Omega \phi^\circ(Dw) \;+\; \frac{1}{2h}(w(x)-d^\phi_E(x))^2\,dx\,, \tag{10}$$

then, $F=\{w\le 0\}$ is a solution of (9), as soon as the domain $\Omega$ is large enough. Clearly, letting $T_h E$ be this solution defines a monotone operator, since $E\subset E'\Rightarrow d^\phi_E\ge d^\phi_{E'}$ so that $w\ge w'$ (being $w'$ the solution of (10) with $E$ replaced with $E'$), and $T_h E\subset T_h E'$. On the other hand, it is also shown in [13, 12, 4] that this choice gives the largest solution, whereas $\{w<0\}$ would be the smallest (yielding uniqueness, up to a negligible set, whenever $|\{w=0\}|=0$, which is "generically" true in some sense). The proof of consistency we will next give would also work with this second choice, yielding convergence of *any* selection of Almgren-Taylor-Wang's scheme to the generalized solution, when unique. We now show:

**Proposition 4.1** *$T_h$, defined as above, is consistent in the sense of Definition 2.2.*

*Proof.* Let $E$ be a superflow on $[t_0,t_1]$, in the sense of Definition 2.1, and let $A$ be the associated neighborhood of $\partial E(t)$, $t\in[t_0,t_1]$.

Observe that as in the previous section, there exists $\rho>0$ such that $\{d(\cdot,t)\le\rho\}\subset A$ at any time $t\in[t_0,t_1]$, and $\partial E(t)$ satisfies both an interior and exterior Wulff shape condition of radius $\rho$.

We fix $t\in[t_0,t_1)$, and let $B=\{d(\cdot,t)<\rho\}$. Consider $\psi:\mathbb{R}\to\mathbb{R}$ a smooth increasing function with $\psi(s)\ge s$ and $\psi(s)=s$ for $|s|\le\varepsilon/2$. We set, for $x\in B$, $v(x):=\psi(d(x,t+h))$. Then, from (5), it follows

$$\frac{v(x)-d_{E(t)}(x)}{h} \;\ge\; \frac{d(x,t+h)-d(x,t)}{h} \;=\; \frac{1}{h}\int_0^h \frac{\partial d}{\partial t}(x,t+\tau)\,d\tau$$
$$\ge\; \frac{1}{h}\int_t^{t+h} \mathrm{div}\,\nabla\phi^\circ(\nabla d)(x,t+\tau)\,d\tau \;+\;\delta\,.$$

Let now $\omega$ be a modulus of continuity for $\operatorname{div} \nabla\phi^\circ(\nabla d)$ in $\{|d| \le \rho\}$: we find

$$\frac{v(x) - d_{E(t)}(x)}{h} \ge \operatorname{div} \nabla\phi^\circ(\nabla d)(x, t+h) + \delta - \omega(h).$$

Observe that for any $x \in B$ it holds $\nabla v(x) = \psi'(d(x, t+h))\nabla d(x, t+h)$, so that (recall that $\nabla\phi^\circ$ 0-homogeneous), $\nabla\phi^\circ(\nabla v(x)) = \nabla\phi^\circ(\nabla d(x, t+h))$ hence $\operatorname{div} \nabla\phi^\circ(\nabla d)(x, t+h) = \operatorname{div} \nabla\phi^\circ(\nabla v)(x)$. Therefore, if $h$ is small enough so that $\omega(h) \le \delta$, we get

$$\frac{v(x) - d_{E(t)}(x)}{h} \ge \operatorname{div} \nabla\phi^\circ(\nabla v)(x).$$

Let $w$ solve (10), with $E = E(t)$. We will show that we may choose $\psi$ in order to have $v \ge w$ on $\partial B$, so that $v$ is a supersolution for the problem

$$\min\left\{ \int_B \phi^\circ(Du) + \frac{1}{2h}\int_B (u(x) - d_{E(t)}(x))^2\, dx \,:\, u = w \text{ on } \partial B \right\} \qquad (11)$$

(which is solved by $w$). We will deduce that $v \ge w$ in $B$, so that $\{w \le 0\} \supseteq \{v \le 0\} = \{d(\cdot, t+h) \le 0\}$, that is, $T_h(E(t)) \supseteq E(t+h)$.

First of all, $d$ is uniformly continuous in time, so that if $h$ is small enough, one has $d(x, t+h) \ge 3\rho/4$ if $d(x,t) = \rho$. If $M > \operatorname{diam}\Omega$, then one shows that $M \ge w$ in $\Omega$. We may choose a function $\psi$ with $\psi(3\rho/4) \ge M$, so that $v(x) \ge M \ge w(x)$ if $d(x,t) = \rho$.

On the other hand, since $E(t)$ satisfies an interior Wulff shape condition of radius $\rho$, one has $d_E^\phi \le \phi(\cdot - x) - \rho$ at any point $x \in \partial B$ with $d(x,t) = -\rho$. The analysis in [12, 15] shows that the solution of (10) with $d_E^\phi$ replaced with $\phi$ takes the value $2N\sqrt{h}/\sqrt{N+1}$ at the origin. We deduce that $w(x) \le 2N\sqrt{h}/\sqrt{N+1} - \rho$: hence, if $h$ is small enough, we get $w(x) \le -3\rho/4$. We can choose $\psi$ such that $\psi(s) \ge -3\rho/4$ for any $s$, so that $v(x) \ge w(x)$ if $d(x,t) = -\rho$. We conclude that $v \ge w$ on $\partial B$. Hence $v$ is a supersolution for (11), which implies $T_{t,t+h}(E(t)) \supseteq E(t+h)$.

If $E(t)$ is a subflow, we can reproduce the same proof to show that $T_{t,t+h}(E(t)) \subseteq E(t+h)$. □

While a (much more difficult) proof of consistency with smooth flows is already found in Almgren, Taylor and Wang's paper [1], our proof is more easily adapted to other situations: in [15], we consider the case of a flow driven by anisotropic curvature with an additional time-dependent forcing term, possibly discontinuous.

## References


[1] F. Almgren, J. E. Taylor, and L.-H. Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993.

[2] L. Ambrosio and Soner H. M. Level set approach to mean curvature flow in arbitrary codimension. *J. Differential Geom.*, 43(4):693–737, 1996.

[3] G. Barles and C. Georgelin. A simple proof of convergence for an approximation scheme for computing motions by mean curvature. *SIAM J. Numer. Anal.*, 32(2):484–500, 1995.

[4] G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga. Crystalline mean curvature flow of convex sets. Preprint Centro De Giorgi, Pisa, 2004.

[5] G. Bellettini and M. Novaga. Minimal barriers for geometric evolutions. *J. Differential Equations*, 139(1):76–103, 1997.

[6] G. Bellettini and M. Novaga. Comparison results between minimal barriers and viscosity solutions for geometric evolutions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):97–131, 1998.

[7] G. Bellettini and M. Novaga. Approximation and comparison for nonsmooth anisotropic motion by mean curvature in $\mathbf{R}^N$. *Math. Models Methods Appl. Sci.*, 10(1):1–10, 2000.

[8] G. Bellettini and M. Paolini. Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 19:43–67, 1995.

[9] G. Bellettini and M. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.*, 25(3):537–566, 1996.

[10] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[11] F. Cao. Partial differential equations and mathematical morphology. *J. Math. Pures Appl. (9)*, 77(9):909–941, 1998.

[12] V. Caselles and A. Chambolle. Anisotropic curvature-driven flow of convex sets. Preprint CMAP #528, Ecole Polytechnique, France, 2004.

[13] A. Chambolle. An algorithm for mean curvature motion. *Interfaces Free Bound.*, 6(2):195–218, 2004.

[14] A. Chambolle and M. Novaga. Convergence of an algorithm for anisotropic mean curvature motion. Preprint CMAP #572, Ecole Polytechnique, France, April 2005.

[15] A. Chambolle and M. Novaga. Implicit time discretization of the mean curvature flow with a discontinuous forcing term. Preprint CMAP #571, Ecole Polytechnique, France, March 2005.

[16] E. De Giorgi. New ideas in calculus of variations and geometric measure theory. In *Motion by mean curvature and related topics (Trento, 1992)*, pages 63–69. de Gruyter, Berlin, 1994.

[17] L. C. Evans. Convergence of an algorithm for mean curvature motion. *Indiana Univ. Math. J.*, 42(2):533–557, 1993.

[18] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. II. *Trans. Amer. Math. Soc.*, 330(1):321–332, 1992.

[19] H. Ishii. A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature. In *Curvature flows and related topics (Levico, 1994)*, volume 5 of *GAKUTO Internat. Ser. Math. Sci. Appl.*, pages 111–127. Gakkōtosho, Tokyo, 1995.

[20] H. Ishii, G. E. Pires, and P. E. Souganidis. Threshold dynamics type approximation schemes for propagating fronts. *J. Math. Soc. Japan*, 51(2):267–308, 1999.

[21] K. Ishii. Optimal rate of convergence of the Bence-Merriman-Osher algorithm for motion by mean curvature. Preprint #16 available at the *Control Theory and PDE* web site `http://cpde.iac.rm.cnr.it`, Nov. 2004.

[22] F. Leoni. Convergence of an approximation scheme for curvature-dependent motions of sets. *SIAM J. Numer. Anal.*, 39(4):1115–1131 (electronic), 2001.

[23] S. Luckhaus and T. Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271, 1995.

[24] B. Merriman, J. K. Bence, and S. J. Osher. Diffusion generated motion by mean curvature. In J. E. Taylor, editor, *"Computational Crystal Growers Workshop"*, Selected Lectures in Math. Amer. Math. Soc., Providence, Rhode Island, 1992.

[25] R. T. Rockafellar. *Convex analysis.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[26] S. J. Ruuth. Efficient algorithms for diffusion-generated motion by mean curvature. *J. Comput. Phys.*, 144(2):603–625, 1998.

[27] S. J. Ruuth and B. Merriman. Convolution-generated motion and generalized Huygens' principles for interface motion. *SIAM J. Appl. Math.*, 60(3):868–890 (electronic), 2000.

[28] J. E. Taylor, J. W. Cahn, and C. A. Handwerker. Geometric models of crystal growth. *Acta Metall.*, 40:1443–1474, 1992.

[29] L. Vivier. Convergence of an approximation scheme for computing motions with curvature dependent velocities. *Differential Integral Equations*, 13(10-12):1263–1288, 2000.