Front propagation in infinite cylinders. I. A variational approach.

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Abstract

In their classical 1937 paper, Kolmogorov, Petrovsky and Piskunov proved that for a particular class of reaction-diffusion equations on the line the solution of the initial value problem with the initial data in the form of a unit step propagates at long times with constant velocity equal to that of a certain special traveling wave solution. This type of a propagation result has since been established for a number of general classes of reaction-diffusion-advection problems in cylinders. Here we show that actually in the problems without advection or in the presence of transverse advection by a potential flow these results do not rely on the specifics of the problem. Instead, they are a consequence of the fact that the considered equation is a gradient flow in an exponentially weighted L^2-space generated by a certain functional, when the dynamics is considered in the reference frame moving with constant velocity along the cylinder axis. We show that independently of the details of the problem only three propagation scenarios are possible in the above context: no propagation, a "pulled" front, or a "pushed" front. The choice of the scenario is completely characterized via a minimization problem.

1 Introduction

This paper is concerned with the study of front propagation in reaction-diffusion-advection problems in cylinders which arise in numerous applications [11, 15, 32]. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain (not necessarily simply connected), and consider $\Sigma = \Omega \times \mathbb{R}$, an infinite cylinder in $\mathbb{R}^n$. In Σ, we shall consider the following parabolic equation

$$u_t + \mathbf{v} \cdot \nabla u = \Delta u + f(u, y). \tag{1.1}$$

Here $u = u(x,t) \in \mathbb{R}$ is the dependent variable (corresponding, e.g., to temperature in combustion problems), $\mathbf{v} = \mathbf{v}(y) \in \mathbb{R}^n$ is an imposed advective flow, and $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is a nonlinear reaction term. By $x = (y, z) \in \Sigma$, we denote a point with coordinate $y \in \Omega$ on the cylinder cross-section and $z \in \mathbb{R}$ along the cylinder axis. We also assume that $u = 0$ is a trivial solution of (1.1).

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We are considering a particular situation in which the flow $\mathbf{v}$ is *transverse* to the axis of the cylinder, i.e., when $\mathbf{v}$ does not have a component along z. Furthermore, we assume that the flow $\mathbf{v}(y)$ is potential:

$$\mathbf{v} = (-\nabla_y \varphi, 0), \qquad \varphi : \overline{\Omega} \to \mathbb{R}. \tag{1.2}$$

Respectively, those parts of $\partial\Omega$, denoted by $\partial\Omega_+$, on which $\nu \cdot \nabla_y \varphi > 0$ are the inlets and those, denoted by $\partial\Omega_-$, where $\nu \cdot \nabla_y \varphi < 0$ are the outlets (of fuel in combustion problems, e.g.). Here and below ν is the outward normal to $\partial\Omega$ (or $\partial\Sigma$). We denote those parts of $\partial\Omega$ on which $\nu \cdot \nabla_y \varphi = 0$ by $\partial\Omega_0$, these are impermeable walls. Consistently with this interpretation of the flow $\mathbf{v}$, we impose the following boundary conditions:

$$u\big|_{\partial\Sigma_\pm} = 0, \qquad \nu \cdot \nabla u\big|_{\partial\Sigma_0} = 0. \tag{1.3}$$

on $\partial\Sigma_\pm = \partial\Omega_\pm \times \mathbb{R}$ and $\partial\Sigma_0 = \partial\Omega_0 \times \mathbb{R}$. Naturally, the case of a purely reaction-diffusion equation ($\varphi = 0$) with either Dirichlet or Neumann boundary conditions is included in our formulation. In fact, it is the diffusion part in combination with Dirichlet boundary conditions and/or inhomogeneous reaction term that present the main difficulties in the analysis of this problem; nevertheless, for the sake of generality and because of the importance to applications (see e.g. [8, 40]) we will treat the case of a general transverse potential flow here.

Equation (1.1) has been the subject of great many studies (see e.g. [2–4,7,18,27, 35,45] and references therein, this list is certainly incomplete), beginning with the pioneering work of Fisher [17] and Kolmogorov, Petrovsky and Piskunov [23]. The celebrated result of Kolmogorov, Petrovsky and Piskunov applied to the Fisher's equation:

$$u_t = u_{xx} + u(1-u), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1.4}$$

states that the solution of the initial value problem for this equation with the initial data $u(x,0) = \theta(-x)$, where θ is the Heaviside step, propagates at long times with the asymptotic speed $c^* = 2$ (in the sense of average velocity of the level sets). The speed c^* is that of a special traveling wave solution and is determined by the linearization of (1.4) around $u = 0$ (the so-called minimal "pulled" front in the terminology of [39]).

One might ask whether this kind of propagation result holds more generally for equations like (1.1). For example, given that the solutions of (1.1) take values in the unit interval, what can one say about propagation of solutions of the initial value problem with front-like initial data, say,

$$u(x,0) = \theta(-z)\tanh(\varepsilon^{-1}\mathrm{dist}(x, \partial\Sigma_\pm \cup \{z=0\})), \tag{1.5}$$

with ε sufficiently small? Previous work on this subject relied heavily on the applications of the Maximum and Comparison Principles which require a rather detailed knowledge of certain special types of solutions of (1.1), in particular, traveling waves [1,2,16,21,27,35] (see also [9,18,28,38] for an alternative approach using probabilistic methods). If no assumptions on the type of the nonlinearity, the geometry of the domain, or the flow are made, then there is no hope to obtain a sharp characterization of propagation within this setup using such techniques.

What we found, however, is that a sharp characterization of propagation can be made *without* relying on any a priori detailed knowledge about the problem using instead a variational approach. This approach relies on the observation, first made in the case of gradient reaction-diffusion systems [30] and generalized here to the considered class of reaction-diffusion-advection problems, that (1.1) written in the reference frame moving with speed c along the axis of the cylinder is a gradient flow in the exponentially weighted L^2-space:

$$u_t = -e^{-cz-\varphi(y)} \frac{\delta \Phi_c[u]}{\delta u}, \tag{1.6}$$

generated by the functional in (2.3). In particular, traveling wave solutions with the right decay at $z = +\infty$ are critical points of Φ_c, and it is natural to look for the minimizers of this functional [24, 25]. It turns out, as we will show below, that the speed of these minimizers, if they exist, in fact determines the asymptotic propagation speed for the solutions of the initial value problem for (1.1) (in the sense of the leading edge, see the following sections for precise definitions) with the initial data from (1.5) for sufficiently small ε. This is the situation of "nonlinear selection" in the terminology of [39], with the minimizer being the fastest "pushed" front.

If, on the other hand, these minimizers do not exist, but at the same time the solution $u = 0$ is linearly unstable, then, as we show below, there exists a certain traveling wave solution (a minimal wave), whose speed is determined by the linearization of the problem around $u = 0$ and governs the asymptotic propagation speed for the initial data from (1.5) with $\varepsilon \ll 1$. This is the situation of "linear selection" in the terminology of [39], with the traveling wave solution in question being the slowest "pulled" front. Finally, if neither the minimizer nor the minimal wave exist, then no propagation is possible.

Thus, we demonstrate that independently of the specifics of the problem under consideration and, in particular, independently of whether the nonlinearity in (1.1) is of KPP, monostable, ignition, bistable, or any other type whatsoever, there are only three scenarios possible for front-like initial data in (1.5) with ε small enough: either no propagation at all, or propagation with the speed of the minimal wave, or propagation with the speed of the minimizer; these statements are a simple consequence of Corollary 5.3 and Theorem 5.8. It appears from our analysis that the phenomenon of propagation from front-like initial data is a consequence of the structure of the equation (1.1) alone, and not the precise details of f, Ω, or φ. This is the main result of this paper. We have also obtained a series of results characterizing the relevant traveling wave solutions, such as their existence, uniqueness (up to translations), monotonicity, asymptotic decay, as well as the way to estimate the propagation speed, together with general statement of propagation results for wide classes of initial data, including localized initial data leading to pairs of counter-propagating fronts. Note that we have not addressed the questions of convergence of solutions of (1.1) to traveling waves, which is part of the conclusions of [16, 23, 27, 35], this will be the subject of future study.

This paper is organized as follows. In Sec. 2, we present all the basic assumptions used throughout the paper. In Secs. 3 and 4, we present our results, given by Theorems 3.3, 3.9, and 4.2 on the existence and properties of certain special traveling wave solutions which play the key role for the propagation results of Sec. 5. In the next section, Sec. 5, we establish general propagation results for the leading edge of the solutions of the initial value problem in (1.1), see Theorem 5.8. Finally, in Sec. 6 we compare the obtained results with other studies in the literature, and then discuss some open problems.

Notation. Throughout the paper C^k, C_0^∞, $C^{k,\alpha}$ denote the usual spaces of continuous functions with k continuous derivatives, smooth functions with compact support, continuously differentiable functions with Hölder-continuous derivatives of order k for $\alpha \in (0,1)$ (or Lipschitz-continuous when $\alpha = 1$), respectively. Unless it is otherwise clear from the context, "$\cdot$" denotes a scalar product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. The symbol ∇ is reserved for the gradient in $\mathbb{R}^n$, while ∇_y stands for the gradient in $\Omega \subset \mathbb{R}^{n-1}$. Similarly, the symbol Δ stands for the Laplacian in $\mathbb{R}^n$, and Δ_y for the Laplacian in Ω. By a classical solution of (3.1) we mean a function $u \in C^2(\Sigma) \cap C^1(\overline{\Sigma})$ that satisfies this equation with a given value of $c > 0$ and the boundary conditions in (1.3). The classical solution of (1.1) is understood to be a $C_1^2(\Sigma \times (0,\infty)) \cap C^0(\overline{\Sigma} \times [0,\infty))$ function [13]. The numbers C, K, M, λ, etc., will denote generic positive constants.

2 Preliminaries and main hypotheses

In this section, we summarize all the hypotheses used in this paper in the analysis of (1.1). Throughout the paper, $\Omega \subset \mathbb{R}^{n-1}$ is assumed to be a bounded, connected (possibly multiply connected) open set with boundary of class C^2. We also assume that $\partial\Omega_\pm$ and $\partial\Omega_0$ are a collection of finitely many (possibly one) closed disjoint portions of $\partial\Omega$.

Now we discuss the assumptions on the nonlinearity $f(u,y)$. Our method is quite general, and so we do not need to explicitly prescribe the type of the nonlinearity in our problem. We basically need to assume that $f(\cdot,y)$ is sufficiently regular on some compact subset of $\mathbb{R}$ which is an invariant set with respect to the evolution governed by (1.1). Since the Maximum Principle holds for (1.1), without the loss of generality we may assume that $u(x,t) \in [0,1]$, as long as:

(H1) The function $f : [0,1] \times \overline{\Omega} \to \mathbb{R}$ satisfies

$$f(0,y) = 0, \qquad f(1,y) \leq 0. \qquad \forall y \in \Omega. \tag{2.1}$$

(H2) For some $\gamma \in (0,1)$

$$f \in C^{0,\gamma}([0,1] \times \overline{\Omega}), \qquad f_u \in C^{0,\gamma}([0,1] \times \overline{\Omega}), \qquad \varphi \in C^{1,\gamma}(\overline{\Omega}), \tag{2.2}$$

where $f_u = \partial f / \partial u$.

The starting point of our variational approach is the functional

$$\Phi_c[u] = \int_\Sigma e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla u|^2 + V(u,y)\right) dx, \tag{2.3}$$

where

$$V(u,y) = \begin{cases} 0, & u < 0, \\ -\int_0^u f(s,y)\,ds, & 0 \le u \le 1, \\ -\int_0^1 f(s,y)\,ds, & u > 1. \end{cases} \tag{2.4}$$

From the definition of V and the assumptions on f it readily follows that $|V(u)| \le Cu^2$, and so $\Phi_c[u]$ is naturally defined in an exponentially weighted Sobolev space $H^1_c(\Sigma)$ (for shortness we do not explicitly mention φ, which is part of the definition of $H^1_c(\Sigma)$). Formally, let $\mathcal{D}(\Sigma)$ be the subspace of $C^\infty(\Sigma)$ defined by the restrictions to Σ of all the functions $u \in C_0^\infty(\mathbb{R}^n)$ which vanish on $\partial\Sigma_\pm$. Then

Definition 2.1. *For $c > 0$, denote by $H^1_c(\Sigma)$ the completion of $\mathcal{D}(\Sigma)$ with respect to the norm*

$$||u||^2_{H^1_c(\Sigma)} = ||u||^2_{L^2_c(\Sigma)} + ||\nabla u||^2_{L^2_c(\Sigma)}, \qquad ||u||^2_{L^2_c(\Sigma)} = \int_\Sigma e^{cz+\varphi(y)}|u|^2 dx. \tag{2.5}$$

These are the spaces in which we will consider both the minimizers of Φ_c and the solutions of the initial value problem for (1.1).

Let us mention an important general property of the spaces $H^1_c(\Sigma)$ which is an analogue of the Poincaré inequality and will be needed to establish the existence result (for the proof we refer to [25]).

Lemma 2.2. *For all $u \in H^1_c(\Sigma)$, we have*

$$\frac{c^2}{4}\int_R^{+\infty}\int_\Omega e^{cz+\varphi(y)}u^2 dydz \le \int_R^{+\infty}\int_\Omega e^{cz+\varphi(y)}u_z^2 dydz, \tag{2.6}$$

$$\int_\Omega e^{\varphi(y)}u^2(y,R)dy \le \frac{e^{-cR}}{c}\int_R^{+\infty}\int_\Omega e^{cz+\varphi(y)}u_z^2\,dydz, \tag{2.7}$$

for any $R \in \mathbb{R} \cup \{-\infty\}$.

We also have the following obvious inclusions for spaces $H^1_c(\Sigma)$ with different values of c:

Lemma 2.3. *Let $c' > c > 0$ then*

$$H^1_{c'}(\Sigma) \cap W^{1,\infty}(\Sigma) \subset H^1_c(\Sigma) \cap W^{1,\infty}(\Sigma). \tag{2.8}$$

We now turn to the hypothesis that is crucial to the existence of the minimizers of Φ_c:

(H3) There exist $c > 0$, satisfying $c^2 + 4\nu_0 > 0$, where

$$\nu_0 = \min_{\substack{\psi \in H^1(\Omega) \\ \psi|_{\partial\Omega_\pm} = 0}} R(\psi), \tag{2.9}$$

with $R(\psi)$ given by (3.5), and $u \in H^1_c(\Sigma)$, such that $\Phi_c[u] \leq 0$ and $u \not\equiv 0$.

This type of condition was already used in [25] in the context of Ginzburg-Landau problems as a sufficient condition for existence of variation traveling waves, and is needed for proving sequential lower semicontinuity of Φ_c in the weak topology of $H^1_c(\Sigma)$. What we will show here, however, is that for scalar equations this condition is also necessary for existence of minimizers of Φ_c.

Let us also introduce an auxiliary functional

$$E[v] = \int_\Omega e^{\varphi(y)} \left(\frac{1}{2} |\nabla_y v|^2 + V(v, y) \right) dy, \tag{2.10}$$

defined for all $v \in H^1(\Omega)$ satisfying Dirichlet boundary conditions on $\partial\Omega_\pm$. By regularity of V and φ the critical points of E satisfy

$$\Delta_y v + \nabla_y \varphi \cdot \nabla_y v + f(v, y) = 0, \qquad v|_{\partial\Omega_\pm} = 0, \quad \nu \cdot \nabla v|_{\partial\Omega_0} = 0. \tag{2.11}$$

Clearly, $v = 0$ is a critical point of E, and in general there may exist a non-trivial minimizer of E over all $v \in H^1(\Omega)$ subject to $v|_{\partial\Omega_\pm} = 0$, call it v_0 [12]. In this case necessarily $v_0 > 0$ and $E[v_0] \leq 0$. Thus, existence of a non-trivial minimizer of E guarantees existence of a critical point of E with negative energy.

3 Existence and properties of the minimizers

In this and the following section we analyze existence of certain traveling wave solutions for (1.1). A traveling wave solution is a pair $(c, \bar{u})$, with $c > 0$, such that $u(x, t) = \bar{u}(y, z - ct)$ solves (1.1). Substituting this form into (1.1), we obtain an elliptic equation for $\bar{u}$ with the respective boundary conditions

$$\Delta \bar{u} + c\bar{u}_z + \nabla_y \varphi \cdot \nabla_y \bar{u} + f(\bar{u}, y) = 0, \qquad \bar{u}\big|_{\partial\Sigma_\pm} = 0, \quad \nu \cdot \nabla \bar{u}\big|_{\partial\Sigma_0} = 0. \tag{3.1}$$

Note that (3.1) may in general have many solutions [7,14,29]. In the context of the initial value problem for (1.1) one is interested in the particular type of traveling waves in the form of fronts that invade the $u = 0$ equilibrium at $z = +\infty$. From the basic energy estimates for (3.1), one expects the solution to connect two distinct equilibria: $v_+ = 0$ at $z = +\infty$ and $v_- = v(y)$ at $z = -\infty$, where v is a solution of (2.11), when $c > 0$ [7,14,43]. The speed c of such a front is part of the problem of finding solutions of (3.1).

Suppose there exists a solution of (3.1) with a particular speed $c > 0$. Linearlizing (3.1) with respect to $u = 0$, we obtain that (here we present a formal discussion of the decay of the solutions in order to clarify the key issues, these statements will be justified later on)

$$\bar{u}(y, z) \sim \sum_k a_k \psi_k(y) e^{-\lambda_k z}, \tag{3.2}$$

which describes the asymptotic behavior of the traveling wave solution at $z = +\infty$, provided that all $\lambda_k > 0$. Here λ_k satisfy a quadratic equation

$$\lambda_k^2 - c\lambda_k - \nu_k = 0. \tag{3.3}$$

where ν_k are the eigenvalues of

$$\Delta_y \psi_k + \nabla_y \varphi \cdot \nabla_y \psi_k + f_u(0, y)\psi_k + \nu_k \psi_k = 0, \tag{3.4}$$

with the same boundary conditions as in (3.1). The eigenvalue problem in (3.4) can be easily characterized.

Proposition 3.1. *There exists a countable set of eigenvalues $\{\nu_k\}$ and a complete set of orthonormal (in $L^2(\Omega; e^{\varphi(y)}dy)$) eigenfunctions ψ_k for problem (3.4). All ν_k are real, and $\nu_0 < \nu_1 \leq \nu_2 \leq \ldots \nu_k \to \infty$. One can choose $\psi_0 > 0$ in Ω, conversely all the other eigenfunctions change sign for $k \geq 1$.*

Proof. The existence of an increasing sequence of real eigenvalues converging to $+\infty$ follows from the spectral representation theorem for compact self-adjoint operators (see for instance [10, Theorem VI.11]). The fact that ν_0 has multiplicity one follows from the characterization of ψ_0 as a minimizer of the Rayleigh quotient

$$R(\psi) = \frac{\int_\Omega e^{\varphi(y)}(|\nabla_y \psi|^2 - f_u(0, y)\psi^2)dy}{\int_\Omega e^{\varphi(y)}\psi^2 dy}, \tag{3.5}$$

which also gives $\psi_0 > 0$, by Strong Maximum Principle. Since the other eigenvectors are orthogonal to ψ_0, they must necessarily change sign. □

Remark 3.2. *Notice that if $\nu_0 \neq 0$, then $v = 0$ is an isolated critical point for the functional E in the cone $C = \{v \in H^1(\Omega) : \ v \geq 0\}$.*

Proof. Assume by contradiction that there exists a sequence of critical points $v_n \to 0$ in $H^1(\Omega)$, such that $v_n \geq 0$. Letting $\tilde{v}_n = v_n/\|v_n\|_{H^1(\Omega)}$, since each v_n solves (2.11), by elliptic regularity we have the estimate

$$\|\tilde{v}_n\|_{H^1(\Omega)} = 1 \qquad \|\tilde{v}_n\|_{H^2(\Omega)} \leq C, \tag{3.6}$$

for some $C > 0$. In particular, there exists a function $\tilde{v} \in H^1(\Omega)$, with $\|\tilde{v}\|_{H^1} = 1$ and $\tilde{v} \geq 0$, such that $\tilde{v}_n \to \tilde{v}$ in $H^1(\Omega)$. Recalling that $\tilde{v}_n$ satisfies the equation

$$\Delta_y \tilde{v}_n + \nabla_y \varphi \cdot \nabla \tilde{v}_n + \frac{1}{\|v_n\|_{H^1(\Omega)}} f\left(\|v_n\|_{H^1(\Omega)}\tilde{v}_n, y\right) = 0, \tag{3.7}$$

passing to the limit as $n \to +\infty$, we obtain that $\tilde{v}$ solves (3.4) with $\nu_k = 0$, thus contradicting Proposition 3.1. □

In the following, we always assume that $\psi_0 > 0$. Then, in order for $\bar{u}$ to remain positive for all $z > 0$ we need $a_0 > 0$ and $\lambda_0 < \lambda_k$ for all $k > 0$. Let us first consider the simpler case of $\nu_0 > 0$, which corresponds to the situation in which $u = 0$ is locally stable with respect to (1.1). In this case (3.3) has a unique positive solution for each k, and, furthermore, λ_k are increasing with k. Therefore, the asymptotic behavior of $\bar{u}$ is given by $a_0\psi_0(y)e^{-\lambda_0^+ z}$, where $a_0 > 0$ and $\lambda_k^\pm = \lambda_\pm(c, \nu_k)$ with

$$\lambda_\pm(c, \nu_k) = \frac{c \pm \sqrt{c^2 + 4\nu_k}}{2}. \tag{3.8}$$

Note that $\lambda_0^+ > \frac{c}{2}$, and so these solutions are expected to lie in the exponentially weighted Sobolev space $H_c^1(\Sigma)$.

On the other hand, the case of $\nu_0 < 0$, when $u = 0$ is unstable, requires a more careful consideration. First of all, it is clear that we should have $c^2 + 4\nu_0 \geq 0$ in order for $\bar{u}$ to remain positive (otherwise the approach to zero is oscillatory due to the imaginary part of λ_k). However, when $c^2 + 4\nu_0 > 0$, there are two positive solutions of (3.3) for λ_0, according to (3.8). In fact, one would generically expect the decay of the solution to be governed by $\lambda_0^- = \lambda_-(c, \nu_0)$, since $\lambda_-(c, \nu_0) < \lambda_+(c, \nu_0)$ in this case. On the other hand, if the solution is also known to lie in $H_c^1(\Sigma)$, then λ_0^- is not allowed, since $\lambda_0^- < \frac{c}{2}$ would make $\bar{u}$ fail to lie in $H_c^1(\Sigma)$. Therefore, those traveling wave solutions that lie in $H_c^1(\Sigma)$ are expected to have a non-generic exponential decay $a_0\psi_0(y)e^{-\lambda_0^+ z}$, with $a_0 > 0$. This is still true in the case $\nu_0 = 0$ for exponentially decaying solutions.

One can repeat the above arguments to study the behavior of the solution at $z = -\infty$. Linearizing around $u = v(y)$, we obtain $u - v \sim \sum_k \tilde{a}_k \tilde{\psi}_k(y) e^{-\tilde{\lambda}_k z}$ and

$$\Delta_y \tilde{\psi}_k + \nabla_y \varphi \cdot \nabla_y \tilde{\psi}_k + f_u(v, y)\tilde{\psi}_k + \tilde{\nu}_k \tilde{\psi}_k = 0. \tag{3.9}$$

Here we should require that $\tilde{\lambda}_k < 0$, where $\tilde{\lambda}_k$ satisfy (3.3) with $\tilde{\nu}_k$ instead of ν_k. Assuming that all $\tilde{\nu}_k \neq 0$, one sees immediately that $\tilde{a}_k = 0$ for all $\tilde{\nu}_k < 0$. If, furthermore, it is known that $\bar{u} - v < 0$ for large negative z, then we must have $\tilde{\nu}_0 > 0$ and $\tilde{a}_0 < 0$, and choose $\tilde{\lambda}_0 = \tilde{\lambda}_0^- = \lambda_-(c, \tilde{\nu}_0)$. In other words, under the assumptions of non-degeneracy of v and approach from below, the equilibrium v is necessarily a local minimum of E.

If $\bar{u} \in H_c^1(\Sigma)$, then, at least formally, $\bar{u}$ is a critical point of the functional Φ_c, since the first variation of Φ_c is

$$\begin{aligned} \delta\Phi_c[u] &= \int_\Sigma e^{cz+\varphi(y)} \left(\nabla u \cdot \nabla \delta u + V_u(u, y)\delta u\right) dx \\ &= -\int_\Sigma e^{cz+\varphi(y)} \Big(\Delta u + c u_z + \nabla_y \varphi \cdot \nabla_y u + f(u, y)\Big) \delta u \, dx, \end{aligned} \tag{3.10}$$

where we integrated by parts, using the boundary conditions from (1.1), and assumed that $0 \leq u(x) \leq 1$. We call this type of traveling wave solutions *variational traveling waves* [25, 30]. Among these solutions, of special interest are the traveling wave solutions which are in fact *minimizers* of Φ_c in $H_c^1(\Sigma)$. Let us note that existence of a minimizer $\bar{u}$ of Φ_c implies that

$$\Phi_c[\bar{u}] = 0. \tag{3.11}$$

This follows immediately from the way the functional Φ_c transforms under translations

$$\Phi_c[u(y, z - a)] = e^{ca}\Phi_c[u(y, z)], \tag{3.12}$$

and the fact that $\Phi_c[\bar{u}]$ should not change under infinitesimal translations of $\bar{u}$. We also point out that for the same reason (3.11) should in fact hold for any critical point of Φ_c and, hence, for any variational traveling wave.

Let us note that not all variational traveling waves can be minimizers of Φ_c, and not all traveling wave solutions, of course, have to be variational. Nevertheless, as was shown in [24, 30], for a large class of nonlinearities and sufficiently rapidly decaying initial data only the variational traveling waves can be selected as the long-time attractors for the initial value problem governed by (1.1). It may also happen that the minimizer of Φ_c is the only variational traveling wave among all traveling wave solutions satisfying $0 < \bar{u}(x) < 1$ in Σ, hence, the only candidate for the long-time asymptotic behavior of the solutions of the initial value problem.

Since the variational traveling waves and minimizers, in particular, play a key role in the propagation phenomena governed by (1.1), we will concentrate our efforts on establishing their existence and uniqueness. Later, in Sec. 5, we will show that their speed in fact determines the asymptotic long time propagation speed for the solutions of the initial value problem for (1.1) with sufficiently rapidly decaying front-like initial data (for more precise definitions and results, see Sec. 5). Below is our main result concerning the existence of minimizers of Φ_c.

Theorem 3.3. *Under hypotheses (H1)–(H3), there exists a unique $c^\dagger \in \mathbb{R}$ such that $c^\dagger \geq c > 0$ (where c is the "trial velocity" given by assumption (H3)), and $\bar{u} \in H^1_{c^\dagger}(\Sigma)$, $\bar{u} \not\equiv 0$, such that*

(i) $\bar{u} \in C^2(\Sigma) \cap W^{1,\infty}(\overline{\Sigma})$, $\bar{u}$ solves (3.1) with $c = c^\dagger$, and $\bar{u}$ is a minimizer of $\Phi_{c^\dagger}$.

(ii) $\bar{u}(y, z)$ is strictly monotone decreasing in z for all $y \in \Omega$, $\lim_{z\to+\infty} \bar{u}(\cdot, z) = 0$ in $C^1(\overline{\Omega})$, and $\lim_{z\to-\infty} \bar{u}(\cdot, z) = v$ in $C^1(\overline{\Omega})$, where v is a critical point of E, with $E[v] < 0$ and $0 < v \leq 1$ in Ω.

(iii) $\bar{u}(y, z) = a_0\psi_0(y)e^{-\lambda_+(c^\dagger,\nu_0)z} + O(e^{-\lambda z})$, with some $a_0 > 0$ and $\lambda > \lambda_+(c^\dagger, \nu_0)$, uniformly in $C^1(\overline{\Omega} \times [R, +\infty))$, as $R \to +\infty$.

(iv) $\tilde{\nu}_0 \geq 0$, moreover, if $\tilde{\nu}_0 > 0$, then $\bar{u}(y, z) = v(y) + \tilde{a}_0\tilde{\psi}_0(y)e^{-\lambda_-(c^\dagger,\tilde{\nu}_0)z} + O(e^{-\lambda z})$, with some $\tilde{a}_0 < 0$ and $\lambda < \lambda_-(c^\dagger, \tilde{\nu}_0)$, uniformly in $C^1(\overline{\Omega} \times (-\infty, R])$, as $R \to -\infty$.

(v) The obtained minimizer $\bar{u}$ of $\Phi_{c^\dagger}$ is unique, up to translations.

Proof of Part (i)

The existence of a speed $c^\dagger \geq c$, a function $\bar{u} \in H^1_{c^\dagger}(\Sigma)$ minimizing $\Phi_{c^\dagger}$, and the regularity of $\bar{u}$ can be proved exactly as in [25, Theorem 1.1]. We will outline the proof of this statement here, modifying it in a few parts (so as to not to rely on regularity of $\bar{u}$), in order to be able to apply it in the sequel to this work [31]. The idea is to consider constrained minimizers of Φ_c, i.e., find $u_c \in \mathcal{B}_c$, where

$$\mathcal{B}_c = \left\{ u \in H^1_c(\Sigma) : \int_\Sigma e^{cz+\varphi(y)} u_z^2 \, dx = 2 \right\}, \tag{3.13}$$

which satisfies $\Phi_c[u_c] = \inf_{u\in\mathcal{B}_c} \Phi_c[u]$. Note that by definition $u_c \neq 0$. Then, by hypothesis (H3), we would necessarily have $\Phi_c[u_c] \leq 0$. Note that $\bar{u}(x) \in [0,1]$ for all $x \in \Sigma$, since for any $u \in H^1_c(\Sigma)$ we have $\Phi_c[\tilde{u}] \leq \Phi_c[u]$, where $\tilde{u}$ is the truncation of u:

$$\tilde{u}(x) = \begin{cases} 0, & u(x) < 0, \\ u(x), & 0 \leq u(x) \leq 1, \\ 1, & u(x) > 1, \end{cases} \tag{3.14}$$

and, in fact, this inequality is strict, unless $\tilde{u} = u$ a.e.

Now, suppose the constrained minimizer u_c exists, and let

$$c^\dagger = c\sqrt{1 - \Phi_c[u_c]}. \tag{3.15}$$

Note that since $\Phi_c[u_c] \leq 0$ we have $c^\dagger \geq c$. For any $u \in H^1_{c^\dagger}(\Sigma), u \neq 0$, define $u_a(y,z) = u\left(y, \frac{c(z-a)}{c^\dagger}\right)$ for all $(y,z) \in \Sigma$. Then clearly $u_a \in H^1_c(\Sigma)$, and it is always possible to choose $a \in \mathbb{R}$ such that $u_a \in \mathcal{B}_c$. Assuming that a is chosen this way, we have

$$\begin{aligned} e^{c^\dagger a}\Phi_{c^\dagger}[u] &= \int_\Sigma e^{c^\dagger z + \varphi(y)} \left(\frac{1}{2}|\nabla u(y,z-a)|^2 + V(u(y,z-a),y) \right) dx \\ &= \left(\frac{c}{c^\dagger}\right) \int_\Sigma e^{cz+\varphi(y)} \left\{ \frac{1}{2}\left(\frac{c^\dagger}{c}\right)^2 \left(\frac{\partial u_a}{\partial z}\right)^2 + \frac{1}{2}|\nabla_y u_a|^2 + V(u_a,y) \right\} dx \\ &= \frac{{c^\dagger}^2 - c^2}{2c^\dagger c} \int_\Sigma e^{cz+\varphi(y)} \left(\frac{\partial u_a}{\partial z}\right)^2 dx + \left(\frac{c}{c^\dagger}\right)\Phi_c[u_a] \\ &= \left(\frac{c}{c^\dagger}\right)(\Phi_c[u_a] - \Phi_c[u_c]), \end{aligned} \tag{3.16}$$

where in the computation of the last line in (3.16) we used (3.13) and (3.15). Now, since $\Phi_c[u_c] \leq \Phi_c[u]$ for any $u \in \mathcal{B}_c$, we have $\Phi_{c^\dagger}[u] \geq 0$ for all $u \in H^1_{c^\dagger}(\Sigma)$ and, furthermore, the minimum is attained on $\bar{u}(y,z) = u_c\left(y, \frac{c^\dagger z}{c}\right)$. In other words, $\bar{u}$ is a non-trivial minimizer of $\Phi_{c^\dagger}$.

To prove existence of a constrained minimizer u_c, one picks a minimizing sequence on $\mathcal{B}_c$. Since $\varphi \in L^\infty(\Omega)$, all the arguments in the proofs of Propositions 5.5 and 5.6 of [25] remain valid. The only difference is that in Lemma 5.4 of [25] one needs to estimate $\Phi_c[u,(R,+\infty)]$ with the help of (3.5). The fact that $\bar{u}$ is a classical solution of (3.1), together with gradient estimates, follows by standard regularity theory [20] (see [25, Prop. 3.3]). Indeed, as a minimizer of $\Phi_{c^\dagger}$ the function $\bar{u}$ solves

$$\int_\Sigma e^{c^\dagger z + \varphi(y)} \left(\nabla\bar{u}\cdot\nabla\phi - f(\bar{u},y)\phi \right) dx = 0, \tag{3.17}$$

where $\phi \in H^1_c(\Sigma)$ is an arbitrary test function. This is the weak form of (3.1).

Finally, to prove uniqueness of $c^\dagger$, suppose there exist $c^\dagger_1 > c^\dagger_2 > 0$, and the corresponding non-trivial minimizers are $\bar{u}_{1,2}$, with $\bar{u}_1 \in H^1_{c^\dagger_2}(\Sigma)$ by Lemma 2.3.

Let $\tilde{u}(y,z) = \bar{u}_1\left(y, \frac{c_2^\dagger z}{c_1^\dagger}\right) \in H^1_{c_2^\dagger}$, then

$$\begin{aligned}
\Phi_{c_2^\dagger}[\tilde{u}] &= \left(\frac{c_1^\dagger}{c_2^\dagger}\right) \int_\Sigma e^{c_1^\dagger z+\varphi(y)} \left\{ \frac{1}{2}\left(\frac{c_2^\dagger}{c_1^\dagger}\right)^2 \left(\frac{\partial \bar{u}_1}{\partial z}\right)^2 + \frac{1}{2}|\nabla_y \bar{u}_1|^2 + V(\bar{u}_1, y) \right\} dx \\
&= \left(\frac{c_1^\dagger}{c_2^\dagger}\right) \left(\Phi_{c_1^\dagger}[\bar{u}_1] - \frac{{c_1^\dagger}^2 - {c_2^\dagger}^2}{2{c_1^\dagger}^2} \int_\Sigma e^{c_1^\dagger z+\varphi(y)} \left(\frac{\partial \bar{u}_1}{\partial z}\right)^2 dx \right) < 0. \qquad (3.18)
\end{aligned}$$

But this contradicts existence of a minimizer for $\Phi_{c_2^\dagger}$, which implies that $\Phi_{c_2^\dagger}[u] \geq \Phi_{c_2^\dagger}[\bar{u}_2] = 0$ for all $u \in H^1_{c_2^\dagger}(\Sigma)$.

Proof of Part (ii)

Let us first prove monotonicity of $\bar{u}$. The idea of the proof is related to the one used to prove uniqueness later on in Part (v). We note that an alternative way of proving monotonicity of the minimizers is via a one-dimensional monotone rearrangement (see, e.g., [33]).

First of all, by repeating the arguments of [25, Proposition 3.3(iii)] we may conclude that $\bar{u}(z,\cdot) \to 0$ in $C^0(\overline{\Omega})$ as $z \to +\infty$ (in fact, $\bar{u}(y,z) \leq Ce^{-\lambda z}$ for some $C > 0$ and $\lambda > 0$). Standard regularity estimates [20] then imply that the convergence of $\bar{u}(y, z+R)$ is in fact in $W^{2,p}(\Omega \times (0,1))$, for all $p > 1$, as $R \to +\infty$. Hence, in particular, $\bar{u}(z,\cdot) \to 0$ in $C^1(\overline{\Omega})$.

Now, for any $a > 0$, let us introduce

$$\bar{u}_1(y,z) = \min(\bar{u}(y,z), \bar{u}(y, z-a)), \qquad (3.19)$$
$$\bar{u}_2(y,z) = \max(\bar{u}(y,z), \bar{u}(y, z-a)). \qquad (3.20)$$

According to (3.11), we have

$$0 = \Phi_{c^\dagger}[\bar{u}(y,z)] + \Phi_{c^\dagger}[\bar{u}(y,z-a)] = \Phi_{c^\dagger}[\bar{u}_1] + \Phi_{c^\dagger}[\bar{u}_2], \qquad (3.21)$$

and since also $\Phi_{c^\dagger}[u] \geq 0$ for all $u \in H^1_{c^\dagger}(\Sigma)$, it follows that

$$\Phi_{c^\dagger}[\bar{u}_1] = 0, \qquad \Phi_{c^\dagger}[\bar{u}_2] = 0. \qquad (3.22)$$

Hence, $\bar{u}_1$ and $\bar{u}_2$ are also non-trivial minimizers. Now, consider $w = \bar{u}_2 - \bar{u}_1 \geq 0$. In view of hypothesis (H2), w satisfies an elliptic equation

$$\Delta w + c^\dagger w_z + \nabla_y \varphi \cdot \nabla_y w + k(y,z) w = 0, \qquad (3.23)$$

for some $k \in L^\infty(\Sigma)$. Then, according to the argument following (6.8) in [25, Proposition 6.4], which is based on the Strong Maximum Principle, we conclude that either $w = 0$ or $w > 0$ in Σ. The first possibility would imply that $\bar{u}$ is independent of z and, hence, is zero, which is impossible. So, $w > 0$, implying that $\bar{u}(y, z-a) > \bar{u}(y,z)$ for all $x = (y,z) \in \Sigma$. In view of the arbitrariness of $a > 0$, this implies that $\bar{u}$ is strictly monotone decreasing.

Now, as was shown in Part (i), the minimizer $\bar{u}$ takes values from the unit interval. Therefore, by monotonicity of $\bar{u}$, there exists a function $v : \Omega \to \mathbb{R}$, with values $v(y) \in [0, 1]$ such that $\bar{u}(y, z) \to v(y)$ for all $y \in \Omega$, hence, again by elliptic regularity, $v \in C^1(\overline{\Omega})$ and $u(\cdot, z)$ converges to v in $C^1(\overline{\Omega})$. For any $R \in \mathbb{R}$, fix a test function $\phi(y, z) = \psi(y)\eta_R(z)$ with arbitrary $\psi \in H^1(\Omega)$, $\psi|_{\partial\Omega_\pm} = 0$ and $\eta_R(z) = \eta_0(z - R) \geq 0$, with $\eta_0 \in C_0^\infty(\mathbb{R})$, then (3.17) reads (here and below the prime denotes differentiation with respect to z)

$$\int_{\mathrm{supp}(\eta)} \int_\Omega e^{c^\dagger z + \varphi(y)} (\psi \bar{u}_z \eta_R' + \eta_R \nabla_y \bar{u} \cdot \nabla_y \psi - f(\bar{u}, y)\eta_R \psi) dy dz = 0. \tag{3.24}$$

Multiplying this equation by $e^{-c^\dagger R}$, passing to the limit $R \to -\infty$ in the integral and using Fubini Theorem, we obtain

$$0 = \int_\Omega e^{\varphi(y)} (\nabla_y v \cdot \nabla_y \psi - f(v, y)\psi) dy, \tag{3.25}$$

which is precisely the Frechet derivative of $E[v]$. Therefore, v is a critical point of E and, furthermore, by standard elliptic regularity, we have $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and v satisfies (2.11).

Let us now show the inequality $E[v] < 0$. First, note that $E[v] = \lim\limits_{z \to -\infty} E[\bar{u}(\cdot, z)]$, and $E[\bar{u}(\cdot, z)]$ is a continuous function of z. Let us show that $E[v] \leq E[\bar{u}(\cdot, z)]$ for all $z \in \mathbb{R}$. Indeed, observe that by (3.11) and Fubini Theorem we have

$$0 = \Phi_{c^\dagger}[\bar{u}] = \int_{-\infty}^{+\infty} e^{c^\dagger z} E[\bar{u}(\cdot, z)] dz + \frac{1}{2} \int_\Sigma e^{c^\dagger z + \varphi(y)} \bar{u}_z^2 dx, \tag{3.26}$$

hence there exists some $z_0 \in \mathbb{R}$ such that $E[\bar{u}(y, z_0)] < 0$. Now, if $E[v] > E[\bar{u}(\cdot, z)]$ for some $z \in \mathbb{R}$, we can choose z_0 to be a minimum of $E[\bar{u}(\cdot, z)]$, in view of the fact that $E[\bar{u}(\cdot, z)] \to 0$ as $z \to +\infty$. Then, taking $\tilde{u}(y, z) = \bar{u}(y, z_0)$ for all $z < z_0$, and $\tilde{u}(y, z) = \bar{u}(y, z)$ for all $z \geq z_0$, for any $y \in \Omega$, we find that $\Phi_{c^\dagger}[\tilde{u}] < 0$, contradicting the fact that $\bar{u}$ is a minimizer. Therefore, $E[v] \leq E[\bar{u}(\cdot, z_0)] < 0$.

Proof of Part (iii)

To obtain the decay of $\bar{u}$ as $z \to +\infty$, we explicitly construct the solution for $z > R$, with R large enough, by expanding it into a Fourier series in terms of the eigenfunctions in (3.4) on the cross sections. The arguments below basically formalize the earlier discussion of the decay of the solution at the beginning of this section (see also [7, 41]).

For any $z \in \mathbb{R}$, introduce

$$a_k(z) = \int_\Omega e^{\varphi(y)} \psi_k(y) \bar{u}(y, z)\, dy. \tag{3.27}$$

By standard $W^{2,p}$ estimates for $\bar{u}$ on slices of Σ [20, 25], we have $a_k \in C^{1,\alpha}(\mathbb{R})$ for any $\alpha \in (0, 1)$ and, furthermore, since by Proposition 3.1 the functions ψ_k form a complete orthonormal basis, we obtain [10, Theorem VI.11]

$$\bar{u}(y,z) = \sum_{k=0}^{\infty} a_k(z)\psi_k(y), \qquad \sum_{k=0}^{\infty} a_k^2(z) = \int_\Omega e^{\varphi(y)}\bar{u}^2(y,z)dy, \tag{3.28}$$

where the first series converges in $L^2(\Omega; e^{\varphi(y)}dy)$ for each z. Testing (3.17) with $\phi(y,z) = \psi_k(y)\eta(z)$, where $\eta \in C_0^\infty(\mathbb{R})$ is arbitrary, applying the Fubini Theorem and performing integration by parts, we obtain

$$\int_{-\infty}^{+\infty} e^{c^\dagger z}(a_k'\eta' + (\nu_k a_k + g_k)\eta)dz = 0, \tag{3.29}$$

where we introduced

$$g_k(z) = \int_\Omega e^{\varphi(y)}(f_u(0,y)\bar{u}(y,z) - f(\bar{u}(y,z),y))\psi_k(y)\,dy. \tag{3.30}$$

Note that $g_k \in C^{0,\gamma}(\mathbb{R})$. Again, by standard regularity theory [20], the functions a_k belong to $C^{2,\gamma}(\mathbb{R})$ and satisfy a second-order ordinary differential equation

$$a_k'' + c^\dagger a_k' - \nu_k a_k = g_k. \tag{3.31}$$

Now, think of $g_k(z)$ as the components of a *linear* operator G in the basis of ψ_k's:

$$g_k(z) = \int_\Omega e^{\varphi(y)}(G\bar{u})(y,z)\psi_k(y)dy, \qquad G\bar{u} = (f_u(0,\cdot) - f_u(\tilde{u},\cdot))\bar{u}, \tag{3.32}$$

for some $0 < \tilde{u} < \bar{u}$ by hypothesis (H2), with $\bar{u}$ given by (3.28). Since $f_u(\cdot,y) \in C^{0,\gamma}(\mathbb{R})$, the operator G is a bounded operator from $L^2(\Omega; e^\varphi dy)$ to itself for any fixed $z \in \mathbb{R}$. In the following we freeze $\tilde{u}$ in (3.32) and treat (3.31) as a system of linear ordinary differential equations with $a_k(z) \in l^2$ for all $z \in \mathbb{R}$ or, equivalently, a dynamical system for $\bar{u}(\cdot,z) \in L^2(\Omega; e^{\varphi(y)}dy)$.

Using variation of parameters and keeping in mind that $c^2 + 4\nu_k > 0$ by hypothesis (H3) and Proposition 3.1, one can write the solution for (3.31) in the form

$$\begin{aligned} a_k(z) = a_k^+(R)&e^{-\lambda_+(c^\dagger,\nu_k)(z-R)} + a_k^-(R)e^{-\lambda_-(c^\dagger,\nu_k)(z-R)} \\ &- \frac{1}{\sqrt{{c^\dagger}^2 + 4\nu_k}} \int_R^z e^{\lambda_+(c^\dagger,\nu_k)(\xi - z)} g_k(\xi)d\xi \\ &+ \frac{1}{\sqrt{{c^\dagger}^2 + 4\nu_k}} \int_R^z e^{\lambda_-(c^\dagger,\nu_k)(\xi - z)} g_k(\xi)d\xi, \end{aligned} \tag{3.33}$$

where $a_k^\pm(R)$ are constants of integration satisfying $a_k^+(R) + a_k^-(R) = a_k(R)$. In particular, if $\lambda_-(c^\dagger,\nu_k) < 0$, we have

$$a_k^-(R) = -\frac{1}{\sqrt{{c^\dagger}^2 + 4\nu_k}} \int_R^{+\infty} e^{\lambda_-(c^\dagger,\nu_k)(\xi - R)} g_k(\xi)d\xi, \tag{3.34}$$

which is obtained by multiplying (3.33) by $e^{\lambda_-(c^\dagger,\nu_k)z}$ and passing to the limit $z \to +\infty$, taking into account boundedness of a_k's and g_k's.

Now, by hypothesis (H2) we have $||G(z)||_{L^2(\Omega;e^{\varphi(y)}dy)} \leq C||\bar{u}(\cdot,z)||^{\gamma}_{L^\infty(\Omega)}$, hence, in particular, $||G(z)||_{L^2(\Omega;e^{\varphi(y)}dy)} = O(e^{-\mu z})$ for some $\mu > 0$ (see the discussion at the beginning of the proof of Part (ii); this condition is only needed if $\lambda_-(c^\dagger,\nu_k) = 0$ for some k). Then, it is easy to see that with $a_k^+(R)$ fixed for all k and with $a_k^-(R)$ fixed whenever $\lambda_-(c^\dagger,\nu_k) \geq 0$, the mapping defined by (3.33) is a contraction for sufficiently large R in the Banach space with the norm

$$||\bar{u}|| = \sup_{z\in[R,+\infty)} ||\bar{u}(\cdot,z)||_{L^2(\Omega;e^{\varphi(y)}dy)}. \tag{3.35}$$

Indeed, denoting the operator generated by the right-hand side of (3.33) as T and introducing u_1, u_2 as described above, after some straightforward calculations we obtain

$$||T(u_1 - u_2)|| \leq Ce^{-\mu R/2}||u_1 - u_2||. \tag{3.36}$$

In arriving at the last estimate we used the fact that the sequences (ν_k), $(\lambda_+(c^\dagger,\nu_k))$, and $(-\lambda_-(c^\dagger,\nu_k))$ are monotone increasing.

So, T is a contraction, and so for any fixed $a_k^+(R)$, and for any fixed $a_k^-(R)$ corresponding to $\lambda_-(c^\dagger,\nu_k) \geq 0$ there is a unique solution whose $L^2(\Omega;e^{\varphi(y)}dy)$ norm is uniformly bounded on $[R,+\infty)$. Moreover, by the estimate (2.7) of Lemma 2.2, we have $|a_k(z)| \leq Ce^{-c^\dagger z/2}$ for all k, which implies that (3.34) in fact holds whenever $\lambda_-(c^\dagger,\nu_k) \geq 0$ as well, since $\lambda_-(c^\dagger,\nu_k) < \frac{c^\dagger}{2}$ by hypothesis (H3).

Let us now show that the value of $\lambda_+(c^\dagger,\nu_0)$ determines the exponential rate of decay of the solution at $z = +\infty$. For that, it is necessary that (3.34) does not hold with $\lambda_-(c^\dagger,\nu_0)$ replaced with $\lambda_+(c^\dagger,\nu_0)$ and $a_k^-(R)$ replaced by $a_k^+(R)$. Otherwise, there exists $k = k_0$ such that this equation does not hold (the opposite implies that $\bar{u} = 0$ in $\Omega \times (R,+\infty)$). Then $a_k = a_k^+(R)e^{-\lambda(z-R)} + O(e^{-(1+\frac{1}{2}\gamma)\lambda z})$ for all $k_0 \leq k \leq k_1$ for which $\lambda = \lambda_+(c^\dagger,\nu_k)$ (with at least one $a_k^+(R) \neq 0$), and $a_k = O(e^{-(1+\frac{1}{2}\gamma)\lambda z})$ for all other k's. That k_1 is finite follows from the fact that $\lambda_+(c^\dagger,\nu)$ is a strictly monotonically increasing function of ν, and that by Proposition 3.1 the eigenvalues of ν_k have finite multiplicities and $\nu_k \to +\infty$. In view of these estimates we have

$$\bar{u}(y,z) = \sum_{k=k_0}^{k_1} a_k^+(R)e^{-\lambda_+(c^\dagger,\nu_{k_0})(z-R)}\psi_k(y) + o(e^{-\lambda_+(c^\dagger,\nu_{k_0})z}), \tag{3.37}$$

Therefore, by orthogonality of all ψ_k's to $\psi_0 > 0$ for $k \geq k_0$ and the fact that these ψ_k's change sign, we see that $\bar{u}(\cdot,z)$ will become negative somewhere on a set of non-zero measure in Ω. This is clearly impossible, and so we finally obtain the estimate

$$\bar{u}(y,z) = a_0^+(R)e^{-\lambda_+(c^\dagger,\nu_0)(z-R)}\psi_0(y) + O(e^{-\lambda z}), \tag{3.38}$$

with $\lambda = \min\{\lambda_+(c^\dagger,\nu_1), (1+\frac{1}{2}\gamma)\lambda_+(c^\dagger,\nu_0)\} > \lambda_+(c^\dagger,\nu_0)$, in $L^2(\Omega, e^{\varphi(y)}dy)$ for each z. By construction $a_0(z) > 0$, and so $a_k^+(R) > 0$ for large enough R.

Finally, consider the function $w(y,z) = \bar{u}(y,z) - a_0^+(R)e^{-\lambda_+(c^\dagger,\nu_0)(z-R)}\psi_0(y)$ which satisfies a linear equation in $\Omega \times (R, +\infty)$

$$\begin{aligned} \Delta w + c^\dagger w_z + \nabla_y \varphi \cdot \nabla_y w + f_u(0,y)w - Gw \\ = a_0^+(R)G\psi_0(y)e^{-\lambda_+(c^\dagger,\nu_0)(z-R)} \end{aligned} \tag{3.39}$$

Since for each z both w and the right-hand side of this equation are $O(e^{-\lambda R})$ in $L^2(\Omega \times (R, R+1), e^{\varphi(y)}dy)$ with $\lambda > \lambda_+(c^\dagger, \nu_0)$, standard elliptic regularity theory [20, Theorem 9.13] implies that w is $O(e^{-\lambda R})$ in $W^{2,2}(\Omega \times (R + \frac{1}{4}, R + \frac{3}{4}))$ and hence, by Sobolev imbedding, in $L^p(\Omega \times (R+\frac{1}{4}, R+\frac{3}{4}), e^{\varphi(y)}dy)$ for some $p > 2$. So, the above estimate in fact holds in $L^p(\Omega \times (R, R+1), e^{\varphi(y)}dy)$. Iterating this argument using $W^{2,p}$ estimates until the space imbeds into $C^1(\overline{\Omega} \times (R+\frac{1}{4}, R+\frac{3}{4}))$, we obtain the result.

Proof of Part (iv)

When $\tilde{\nu}_0 > 0$, the proof follows exactly as in Part (iii), where we do not need an a priori estimate on the exponential decay of $\bar{u}(\cdot, z)$ to v as $z \to -\infty$ any more, since all $\tilde{\nu}_k > 0$, and hence $\lambda_-(c^\dagger, \tilde{\nu}_k) < 0$ and $\lambda_+(c^\dagger, \tilde{\nu}_k) > 0$ for all k.

To prove that $\tilde{\nu}_0 < 0$ is impossible, consider the analog of (3.31) with $k = 0$:

$$\tilde{a}_0'' + c^\dagger \tilde{a}_0' - \tilde{\nu}_0 \tilde{a}_0 = \tilde{g}_0. \tag{3.40}$$

Observe that since $\bar{u}(\cdot, z) \to v$ uniformly as $z \to -\infty$, by hypothesis (H2) and the fact that $\tilde{\psi}_0 > 0$ and $\bar{u} - v < 0$ we have

$$|\tilde{g}_0(z)| \le C \int_\Omega e^{\varphi(y)}(v(y) - \bar{u}(y,z))^{1+\gamma}\tilde{\psi}_0(y)dy \le \varepsilon |a_0(z)|, \tag{3.41}$$

for any $\varepsilon > 0$, as long as z is sufficiently large negative. It is then easy to see (using e.g. variation of parameters) that (3.40) does not have bounded solutions for $z < R$ when ε is small enough.

Proof of Part (v)

Our proof of uniqueness is based on the argument due to Heinze [21]. Suppose that $\bar{u}_1$ and $\bar{u}_2$ are two non-trivial minimizers of Φ_c. Then, there exists a translation a such that $\bar{u}_1(y^*, z^*) = \bar{u}_2(y^*, z^* - a)$ at some point $x^* = (y^*, z^*) \in \Sigma$. Indeed, if not, then without loss of generality we can assume that $\bar{u}_1(y,z) < \bar{u}_2(y, z-a)$ for all $x = (y,z) \in \Sigma$ and all $a \in \mathbb{R}$. Also, by the result of Part (iii), $\bar{u}_2(y, z-a) \to 0$ as $a \to -\infty$, hence, $\bar{u}_1 = 0$, contradicting the assumption that $\bar{u}_1$ is a non-trivial minimizer. So, $\bar{u}_1(y^*, z^*) = \bar{u}_2(y^*, z^* - a)$, and let us introduce

$$\bar{u}_3(y,z) = \min(\bar{u}_1(y,z), \bar{u}_2(y, z-a)), \tag{3.42}$$

$$\bar{u}_4(y,z) = \max(\bar{u}_1(y,z), \bar{u}_2(y, z-a)). \tag{3.43}$$

Arguing as in Part (ii), we have

$$0 = \Phi_c[\bar{u}_1] + \Phi_c[\bar{u}_2] = \Phi_c[\bar{u}_3] + \Phi_c[\bar{u}_4] \Rightarrow \Phi_c[\bar{u}_3] = \Phi_c[\bar{u}_4] = 0. \tag{3.44}$$

Therefore, $\bar{u}_3$ and $\bar{u}_4$ are also minimizers of Φ_c, and $w = \bar{u}_4 - \bar{u}_3 \geq 0$. Once again, using the arguments following (3.23) and taking into account that $w(x^*) = 0$, from Strong Maximum Principle we conclude that $w(x) = 0$ in all of Σ. So, $\bar{u}_1(y, z) = \bar{u}_2(y, z - a)$ for all $x = (y, z) \in \Sigma$.

This completes the proof of Theorem 3.3. $\square$

Let us note that if the nonlinearity f is independent of y and the boundary conditions are Neumann, then the solution is essentially one-dimensional.

Proposition 3.4. *Let $\bar{u}$ be a solution obtained in Theorem 3.3, and assume that $\nabla_y f = 0$ and $\partial\Sigma_\pm = \varnothing$. Then $\bar{u}$ depends only on the variable z.*

Proof. The proof follows directly from the argument of Proposition 6.3 of [25]. $\square$

In view of Proposition 3.4, the planar front solutions of Proposition 3.4 are also the fastest variational traveling waves among all waves with fixed y-independent nonlinearity and different choices of the boundary conditions.

The proof of Parts (iii), (iv) of Theorem 3.3 relied only on the fact that the minimizer is sandwiched between the two equilibria it connects. Using the same arguments as in Part (iii) of Theorem 3.3, it is also easy to show that for a variational traveling wave one should have $c^2 + 4\nu_0 > 0$ in order for the wave to have the right decay. So we have

Proposition 3.5. *Let $u_c \in H^1_c(\Sigma)$ be a solution of (3.1) which also satisfies $0 < u_c < v$, where $v = \lim_{z\to-\infty} u_c(\cdot, z)$ uniformly in Ω. Then, $c^2 + 4\nu_0 > 0$, and statement (iii) of Theorem 3.3 holds for u_c. If, in addition, $\tilde{\nu}_0 \neq 0$, statement (iv) of Theorem 3.3 holds for u_c as well.*

More generally, since, according to Proposition 3.5 and (3.11), any constant sign variational traveling wave is a trial function satisfying hypothesis (H3), the minimizer obtained in Theorem 3.3 is the fastest variational traveling wave. In other words, we have

Proposition 3.6. *If $u_c \in H^1_c(\Sigma)$ is a solution of (3.1), as in Proposition 3.5, then $c \leq c^\dagger$.*

In fact, the following stronger statement concerning *all* variational traveling waves that connect the same equilibria as the minimizer holds.

Proposition 3.7. *Let $u_c \in H^1_c(\Sigma)$ be a solution of (3.1), and let $0 < u_c < v$, where $v = \lim_{z\to-\infty} u_c(y, z)$ is the same as in Theorem 3.3, and $\tilde{\nu}_0 \neq 0$. Then $(c, u_c) = (c^\dagger, \bar{u})$, where the pair $(c^\dagger, \bar{u})$ is the solution obtained in Theorem 3.3, up to translation.*

Proof. First of all, in view of Proposition 3.5 the fact that $u_c \in H^1_c(\Sigma)$ implies that u_c has the decay specified in Part (iii) of Theorem 3.3. By direct inspection, $\lambda_+(c, \nu_0) > 0$ and $\lambda_-(c, \tilde{\nu}_0) < 0$ are both increasing functions of c. Therefore, when $c < c^\dagger$, the solution u_c decays to zero slower exponentially than $\bar{u}$ as $z \to +\infty$, and faster to v as $z \to -\infty$. So, it is possible to translate $\bar{u}$ sufficiently far towards $z = -\infty$ to achieve $\bar{u} < u_c$ in Σ. Then, using Comparison Principle for parabolic equations [34] applied to the corresponding traveling wave solutions of (1.1), we see that u_c must move no slower than $c^\dagger$, which is impossible. Repeating this argument for $c > c^\dagger$, except now one has to translate $\bar{u}$ towards $z = +\infty$ to get an appropriate supersolution, we obtain a contradiction once more. □

In other words, there are no other variational traveling waves which are sandwiched between the same equilibria as the minimizer of $\Phi_{c^\dagger}$ obtained in Theorem 3.3. We also point out that under an assumption of non-degeneracy and uniqueness of the local minimizer $v_0 > 0$ with $E[v_0] < 0$ (which is then the global minimizer, see the discussion at the end of Section 2), the pair $(c^\dagger, \bar{u})$ from Theorem 3.3 is in fact the *only* variational traveling wave solution. In general, however, there may exist other variational traveling waves with speeds $c < c^\dagger$ which connect $u = 0$ to a local minimum of E other than v in Part (ii) of Theorem 3.3.

Remark 3.8. *In view of Proposition 3.5 and equation (3.11), existence of a minimizer necessarily implies that hypothesis (H3) is true. Thus, hypothesis (H3) is both necessary and sufficient for existence of variational traveling waves (this fact was already pointed out in [24] in the case $\Sigma = \mathbb{R}$).*

Now we would like to get back to considering an important special case $\nu_0 \geq 0$, i.e. the case when $u = 0$ is a locally stable (or marginally stable) solution of (1.1). Here, to satisfy hypothesis (H3) we just need to find a non-trivial trial function $u \in H^1_c(\Sigma)$ for which $\Phi_c[u] \leq 0$ for some small enough $c > 0$. In fact, the following stronger version of Theorem 3.3 holds.

Theorem 3.9. *Assume that hypotheses (H1) and (H2) hold, and that $\nu_0 \geq 0$ in (2.9). Then the statements of Theorem 3.3 remain true, if and only if*

$$\inf_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega_\pm} = 0}} E[v] < 0. \tag{3.45}$$

Proof. The proof follows from a straightforward extension of the arguments of Proposition 6.2 of [25]. □

In other words, the statement of Theorem 3.9 holds if and only if there exists a non-trivial minimizer of E in the admissible class. In particular, if v_0 is the unique critical point with negative energy (necessarily the minimizer) and $\nu_0 > 0$, then there is a unique pair $(c^\dagger, \bar{u})$ solving (3.1). Indeed, by Proposition 3.7 the minimizer in Theorem 3.9 is the only variational traveling wave, and there are no other traveling wave solutions for $\nu_0 > 0$. Note that the same statement is true for the ignition-type nonlinearity from combustion theory under the assumption of uniqueness of v_0.

4 Minimal waves

What if hypothesis (H3) is *not* satisfied? In this case, there are obviously no variational traveling wave solutions. However, if $\nu_0 < 0$, then it is possible to have traveling wave solutions which do not lie in $H^1_c(\Sigma)$. The following proposition gives a general sufficient condition for non-existence of minimizers for Φ_c and is a generalization of the earlier results of [24, 25].

Proposition 4.1. *Under hypotheses (H1) and (H2), assume that $\nu_0 < 0$ and*

$$\frac{2}{u^2}\int_0^u f(s,y)ds \le f_u(0,y), \qquad \forall y \in \Omega. \tag{4.1}$$

Then the functional Φ_c has no non-trivial minimizers.

Proof. Let us first show that under this assumption $\Phi_c[u] = 0$ implies $u = 0$ for all $c \ge c_0$, where c_0 is given by (4.6). After an integration by parts, we can write

$$\Phi_c[u] = \int_\Sigma e^{cz+\varphi(y)} \left\{ \frac{1}{2}\left(u_z + \frac{c}{2}u\right)^2 + \frac{1}{2}|\nabla_y u|^2 + \frac{c^2}{8}u^2 + V(u,y) \right\} dx. \tag{4.2}$$

By the assumption of the proposition we have $V(u,y) \ge -\frac{1}{2}f_u(0,y)u^2$, and so

$$\begin{aligned}
\Phi_c[u] &\ge \frac{1}{2}\int_\Sigma e^{cz+\varphi(y)} \left\{ \left(u_z + \frac{c}{2}u\right)^2 + |\nabla_y u|^2 + \left(\frac{c^2}{4} - f_u(0,y)\right)u^2 \right\} dx \\
&\ge \frac{1}{2}\int_\Sigma e^{cz+\varphi(y)} \left\{ \left(u_z + \frac{c}{2}u\right)^2 + \left(\frac{c^2}{4} + \nu_0\right)u^2 \right\} dx. \qquad (4.3)
\end{aligned}$$

The second term in the integrand above is non-negative for $c \ge c_0$, so $\Phi_c[u] = 0$ would imply that u is a minimizer and that the first term in the integrand is equal to zero. That, in turn, means that $u = v(y)e^{-cz/2}$ for some $v : \Omega \to \mathbb{R}$, and, in view of boundedness of the minimizers of Φ_c, we have $v \equiv 0$.

Let us now show that when $c < c_0$, the functional Φ_c is not bounded from below. For that, consider a trial function

$$u_\lambda(y,z) = \begin{cases} a\psi_0(y)e^{-\lambda z}, & z > 0, \\ a\psi_0(y), & z \le 0, \end{cases} \tag{4.4}$$

where $\psi_0 > 0$ is the zeroth eigenfunction of the operator in (3.4) and $\lambda > \frac{c}{2}$. Choosing $a > 0$ small enough, we can always make $V(u_\lambda(y,z),y) \le -\frac{1}{2}(f_u(0,y) - \varepsilon)u_\lambda^2$ for any $\varepsilon > 0$. Plugging u_λ into the functional, we obtain

$$\begin{aligned}
\Phi_c[u_\lambda] &\le \frac{1}{2}\int_{-\infty}^0 \int_\Omega e^{cz+\varphi(y)} \{|\nabla_y u_\lambda|^2 - (f_u(0,y) - \varepsilon)u_\lambda^2\}\, dydz \\
&\quad + \frac{1}{2}\int_0^{+\infty} \int_\Omega e^{cz+\varphi(y)} \{|\nabla_y u_\lambda|^2 + (\lambda^2 + \varepsilon - f_u(0,y))u_\lambda^2\}\, dydz \\
&= \frac{a^2}{2c}\int_\Omega e^{\varphi(y)} (|\nabla\psi_0|^2 - (f_u(0,y) - \varepsilon)\psi_0^2)\, dy \\
&\quad + \frac{a^2}{2(2\lambda - c)}\int_\Omega e^{\varphi(y)} (|\nabla_y\psi_0|^2 + (\lambda^2 + \varepsilon - f_u(0,y))\psi_0^2)\, dy \\
&= \frac{a^2}{2}\left(\frac{\nu_0 + \varepsilon}{c} + \frac{\lambda^2 + \varepsilon + \nu_0}{2\lambda - c}\right)\int_\Omega e^{\varphi(y)}\psi_0^2\, dy. \qquad (4.5)
\end{aligned}$$

It is then easy to see that the last line in the expression above can be made arbitrarily large negative for sufficiently small ε and $c < c_0$ by choosing λ sufficiently close to $\frac{c}{2}$. Therefore, in this case the functional Φ_c has no minimizers. $\square$

A typical example of the situation in which Φ_c has no non-trivial minimizers is the KPP-type nonlinearity of (1.4). Non-existence of variational traveling waves in this case follows from the above Proposition. However, our variational procedure allows us to establish existence of an important class of traveling wave solutions, having the speed which is equal to the minimal speed $c = c_0$, where c_0 is defined in (4.6), allowed for a positive traveling wave solution. As in the case of the minimizers, this solution turns out to determine the asymptotic propagation speed for sufficiently rapidly decaying front-like initial data (see Sec. 5). Thus, to summarize the results of Theorem 3.3 and Theorem 4.2 below, under the condition in (3.45) there always exists a positive monotone traveling wave solution with speed satisfying $c^2 + 4\nu_0 \geq 0$ which decays exponentially at $z = +\infty$.

Theorem 4.2. *Assume that hypotheses (H1) and (H2) hold, whereas hypothesis (H3) is not satisfied. Assume in addition that $\nu_0 < 0$. Then, there exists $\bar{u}_0 \in C^2(\Sigma) \cap W^{1,\infty}(\Sigma)$ which solves (3.1) with $c = c_0$, where*

$$c_0 = 2\sqrt{-\nu_0}. \tag{4.6}$$

Furthermore, $\bar{u}_0$ has the limiting behavior

$$\bar{u}_0(y, z) = (a_0 + b_0 z)\, e^{-\frac{1}{2}c_0 z}\psi_0(y) + O(e^{-\lambda z}), \tag{4.7}$$

for some $\lambda > \frac{c_0}{2}$ and either $b_0 > 0$ or $b_0 = 0, a_0 > 0$, as $z \to +\infty$, and assertions (ii) and (iv) of Theorem 3.3 still hold for u_0.

Proof. We prove this theorem by approximating the solution $(c_0, \bar{u}_0)$ of (3.1) with pairs $(c_\varepsilon, \bar{u}_\varepsilon)$ solving

$$\Delta\bar{u}_\varepsilon + c_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial z} + \nabla_y\varphi \cdot \nabla_y \bar{u}_\varepsilon + f_\varepsilon(\bar{u}_\varepsilon, y) = 0, \tag{4.8}$$

$$\bar{u}_\varepsilon\big|_{\partial\Sigma_\pm} = 0, \qquad \nu \cdot \nabla\bar{u}_\varepsilon\big|_{\partial\Sigma_0} = 0, \tag{4.9}$$

where

$$f_\varepsilon(u, y) = f(u, y) - \frac{\varepsilon^\gamma K u}{\varepsilon^\gamma + u^\gamma}, \qquad K = \max_{y\in\overline{\Omega}} f_u(0, y) > 0. \tag{4.10}$$

Associated with f_ε are the function V_ε and the functionals Φ_c^ε, E_ε, and R_ε, defined with f_ε in place of f. Note that by the definition of f_ε we have

$$0 \leq f(u, y) - f_\varepsilon(u, y) \leq \varepsilon^\gamma K, \qquad \forall u \in [0, 1], \quad \forall y \in \Omega, \tag{4.11}$$

and $f_\varepsilon(u, y)$ is a monotonically decreasing function of ε.

Observe that the assumption $\nu_0 < 0$ implies (3.45), since for $a > 0$ sufficiently small we have

$$\inf E \leq E[a\psi_0] = \tfrac{1}{2}\nu_0 a^2 + o(a^2) < 0. \tag{4.12}$$

So, by continuity $\inf E_\varepsilon < 0$ for sufficiently small ε. We also have $\left.\frac{\partial f_\varepsilon(u,y)}{\partial u}\right|_{u=0} \leq 0$ for all $y \in \Omega$. Hence $\nu_0^\varepsilon \geq 0$, where ν_0^ε is defined as the minimum of R_ε. Then, by Theorem 3.9 there exists a pair $(c_\varepsilon, \bar{u}_\varepsilon)$, with $\bar{u}_\varepsilon \in H^1_{c_\varepsilon}(\Sigma)$, which is the minimizer of $\Phi^\varepsilon_{c_\varepsilon}$, with all the properties given by Theorem 3.3. In particular, we have

$$\lim_{z\to-\infty} \bar{u}_\varepsilon(y,z) = v_\varepsilon(y) \quad \text{in } C^1(\overline{\Omega}), \tag{4.13}$$

$$\Delta_y v_\varepsilon + \nabla_y \varphi \cdot \nabla v_\varepsilon + f_\varepsilon(v_\varepsilon, y) = 0, \quad E_\varepsilon[v_\varepsilon] < 0. \tag{4.14}$$

We are now going to demonstrate that v_ε are uniformly bounded away from zero as $\varepsilon \to 0$.

The proof comes in two steps. First, we show that any critical point v_ε of E_ε such that $E_\varepsilon[v_\varepsilon] < 0$ has $||v_\varepsilon||_{L^\infty(\Omega)} \geq C\varepsilon^{1/2}$, with some $C > 0$, for small enough ε. Indeed, multiplying (4.14) by $e^{\varphi(y)} v_\varepsilon$ and integrating over Ω, and then substituting the result into the definition of E_ε, we get

$$\begin{aligned} E_\varepsilon[v_\varepsilon] = \int_\Omega e^{\varphi(y)} \left\{ V_\varepsilon(v_\varepsilon, y) + \tfrac{1}{2} v_\varepsilon f_\varepsilon(v_\varepsilon, y) \right\} dy = \\ \frac{1}{2} \int_\Omega e^{\varphi(y)} \int_0^{v_\varepsilon} \left(s \frac{\partial f_\varepsilon(s,y)}{\partial s} - f(s,y) \right) ds\, dy = \\ \frac{1}{2} \int_\Omega e^{\varphi(y)} \int_0^{v_\varepsilon} s^2 \frac{\partial}{\partial s} \left(\frac{f_\varepsilon(s,y)}{s} \right) ds \geq 0, \end{aligned} \tag{4.15}$$

whenever $v_\varepsilon \leq C\varepsilon^{1/2}$ for some $C > 0$ and ε small enough. The last inequality follows by direct computation and taking into account that $f(\cdot, y) \in C^{1,\gamma}([0,1])$, uniformly for all $y \in \Omega$.

In the second step, we demonstrate that $||v_\varepsilon||_{L^\infty(\Omega)} \geq C\varepsilon^{1/2}$ implies $||v_\varepsilon||_{L^\infty(\Omega)} \geq \delta$ for some $\delta > 0$ independent of ε, for small enough ε. Indeed, assume that $||v_\varepsilon||_{L^\infty(\Omega)} \to 0$ and define $\tilde{v}_\varepsilon = v_\varepsilon / ||v_\varepsilon||_{L^\infty(\Omega)}$. Then, from (4.14) $\tilde{v}_\varepsilon$ satisfies

$$\Delta_y \tilde{v}_\varepsilon + \nabla_y \varphi \cdot \nabla \tilde{v}_\varepsilon + f_u(0,y) \tilde{v}_\varepsilon = g_\varepsilon, \tag{4.16}$$

where the right-hand side may be estimated as

$$||g_\varepsilon||_{L^\infty(\Omega)} \leq C ||v_\varepsilon||^\gamma_{L^\infty(\Omega)} + \varepsilon^\gamma K ||v_\varepsilon||^{-\gamma}_{L^\infty(\Omega)}. \tag{4.17}$$

By previous result, we have $\varepsilon ||v_\varepsilon||^{-1}_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$, hence by assumption $||g_\varepsilon||_{L^\infty(\Omega)} \to 0$ as well. So, since by standard elliptic regularity theory the functions v_ε are uniformly bounded in $W^{2,p}(\Omega)$ for all $p > n$, on a suitable sequence of $\varepsilon \to 0$ we have $\tilde{v}_\varepsilon \to \tilde{v}_0$, where $\tilde{v}_0$ solves (4.16) with $g_\varepsilon = 0$, and, furthermore, $||\tilde{v}_0||_{L^\infty(\Omega)} = 1$ and $\tilde{v}_0 \geq 0$ (in fact, by Strong Maximum Principle $\tilde{v}_0 > 0$) [20]. This gives a contradiction, since, recalling the fact that $\nu_0 < 0$, the kernel of the linear operator in the left-hand side of (4.16) does not contain functions which are positive throughout Ω.

Now, observe that in view of monotonicity of V_ε as a function of ε the sequence of c_ε is monotone increasing. Furthermore, it is bounded from above by c_0. Indeed, $\Phi_{c_\varepsilon}[\bar{u}_\varepsilon] \leq \Phi^\varepsilon_{c_\varepsilon}[\bar{u}_\varepsilon] = 0$, and if $c_\varepsilon > c_0$, then the pair $(c_\varepsilon, \bar{u}_\varepsilon)$ satisfies hypothesis

(H3) for the original problem, which is false. So, $c_\varepsilon \le c_0$. Let us show that in fact $c_0 = \lim_{\varepsilon\to 0} c_\varepsilon$. Indeed, for $a > 0$ fixed consider a trial function

$$\tilde{u}_\varepsilon(y,z) = \begin{cases} a\psi_0(y), & z < 0, \\ ae^{-cz/2}\psi_0(y), & 0 \le z \le R, \\ ae^{-cR/2}\left(1 - \frac{c(z-R)}{2}\right)\psi_0(y), & R \le z \le R + \frac{2}{c}, \\ 0, & z > R + \frac{2}{c}. \end{cases} \tag{4.18}$$

for some $R > 0$. By construction, $\tilde{u}_\varepsilon \in H^1_c(\Sigma)$. Then

$$\begin{aligned} \Phi^\varepsilon_c[\tilde{u}_\varepsilon] &= \int_{-\infty}^0 \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V_\varepsilon(\tilde{u}_\varepsilon, y)\right) dy\, dz \\ &\quad + \int_0^{+\infty} \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V_\varepsilon(\tilde{u}_\varepsilon, y)\right) dy\, dz \\ &= \frac{1}{c}\int_\Omega e^{\varphi(y)} \left(\frac{a^2}{2}|\nabla_y \psi_0|^2 + V_\varepsilon(a\psi_0(y), y)\right) dy \\ &\quad + \int_0^R \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V(\tilde{u}_\varepsilon, y)\right) dy\, dz \\ &\quad + \int_0^R \int_\Omega e^{cz+\varphi(y)} \left(V_\varepsilon(\tilde{u}_\varepsilon, y) - V(\tilde{u}_\varepsilon, y)\right) dy\, dz \\ &\quad + \int_R^{R+\frac{2}{c}} \int_\Omega e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla \tilde{u}_\varepsilon|^2 + V_\varepsilon(\tilde{u}_\varepsilon, y)\right) dy\, dz. \end{aligned} \tag{4.19}$$

By hypothesis (H2), it is possible to choose the constant a such that

$$V(\tilde{u}_\varepsilon, y) \le -\tfrac{1}{2}\left(f_u(0,y) - \delta\right)\tilde{u}_\varepsilon^2, \qquad \forall \delta > 0. \tag{4.20}$$

Also, we have uniform estimates $|V_\varepsilon(u,y)| \le Cu^2/2$, where C is independent of ε or y, and $V_\varepsilon(u,y) - V(u,y) \le \varepsilon^\gamma K u$, in view of (4.11). Therefore, continuing the argument above, we can write

$$\begin{aligned}
\Phi_c^\varepsilon[\tilde{u}_\varepsilon] &\le \frac{a^2}{2c}\int_\Omega e^{\varphi(y)}\left(|\nabla_y\psi_0|^2 + C\psi_0^2\right)dy \\
&+\frac{a^2}{2}\int_0^R\int_\Omega e^{\varphi(y)}\left(\frac{c^2}{4}\psi_0^2 + |\nabla_y\psi_0|^2 - f_u(0,y)\psi_0^2 + \delta\psi_0^2\right)dy\,dz \\
&+K\varepsilon^\gamma a e^{cR/2}\int_0^R\int_\Omega e^{\varphi(y)}\psi_0 dy\,dz \\
&+\frac{a^2}{2}\int_R^{R+\frac{2}{c}}\int_\Omega e^{c(z-R)+\varphi(y)}\left(\frac{c^2}{4}\psi_0^2 + |\nabla_y\psi_0|^2 + C\psi_0^2\right)dy\,dz. \\
&\le \frac{a^2}{2c}\int_\Omega e^{\varphi(y)}\left(|\nabla_y\psi_0|^2 + C\psi_0^2\right)dy \\
&+\frac{a^2R}{8}(c^2+4\nu_0-\delta)\int_\Omega e^{\varphi(y)}\psi_0^2\,dy + K\varepsilon^\gamma aRe^{cR/2}\int_\Omega e^{\varphi(y)}\psi_0 dy \\
&+\frac{9a^2}{c}\int_\Omega e^{\varphi(y)}\left(\frac{c^2}{4}\psi_0^2 + |\nabla_y\psi_0|^2 + C\psi_0^2\right)dy \\
&= a^2M_1R(c^2+4\nu_0+\delta) + \varepsilon^\gamma aM_2Re^{cR/2} + a^2M_3, \qquad (4.21)
\end{aligned}$$

where M's are positive constants independent of $\varepsilon, \delta, a, R$.

Now, for any positive $c < c_0$ it is possible to choose $\delta > 0$ small enough such that $c^2 + 4\nu_0 + \delta < 0$. This fixes the value of a. Then, choose R large enough, so that $M_1R(c^2 + 4\nu_0 + \delta) + M_3 < 0$. Finally, there exists $\varepsilon > 0$ small enough such that the term multiplying M_2 is sufficiently small, so that the expression in the last line of (4.21) remains negative. This implies that $\Phi_c^\varepsilon[\tilde{u}_\varepsilon] \le 0$, so that $c_\varepsilon \ge c$ for ε small enough. In view of the arbitrariness of c, this implies $c_\varepsilon \to c_0$.

Now we construct the limit function $\bar{u}_0$ and show that it satisfies (3.1) with $c = c_0$. Recalling Remark 3.2, the assumption $\nu_0 < 0$ implies that there exists $\delta > 0$ such that 0 is the only critical point of E taking values in $[0, \delta]$. In particular, we can choose δ small enough, so that $\max_{\overline{\Omega}} v_\varepsilon > \delta$ for ε small enough. Recalling the monotonicity and the limit behavior of $\bar{u}_\varepsilon$, as $z \to \pm\infty$, after an appropriate translation we can assume that $\bar{u}_\varepsilon$ satisfies

$$\max_{y\in\overline{\Omega}} \bar{u}_\varepsilon(y, 0) = \delta \qquad (4.22)$$

for small enough ε. As the functions $\bar{u}_\varepsilon$ are uniformly bounded in $W^{1,\infty}(\Sigma)$ and in $W^{2,p}_{\text{loc}}(\Sigma)$, we can pass to the limit as $\varepsilon \to 0$, and obtain a function $\bar{u}_0 \in C^2(\Sigma) \cap W^{1,\infty}(\Sigma)$ which solves (3.1) with speed $c = c_0$. Moreover, we have $0 \le \bar{u}_0 \le 1$, and $\bar{u}_0$ satisfies (4.22) and, hence, is not identically zero. Furthermore, $\bar{u}_0$ is non-increasing in the z-variable, and so by Strong Maximum Principle we have $0 < \bar{u}_0 < 1$ and $\partial\bar{u}_0/\partial z < 0$ in Σ.

Reasoning as in the proof of [25, Proposition 6.6], we can show that $\bar{u}_0$ connects two critical points $v_\pm$ of E, for $z \to \pm\infty$ respectively, with $0 \le v_+ \le \delta$ and $E[v_-] < E[v_+]$. By (4.22) and Remark 3.2 it follows that $v_+ = 0$ and $v_- = v$, where $0 < v \le 1$, and hence $E[v] < 0$.

The asymptotic expansion in (4.7) follows from exactly the same arguments as in the proof of Part (iii) of Theorem 3.3. The only ingredient that is missing here is an a priori estimate of exponential decay of the solution (since it may no longer lie in any of the exponentially weighted Sobolev spaces $H^1_c(\Sigma)$). To overcome this difficulty, consider (3.31) with $k = 0$:

$$a_0'' + c_0 a_0' + \tfrac{1}{4} c_0^2 a_0 = g_0. \tag{4.23}$$

By the same argument as the one leading to (3.41), we have $|g_0(z)| \leq \varepsilon a_0(z)$ with arbitrary $\varepsilon > 0$ when z is large enough. Therefore, it is easy to see that $\bar{a}_0(z) = a_0(R) e^{-\lambda(z-R)}$ with $\lambda = \frac{1}{2} c_0 - \sqrt{\varepsilon}$ is a supersolution for large enough R, implying that $a_0(z)$ decays exponentially to zero as $z \to +\infty$.

This, in turn, implies exponential decay of u_0. Indeed, let $u_m(z) = \max_{y \in \overline{\Omega}} \bar{u}_0(y, z)$ and $y_m(z)$ the location of this maximum in $\overline{\Omega}$. By previous results we have $u_m \to 0$ as $z \to +\infty$. Now, by regularity of $\partial\Omega$, there exists a closed cone $\mathcal{C}_\Omega$ (with finite height) such that each point $y \in \overline{\Omega}$ is a vertex of a cone $\mathcal{C}_y \subset \overline{\Omega}$ congruent to $\mathcal{C}_\Omega$. Therefore, by the uniform estimate on $|\nabla u_0|$ for each z sufficiently large there exists a cone $\tilde{\mathcal{C}}_{y_m} \subseteq \mathcal{C}_{y_m}$ similar to $\mathcal{C}_{y_m}$ such that $u_0(y, z) \geq \frac{1}{2} u_m(z)$ for all $y \in \tilde{\mathcal{C}}_{y_m}$ and $|\tilde{\mathcal{C}}_{y_m}| = C_1 u_m^{n-1}(z)$, with some $C_1 > 0$ independent of z. Also, since $\mathrm{dist}(\tilde{\mathcal{C}}_{y_m}, \partial\Omega_\pm) \geq C_2 u_m$ and by Hopf Lemma the normal derivative of ψ_0 on $\partial\Omega_\pm$ is bounded from below [20, Lemma 3.4], we also have $\psi_0(y) \geq C_3 u_m$ for all $y \in \mathcal{C}_{y_m}$. Using these estimates in the definition of a_0, we get

$$a_0(z) = \int_\Omega e^{\varphi(y)} \bar{u}_0(y, z) \psi_0(y) dy \geq \int_{\tilde{\mathcal{C}}_{y_m}} e^{\varphi(y)} \bar{u}_0(y, z) \psi_0(y) dy \geq C u_m^{n+1}, \tag{4.24}$$

and so $u_m \leq C a_0^{\frac{1}{n+1}}(z) \leq C e^{-\mu z}$, with some $\mu > 0$, for large enough z. □

Note that in general there is no uniqueness in Theorem 4.2. This can be easily seen from the phase plane analysis already in the case $\Sigma = \mathbb{R}$ in the presence of multiple equilibria. On the other hand, under an extra assumption of non-degeneracy of $v = \lim_{z \to -\infty} \bar{u}_0(\cdot, z)$, uniqueness follows from the sliding domain method of Berestycki and Nirenberg [6] in the class of solutions with the same limit at $z = -\infty$.

Remark 4.3. *As follows from the argument of Kawohl [22], if $\nu_0 < 0$ and $s \mapsto V(\sqrt{s}, y)$ is a strictly convex function of s for any fixed $y \in \Omega$, there exists a unique positive critical point of E (necessarily the minimizer with negative energy). Since the assumption of convexity above implies the condition in Proposition 4.1, we are automatically dealing with the case covered by Theorem 4.2.*

We illustrate this situation with an example of the Allen-Cahn equation, for which $f(u) = u(1 - u^2)$. Letting $w = v^2$, we can rewrite the functional E as

$$E[v] = \widetilde{E}[w] = \int_\Omega e^{\varphi(y)} \left(\frac{|\nabla w|^2}{8w} - \frac{w}{2} + \frac{w^2}{4} \right) dy. \tag{4.25}$$

By inspection, $\widetilde{E}$ is strictly convex on $w > 0$ (which corresponds to $v > 0$), and so it admits at most one critical point with negative energy. If it does, there exists a unique (up to translations) traveling wave solution of (3.1) with speed c_0. Let us also point out that the classical Fisher nonlinearity $f(u) = u(1-u)$ satisfies the assumption of Remark 4.3.

5 Propagation

We are now going to study the role the traveling waves constructed in the preceding sections play for the initial value problem governed by (1.1). Following [24, 30], we introduce the concept of the solution's *leading edge* to study the notion of propagation:

$$R_\delta(t) = \sup\{z \in \mathbb{R} : u(y,z,t) > \delta,\ \forall y \in \Omega\}, \tag{5.1}$$

where u solves (1.1), $\delta > 0$ is small enough, setting $R_\delta = -\infty$ if the set in (5.1) is empty. Our main result in this section is that, under certain generic assumptions on the front-like initial data, the solutions of the initial value problem propagate asymptotically with the speed $c^\dagger$ of the minimizers or, if these do not exist, with speed c_0 of the minimal waves.

In order to proceed, we first need to set up a suitable existence theory for the initial value problem associated with (1.1). This is relatively standard, except for the fact that we want also to have control on the behavior of solutions at $z = +\infty$ to ensure that the solutions stay in the spaces $H^1_c(\Sigma)$ with appropriate values of c. This is needed in order to be able to apply the energy methods associated with the functional Φ_c evaluated on the solutions of (1.1).

We start with the following basic result that guarantees existence of solutions for the initial value problem in (1.1) for initial data with sufficiently rapid exponential decay.

Proposition 5.1. *Let $c > 0$ and let $u_0 \in UC(\overline{\Sigma})$, where $UC(\overline{\Sigma})$ denotes the space of uniformly continuous functions on $\overline{\Sigma}$. Let also u_0 satisfy the boundary conditions in (1.3) and assume $u_0(x) \in [0,1]$ for all $x \in \Sigma$. Then there exists a unique solution $u \in C^2_1(\Sigma \times (0,\infty)) \cap C^0(\overline{\Sigma} \times [0,+\infty))$ of (1.1) with boundary conditions from (1.3), which satisfies $u(\cdot,0) = u_0$, $u(x,t) \in [0,1]$, for all $x \in \Sigma$ and $t > 0$, and $\|\nabla u\|_{L^\infty(\Sigma\times(t_0,+\infty))} < \infty$, for all $t_0 > 0$. Moreover, if $u_0 \in L^2_c(\Sigma)$, we also have $u \in C^\alpha((0,+\infty); H^2_c(\Sigma)) \cap C^{1,\alpha}((0,+\infty); L^2_c(\Sigma))$, for all $\alpha \in (0,1)$, where $H^2_c(\Sigma)$ denotes the space of functions with up to second derivatives in $L^2_c(\Sigma)$.*

Proof. The result follows by standard theory of analytic semigroups (see e.g. [26]). The existence of a unique solution $u \in C^2_1(\Sigma \times (0,+\infty)) \cap C^0(\overline{\Sigma} \times [0,+\infty))$ follows as in [26, Proposition 7.3.1], which can be extended to a cylindrical domain with boundary of class C^2. The estimate $u(\cdot,t) \in [0,1]$ follows from the Comparison Principle for parabolic equations [34, Chapter 3], since $u \equiv 0$ and $u \equiv 1$ are sub- and supersolution of (1.1), respectively. As a consequence we obtain that $\|\nabla u\|_{L^\infty(\Sigma\times(t_0,+\infty))} < \infty$, for all $t_0 > 0$.

Let now $u_0 \in L^2_c(\Sigma)$, and denote by $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L^2_c(\Sigma)$ the linear operator $\mathcal{A}u = \Delta u + cu_z + \nabla_y \varphi \cdot \nabla_y u$, with $u \in \mathcal{D}(\mathcal{A}) = H^2_c(\Sigma) \subset L^2_c(\Sigma)$. Since $\mathcal{A}$ is a sectorial operator in $L^2_c(\Sigma)$, and $u \mapsto f(u,y)$ is (after an appropriate extension outside $[0,1]$) a Lipschitz map from $L^2_c(\Sigma)$ into itself, it follows from [26, Proposition 7.1.10] extended to a cylindrical domain that $u \in C^\alpha((0,+\infty); H^2_c(\Sigma)) \cap C^{1,\alpha}((0,+\infty); L^2_c(\Sigma))$, for all $\alpha \in (0,1)$. $\square$

Note that $\tilde{u}(y,z,t) = u(y, z+ct, t)$ satisfies the equation

$$\tilde{u}_t = \Delta \tilde{u} + c\tilde{u}_z + \nabla_y \varphi \cdot \nabla_y \tilde{u} + f(\tilde{u}, y), \tag{5.2}$$

and as was already noted in the Introduction, (5.2) is a gradient flow generated by Φ_c on $L^2_c(\Sigma)$. Therefore,

$$\frac{d\Phi_c[\tilde{u}(\cdot,t)]}{dt} = -\int_\Sigma e^{cz+\varphi(y)} \tilde{u}_t^2(\cdot,t)\, dx \le 0, \tag{5.3}$$

which helps to establish $c^\dagger$ as the upper bound for the speed of the leading edge for the initial data with sufficiently fast decay (see also [25, 30]):

Proposition 5.2. *Under the assumptions of Theorem 3.3, let u_0 satisfy the assumptions of Proposition 5.1 with some $c > c^\dagger$. Then, for any $\delta > 0$ we have $R_\delta(t) < c't$ for any $c' > c^\dagger$ and for all $t \ge T$, where $T = T(c') \ge 0$.*

Proof. First fix any $c' \in (c^\dagger, c)$ and $c'' \in (c^\dagger, c')$, then, according to Lemma 2.3, $u(\cdot,t) \in H^1_{c''}(\Sigma)$ as well. According to (5.3) with $c = c''$, the function $t \mapsto \Phi_{c''}[u(y, z+c''t, t)]$ is non-increasing, and so $0 \le \Phi_{c''}[u(y,z+c''t,t)] \le C$ for all $t \ge t_0 > 0$. This, in turn, implies that $\Phi_{c''}[u(y, z+c't, t)] = e^{-c''(c'-c'')t}\Phi_{c''}[u(y, z+c''t, t)] \to 0$. Arguing as in [25, Proposition 6.10], we conclude that $u(y, z+c't, t) \to 0$ in $L^2_{c''}(\Sigma)$, and in view of the uniform gradient estimate of Proposition 5.1 this means that $u(\cdot,t)$ converges to zero uniformly on the set $\overline{\Omega} \times [c't, +\infty)$ as $t \to \infty$. Therefore, there exists $T \ge 0$ such that $R_\delta(t) < c't$ for all $t > T$. Since this statement remains true also for all $c' > c$, this completes the proof. $\square$

We point out that the proof of Proposition 5.2 relied only on the property $\Phi_c[u] \ge 0$ for all $u \in H^1_c(\Sigma)$ for all $c > c^\dagger$. So, as a simple extension of the argument above, we have the following propagation failure result, consistent with the conclusion of [30]:

Corollary 5.3. *Assume $\nu_0 \ge 0$ and hypothesis (H3) is false, and let u_0 satisfy the assumptions of Proposition 5.1 with some $c > 0$. Then, for any $\delta > 0$ we have $R_\delta(t) < c't$ for any $c' > 0$ and all $t \ge T$, where $T = T(c') \ge 0$.*

On the other hand, if hypothesis (H3) is false, but $\nu_0 < 0$, then the same is true for all $c > c_0$, with c_0 given by (4.6):

Corollary 5.4. *Under the assumptions of Theorem 4.2, let u_0 satisfy the assumptions of Proposition 5.1 with $c > c_0$. Then, for any $\delta > 0$ we have $R_\delta(t) < c't$ for any $c' > c_0$ and all $t \ge T$, where $T = T(c') \ge 0$.*

Now we are going to study sufficient conditions for propagation. We point out right away that in general it is not clear whether a particular initial condition will result in a solution which propagates with non-zero velocity at long times.

For example, if f is of bistable type, then propagation is clearly impossible for sufficiently small initial data, since they will rather decay to zero. In their classical work, Aronson and Weinberger presented a comprehensive study of propagation phenomena for scalar reaction-diffusion equations under various assumptions on the nonlinearity f [1, 2]. Their results depend quite delicately on the properties of the traveling wave solutions admitted by these equations and involve extensive applications of Maximum and Comparison Principles. Recently, a general notion of *wave-like solutions* of (1.1) was introduced in [30] that identifies a large class of solutions of gradient reaction-diffusion systems which are propagating in a certain generalized sense (see Theorem 4.7 of [30]). Under some extra assumptions on the nonlinearity, propagation in this generalized sense implies propagation in the sense similar to the one used by Aronson and Weinberger [24, 30].

Generally, different modes of propagation can occur in the presence of multiple traveling wave solutions. Therefore, it is reasonable to ask what part of the initial condition determines the final propagation speed when propagation does occur. What we will show below is that for sufficiently rapidly decaying initial data the propagation speed can be controlled by the behavior of the initial data at $z = -\infty$ for front-like initial data.

We first give the result under the assumption of existence of minimizers of Φ_c.

Proposition 5.5. *Under the assumptions of Theorem 3.3, let u_0 satisfy the assumptions of Proposition 5.1 with $c = c^\dagger$, and also assume that $\liminf_{z\to-\infty} u_0(\cdot, z) \geq v$ of Theorem 3.3 uniformly in Ω. Then, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have $R_\delta(t) > ct$ for any $c \in (0, c^\dagger)$ and all $t \geq T$, where $T = T(c) \geq 0$.*

Proof. Let us first show that for any $c < c^\dagger$ there exists a function $u_c \in C^1(\overline{\Sigma})$ with compact support such that $\Phi_c[u_c] < 0$. Indeed, multiplying (3.1) by $e^{cz+\varphi(y)}\bar{u}_z$ and integrating over Σ, after a number of integrations by parts we obtain (see also [25, 30])

$$\Phi_c[\bar{u}] = \frac{c - c^\dagger}{c} \int_\Sigma e^{cz+\varphi(y)} \bar{u}_z^2 \, dx < 0. \tag{5.4}$$

Therefore, approximating $\bar{u}$ by a function from $C^1(\overline{\Sigma})$ with compact support and taking into account continuity of Φ_c in $H^1_c(\Sigma)$, we obtain the desired function u_c.

Observe also that since $\bar{u}(\cdot, z) < v$ for all $z \in \mathbb{R}$, we can choose u_c such that $0 \leq u_c(\cdot, z) \leq v$. Define $\Sigma_R = \Omega \times (-R, R)$, where $R > 0$ is big enough, so that $\operatorname{supp}(u_c) \subseteq \Sigma_R$. We now consider a minimizer $0 \leq \bar{u}_c \leq v$ of Φ_c over all functions in $H^1(\Sigma)$ vanishing outside Σ_R and on $\partial\Sigma_\pm$. By elliptic regularity theory [20], the minimizer $\bar{u}_c$ is a classical solution of (3.1) in Σ_R, and by the standard reflection argument $\bar{u}_c \in W^{1,\infty}(\Sigma_R)$. Furthermore, since $\Phi_c[\bar{u}_c] \leq \Phi_c[u_c] < 0 = \Phi_c[0]$ and $\bar{u}_c$ has compact support, we have $0 < \bar{u}_c \leq v - \varepsilon$, for some $\varepsilon > 0$, by Strong Maximum Principle.

Therefore, in view of the uniformity of $\liminf_{z\to-\infty} u_0(\cdot, z)$ in Ω, there exists $a \in \mathbb{R}$ such that $u_0(y, z) \geq \bar{u}_c(y, z - a)$ for all $(y, z) \in \Sigma$. Choosing $\tilde{u}(y, z, 0) = \bar{u}_c(y, z - a)$, we obtain a subsolution of (5.2) that propagates as $\tilde{R}_\delta(t) \geq ct + a - R$.

Therefore, $R_\delta(t) \geq \tilde{R}_\delta(t) > c' t$ for any $c' < c$ and $t \geq T(c')$, and the statement follows by the fact that c can be chosen arbitrarily close to $c^\dagger$. $\square$

Remark 5.6. *Note that in general the dependence of $R_\delta(t)$ on δ cannot be removed because of the possibility of stacked waves moving with different speeds in the presence of multiple equilibria of E [36, 44].*

Now, if the minimizers do not exist for Φ_c, then, as expected, a similar result holds for u as in Proposition 5.5.

Proposition 5.7. *Under the assumptions of Theorem 4.2, let u_0 satisfy the assumptions of Proposition 5.1 with $c = c_0$ and let $\liminf_{z\to\infty} u_0(\cdot, z) \geq v$ of Theorem 4.2 uniformly in Ω. Then, there exists $\delta_0 > 0$ such that for any $c < c_0$ we have $R_\delta(t) > ct$, for all $\delta \in (0, \delta_0)$ and for all $t \geq T$, where $T = T(c) \geq 0$.*

Proof. The proof is a slight modification of the proof of Proposition 5.5. We first use the same approximation as in Theorem 4.2 to show that for any $c \in (0, c_0)$ there exists a function $u_c \in C^1(\Sigma)$ with compact support such that $\Phi_c[u_c] < 0$. Indeed, if u_ε is a minimizer of the approximating functional $\Phi^\varepsilon_{c^\dagger_\varepsilon}$ from the proof of Theorem 4.2, then by (5.4) we have $\Phi^\varepsilon_c[u_\varepsilon] < 0$ for all $c < c^\dagger_\varepsilon$. In view of the fact that $c^\dagger_\varepsilon \to c_0$ from below, and in view of the continuity of Φ^ε_c in $H^1_c(\Sigma)$, for any $c \in (0, c_0)$ there exists $\varepsilon > 0$ and $u_c \in C^1(\Sigma)$ with compact support such that $\Phi^\varepsilon_c[u_c] < 0$ also. Then, we prove the claim above by observing that $V \leq V_\varepsilon$, and so $\Phi_c[u_c] \leq \Phi^\varepsilon_c[u_c]$. Now the conclusion follows exactly as in Proposition 5.5. $\square$

Summarizing all the results obtained above, we have the following

Theorem 5.8. *Assume hypotheses (H1) and (H2) are satisfied. Let u_0 satisfy the assumptions of Proposition 5.1 with some $c > c^*$, where $c^* = c^\dagger$ if hypothesis (H3) is true, or $c^* = c_0$ if hypothesis (H3) is false and $\nu_0 < 0$. In addition, assume that $\liminf_{z\to-\infty} u_0(\cdot, z) \geq v$ uniformly in Ω, where v is defined in Theorem 3.3 or Theorem 4.2. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ and any $\varepsilon > 0$ it holds*

$$(c^* - \varepsilon)t < R_\delta(t) < (c^* + \varepsilon)t, \tag{5.5}$$

for all $t \geq T$, where $T = T(\varepsilon) \geq 0$.

Thus, the speed c^* in Theorem 5.8 has a meaning of the propagation speed for the solutions of (3.1) with the initial data whose decay is governed by the L^2_c-norm with c sufficiently large. These are the data that we call "decaying sufficiently rapidly"; in particular, initial data that equal zero identically for large enough z automatically fall in this class. Let us mention here that this assumption on the decay of the initial data is in fact crucial: as is well-known, one can construct solutions which propagate faster than c^* when $\nu_0 < 0$, if the initial data are allowed to decay slower [9, 27, 28, 30, 37, 38].

Let us conclude by briefly discussing the situation in which u_0 is not a front-like function, contrary to the lim inf assumption of Theorem 5.8, but instead is sufficiently localized in both positive and negative z-directions, e.g. compactly supported. For this type of initial data it is a generic property of (1.1) that $u(x,t)$ approaches a z-independent limit on compact sets as $t \to \infty$ (see e.g. [2]). Then, if this limit is in fact v of Proposition 5.5, we can once again use the function u_c constructed in its proof as a subsolution for large enough times for the solutions of (1.1). Therefore, we get the following more general version of Theorem 5.8 which, in particular, applies to localized initial data:

Corollary 5.9. *The statement of Theorem 5.8 remains true, if the the condition* $\liminf_{z\to-\infty} u_0(y,z) \geq v(y)$ *is replaced with* $\liminf_{t\to\infty} u(y,z,t) \geq v(y)$ *uniformly on compact subsets of* Σ. *If, in addition,* $u_0(y,-z) \in L^2_c(\Sigma)$ *the same statement holds for* $u(y,-z,t)$.

The last statement in Corollary 5.9 implies that when the initial data are sufficiently localized, a *pair* of counter-propagating fronts will develop, moving with the same speed c^* in both positive and negative z-directions.

6 Discussion

Let us finally comment on the relationship of our results with those available in the literature and discuss some open problems. Equation (1.1) has been studied in an enormous number of works (for references, see Introduction). Let us point out, however, that the main thrust of research on the reaction-diffusion-advection equation in (1.1) has been towards problems with shear flows (e.g. when $\mathbf{v}$ has a z-component which depends on y [7,45,46]). Such problems are motivated by, e.g., considering a Poisueille flow of premixed fuel-oxidizer mixture inside an insulated pipe which can sustain wrinkled deflagration fronts in combustion. We, on the other hand, considered a different setup, in which the flow is perpendicular to the cylinder axis. Also, we are constrained to considering only potential flows because of the limitations of our variational approach. So, we cannot readily treat problems of front propagation in shear flows considered in the majority of the literature.

Nevertheless, our results can be compared to previous results on existence of traveling wave solutions in the absence of the flow, $\mathbf{v} \equiv \mathbf{0}$, i.e. for purely reaction-diffusion problems. For Neumann boundary conditions our results naturally extend a number of results of Berestycki and Nirenberg [7] to arbitrary types of nonlinearities and, in particular, to nonlinearities which change type in different portions of the cylinder cross-section. Moreover, under the assumption of uniqueness of the local (hence global) non-degenerate minimizer v of $E[v]$ with negative energy we obtain a unique, monotone, exponentially decaying traveling wave solution which determines the asymptotic propagation speed for a large class of sufficiently rapidly decaying front-like initial data. This is even true when $\nu_0 = 0$, the case that was left open in the study of [7] and consequent studies (apart from the case of ignition nonlinearities). Also, as was already mentioned earlier, in the case $\nu_0 < 0$ existence of minimizers gives a sharp criterion for linear vs. nonlinear selection, i.e. whether $c^* = c_0$ or $c^* > c_0$ in [7]. In fact, we have proposed a new sufficient condition for linear selection (Proposition 4.1), which is more general than a commonly used assumption $f(u,y) \leq u f_u(0,y)$, which is usually referred to as the KPP-type nonlinearity [5]. Moreover, a more restrictive condition from Remark 4.3 would guarantee both the KPP-type behavior of the traveling waves and the uniqueness of the minimizer of $E[v]$, also for combinations of Neumann and Dirichlet boundary conditions. We also note that the existence of a critical speed c^* in the case $\nu_0 < 0$ which is established by our analysis does not rely on positivity of f any more and, together with Remark 4.3, applies e.g. to nonlinearities like $f(u, y) = u(\mu(y) - u)$ considered in [5].

Our method also easily treats various boundary conditions, in particular, Dirichlet boundary conditions. The papers that are most relevant to our results here are those of Vega [41–43] (see also [19,21]). Vega constructed the unique solution connecting two stable (in a certain sense) critical points of the functional E, provided there are no other critical points of E with negative energy that are sandwiched between them. He also constructed a family of solutions connecting an unstable equilibrium of E at $z = +\infty$ with the stable one at $z = -\infty$ under a similar assumption on other critical points of E. Our analysis generalizes these results by weakening the assumptions on the nonlinearity, if we redefine the potential in (2.4) to be the negative antiderivative of f only for $0 \leq u(y, z) \leq v(y)$, where v is a stable equilibrium of E with negative energy such that there are no other such equilibria sandwiched between 0 and v. We also only need to verify that $\nu_0 \geq 0$ to ensure existence of a unique, monotone traveling wave solution connecting 0 and v. In the case $\nu_0 < 0$, we obtain existence of the minimal speed front characterized by the fast exponential decay (see also [35]). Let us point out that with our variational approach we are able to obtain various estimates on the traveling wave speed, as well as distinguish between linear and nonlinear selection mechanism. We also similarly can obtain various monotonicity properties of the speed with respect to changes in the nonlinearity or the shape of Ω.

Let us also point out an important limitation of the approach of Vega. Consider a situation in which the domain Ω consists of two sufficiently large mirror-symmetric regions connected by a thin neck, with Dirichlet boundary conditions. Clearly, with a bistable y-independent nonlinearity f (i. e. satisfying (3.45) and $\nu_0 > 0$) one could have three positive local minimizers for the functional E: two that are localized in each of the halves of Ω and one which is localized in both halves. With no other local minima of E one can use the method of Vega to construct the traveling wave solutions which are localized in one of the corresponding halves of the cylinder Σ. Our method, on the other hand, will always pick the (faster) traveling wave solution that is localized in both halves of the cylinder. Indeed, by mirror symmetry the minimizer of Φ_c must also be symmetric, hence the traveling wave will necessarily connect zero with the symmetric local (also global) minimizer of E. In view of the discussion in the previous paragraph, in this situation our method will yield all the traveling wave solutions in the problem after a suitable redefinition of V in each case. We emphasize that in general our method does not rely on the knowledge of the global picture for the critical points of the functional E, in contrast to the approach of Vega.

In short, we have obtained a characterization of propagation in the spirit of Aronson and Weinberger [2] for the considered problem (see also [18]). It would be interesting to see how the notion of propagation related to the motion of the leading edge used here relates to the generalized notion of propagation which was recently introduced for a class of the so-called wave-like solutions in [30, Theorem 4.7]. Let us point out that both definitions of propagation velocity have $c^\dagger$ (or c_0 in the absence of minimizers of Φ_c) as the upper bound. Similarly, the asymptotic propagation speed from [30, Theorem 4.11] gives a lower bound for the propagation speed of the leading edge. One may naturally ask when these two asymptotic propagation speeds are actually the same. We have not yet been able to answer this question. One way to proceed here would be to apply Theorem 4.8 of [30] under the assumption that there are no variational traveling waves other than the minimizer. This, however, seems to be difficult to do, since one needs some a priori information on the exponential decay of the solution of the initial value problem in the reference frame associated with the leading edge. We note that this would also, in turn, imply a much stronger result of convergence of the solution to the initial value problem to a minimizer as $t \to \infty$, in view of the linear stability of the minimizer (by monotonicity, zero is the smallest eigenvalue of the linearization around $\bar{u}$, all other eigenvalues and the essential spectrum are strictly above zero in the weighted space [37]). Alternatively, one could use positivity of Φ_c evaluated on the solution of the initial value problem for $c = \lim_{t\to\infty} \bar{c}(t)$, see [30, Definition 4.5], to interpret Φ_c as a Lyapunov functional for the problem in the reference frame moving with speed c. Here, however, one faces a difficulty associated with the lack of compactness in the problem. In sum, a more precise characterization of propagation in the considered problem is still open.

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