On the existence of heteroclinic connections

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Abstract

Assume that $W : \mathbb{R}^m \to \mathbb{R}$ is a nonnegative potential that vanishes only on a finite set A with at least two elements. By direct minimization of the action functional on a suitable set of maps we give a new elementary proof of the existence of a heteroclinic orbit that connects any given $a_- \in A$ to some $a_+ \in A \setminus \{a_-\}$.

1 Introduction

Let $W : \mathbb{R}^m \to \mathbb{R}$ be a smooth nonnegative function that vanishes on a finite set A, with $\#A \geq 2$, Given two distinct points $a_-, a_+ \in A$ we can ask about the existence of a solution $u^* : \mathbb{R} \to \mathbb{R}^m$ of the equation

$$\ddot{u} = W_u(u), \ x \in \mathbb{R}, \tag{1.1}$$

with the conditions

$$\lim_{x \to \pm\infty} u(x) = a_\pm. \tag{1.2}$$

If a solution u^* of (1.1), (1.2) does exist we say that there is a *heteroclinic connection* between a_- and a_+.

A first motivation for studying connections comes from the mathematical theory of phase transitions where a widely used model is the Allen-Cahn equation

$$\begin{cases} u_t = \epsilon^2 \Delta u - W_u(u), & x \in \Omega, \\ \partial_\nu u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where u is an order parameter, ν the unit exterior normal and $\epsilon > 0$ a small parameter. Equation (1.3) describes the evolution of a substance which may appear in two or more preferred phases and is contained in a region $\Omega \subset \mathbb{R}^n$. In this context a_- and a_+ represent different phases in which the specific substance may exist. For small $\epsilon > 0$ typical solutions u^ϵ of (1.3) divide Ω as $\Omega = \Omega_- \cup \Gamma \cup \Omega_+$ with $\Omega_\pm = \{u^\epsilon \approx a_\pm\}$ and Γ an interface of thickness $\mathrm{O}(\epsilon)$ that separates the regions Ω_- and Ω_+ where the substance is in phase a_- or in phase a_+. Heteroclinic connections describe the behavior of u^ϵ across the interface. Indeed it results

$$u^\epsilon(x) \approx u^*(\frac{d(x)}{\epsilon}),$$

where $d(x)$ is the signed distance from the interface and $u^* : \mathbb{R} \to \mathbb{R}^m$ is a connection between a_- and a_+. For multi-phase systems, the description of u^ϵ in a neighborhood of multiple points where three

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or more regions $\{u^\epsilon \approx a_j\}$ meet, requires the consideration of connections between three or more of the $a_j \in A$, see [3], [7].

If x is interpreted as time, equation (1.1) can be seen as the Newton equation of a particle of unit mass moving in $m-$ dimensional space under a conservative field of force of potential W. Then problem (1.1), (1.2) is the same as to show that one can choose position and velocity of the particle at time 0 in such a way that the asymptotic fate of the particle in the future and in the past are a_+ and a_- respectively. From the mechanical point of view the understanding of the connections that exist between elements of A is a significant step toward a description of the global dynamics of equation (1.1).

In the scalar case ($m = 1$) existence of connections between neighboring zeros of W can be established via the method of phase plane analysis. In the vector case ($m > 1$) this approach is not available and, since solutions of (1.1) are, in each bounded interval (x_1, x_2), stationary points of the action functional

$$J(u) = \int_{x_1}^{x_2} \Big(\frac{1}{2}|u_x|^2 + W(u)\Big)dx, \tag{1.4}$$

a variational approach is generally used. Existence of vector-valued heteroclinic connections as minimizers of J on suitable sets of maps and under different assumption on W has been established by various authors either by direct minimization of J [1], [11], [4] or by minimizing the associate Jacobi functional

$$L(u) = \int_{x_1}^{x_2} \sqrt{2W(u)}dx, \tag{1.5}$$

as in [10], [5], [12]. In [1] W was assumed to satisfy a mild monotonicity condition at $a_\pm$. This condition was later removed in [11]. The minimization of (1.5) for proving the existence of connections was first used in [10] under restrictive assumptions on the behavior of W in a neighborhood of $a_\pm$. In [5] and [12] the idea is to show that, in spite of the fact that W vanishes at $a_\pm$, the connection problem can be seen as the problem of the existence of a geodesic connecting a_- to a_+ for the metric induced by (1.5). Aside from different requirements on the smoothness and on the behaviour of W at infinity, the only assumption in [11], [4], [5] and [12] is that W is nonnegative and vanishes in a finite set. For connections and related questions see also [2], [8], [9].

The scope of the present paper is to present a new elementary proof of the existence of heteroclinic connections under minimal assumption on W and by direct minimization of the functional (1.4). Our proof is a by product of the analysis developed in [6].

While for a classical solution of equation (1.1) we need W to be a C^1 function, the variational problem can be formulated under the assumption that W is merely continuous. As we shall see, with W continuous it is not guaranteed that the time interval required to a minimizer to travel from a_- to a_+ be infinite and therefore the function space where we minimize J has to include maps defined on bounded or semi-bounded intervals. We shall show that each $a_- \in A$ is connected to some other $a_+ \in A$ by minimizing J on the set of maps $u : (l_-^u, l_+^u) \to \mathbb{R}^m$ defined by

$$\tag{1.6}$$

$$\begin{aligned}\mathcal{A} = \{u \in W^{1,2}_{\text{loc}}((l_-^u, l_+^u); \mathbb{R}^m) : -\infty \le l_-^u < l_+^u \le +\infty,\\ \lim_{x \to l_-^u} u(x) = a_-, \lim_{x \to l_+^u} u(x) \in A \setminus \{a_-\}, u((l_-^u, l_+^u)) \subset \mathbb{R}^m \setminus A\}.\end{aligned} \tag{1.7}$$

Note that in (1.7) the interval (l_-^u, l_+^u) associated to u is not fixed but is free to change with u.

Without some condition on the behavior of W at infinity a minimizer of J on $\mathcal{A}$ may not exist. The problem is that J may be not coercive on $\mathcal{A}$ in the sense that there exist minimizing sequences $\{u_j\} \subset \mathcal{A}$ such that $\|u_j\|_{W^{1,2}} \to +\infty$ as $j \to +\infty$ while $J(u_j)$ remains bounded. A sufficient condition for coerciveness is

$$\limsup_{|u| \to +\infty} W(u) > 0,$$

but it is possible to allow potentials W that decay to 0 at infinity provided the decaying is not too fast. As observed in [5] it suffices to assume

(H)

$$\sqrt{W(u)} \geq \rho(|u|), \quad |u| \geq r_0$$

for some $r_0 > 0$ and a nonnegative function $\rho : [r_0, +\infty) \to \mathbb{R}$ such that $\int_{r_0}^{+\infty} \rho(r)dr = +\infty$.

We have

Theorem 1.1. *Assume that $W : \mathbb{R}^m \to \mathbb{R}$ is a continuous function that satisfies (H). Then, given $a_- \in A$, there exist $a_+ \in A \setminus \{a_-\}$ and a Lipschitz-continuous map $u : (l_-, l_+) \to \mathbb{R}^m$, with $-\infty \leq l_- < 0 < l_+ \leq +\infty$, which minimizes $J : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ and satisfies*

$$\frac{1}{2}|\dot{u}|^2 - W(u) = 0, \quad a.e. \ \ in \ \ (l_-, l_+). \tag{1.8}$$

In particular

(i)

$$\lim_{x \to l_\pm} u(x) = a_\pm, \tag{1.9}$$

(ii)

$$W(u(x)) > 0 \ \ x \in (l_-, l_+). \tag{1.10}$$

If W is continuously differentiable in $\mathbb{R}^m \setminus A$, then u is a classical solution of (1.1).

Before giving the proof of Theorem 1.1 we make some observations and present some related results.

2 Observations and related results

From Theorem 1.1 we have that, under the assumption that $W \in C^1(\mathbb{R}^m \setminus A; \mathbb{R})$, for each $a_- \in A$ there is an orbit of (1.1) that starts in a_- and ends up in some $a_+ \in A \setminus \{a_-\}$ without any other intersection with A. It follows that there are at least $\frac{\#A}{2}$ such orbits if $\#A$ is even and $\frac{\#A+1}{2}$ if $\#A$ is odd.

Given $a_i \neq a_j \in A$, a sufficient condition for the existence of an orbit that connects a_i to a_j and satisfies (1.10) is

$$\sigma_{ij} < \sigma_{ih} + \sigma_{hj}, \quad \text{for} \ \ a_h \in A \setminus \{a_i, a_j\},$$

where

$$\begin{aligned} &\sigma_{ij} = \inf_{u \in \mathcal{A}_{ij}} J(u), \\ &\mathcal{A}_{ij} = \{u \in W^{1,2}_{\text{loc}}((l_-^u, l_+^u); \mathbb{R}^m) : -\infty \leq l_-^u < l_+^u \leq +\infty, \\ &\lim_{x \to l_-^u} u(x) = a_i, \lim_{x \to l_+^u} u(x) = a_j\}. \end{aligned}$$

In the scalar case $m = 1$ from (1.8) and (1.10) it follows that the minimizer u given by Theorem 1.1 is a solution of

$$\dot{u} = \sqrt{2W(u)} > 0, \ \ x \in (l_-, l_+). \tag{2.1}$$

If a_- and a_+ are two neighboring zeros of $W \in C^1(\mathbb{R} \setminus A; \mathbb{R})$ this equation has a unique solution u that satisfies (1.9) and $u(0) = \frac{a_- + a_+}{2}$, therefore u is the minimizer in Theorem 1.1. For instance if $W(u) = \frac{1}{2}(1 - u^2)^2$ this solution is given by $u(x) = \tanh x$, $x \in \mathbb{R}$ and satisfies $\lim_{x \to \pm\infty} u(x) = \pm 1$. Note that, if W vanishes at a point a between a_- and a_+, there is no minimizer. Indeed any continuous function u that travels from a_- to a_+ has to assume the value a violating (1.10).

We give a simple criterion to have $l_\pm = \pm\infty$.

Proposition 2.1. *Assume there exist $c > 0$ and $r_0 > 0$ such that*

$$W(u) \leq c|u - a_+|^2, \quad \textit{for} \ \ |u - a_+| \leq r_0.$$

Then $l_+ = +\infty$ and an analogous statement applies to l_-.

Proof. From (i) there is $x_0 \in (l_-, l_+)$ such that $|u - a_+| \leq r_0$ for $x \in [x_0, l_+)$. This and the assumption on W imply

$$\frac{d}{dx}|u - a_+| \geq -|\dot{u}| = -\sqrt{2W(u)} \geq -\sqrt{2c}|u - a_+|, \quad \text{for} \ \ x \in [x_0, l_+)$$

which yields

$$|u(x) - a_+| \geq |u(x_0) - a_+|e^{-\sqrt{2c}(x - x_0)}, \quad \text{for} \ \ x \in [x_0, l_+).$$

This is compatible with (1.9) only if $l_+ = +\infty$. □

Proposition 2.2. *Assume that $W \in C^2(\mathbb{R}^m; \mathbb{R})$ and that the Jacobian matrix $j(a)$ is positive definite for $a \in A$. Let u be as in Theorem 1.1. Then $l_\pm = \pm\infty$ and there are positive constants k, K such that*

$$|u(x) - a_+| \leq Ke^{-kx} \ \ \textit{and} \ \ |u(x) - a_-| \leq Ke^{kx}, \ \forall x \in \mathbb{R}. \tag{2.2}$$

Proof. $l_\pm = \pm\infty$ follows from Proposition 2.1. To prove the exponential estimates (2.2) note that from $W_u(u) = j(a)(u - a) + \mathrm{o}(|u - a|)$ and the assumption on $j(a)$ it follows

$$W_u(u) \cdot (u - a) \geq c^2|u - a|^2, \quad \text{for} \ \ |u - a| \leq r_0, \ a \in A, \tag{2.3}$$

for some positive constants r_0 and c. Set $\phi(x) := |u - a_+|^2$. From (1.9) there is $x_0 > 0$ such that $x \geq x_0$ implies $\phi(x) \leq r_0^2$. This inequality, (1.1) and (2.3) yield

$$\begin{aligned} \ddot{\phi}(x) &= 2|\dot{u}(x)|^2 + 2(u(x) - a_+) \cdot W_u(u(x)) \\ &\geq 2c^2\phi(x), \ \text{for} \ x \geq x_0. \end{aligned} \tag{2.4}$$

Since we have $\phi(x) \leq r_0^2$ for $x \geq x_0$, from (2.4) and the maximum principle we get, for every $l > 0$

$$\phi(x) \leq \varphi_l(x), \ \ x \in [x_0, x_0 + 2l], \tag{2.5}$$

where

$$\varphi_l(x) := r_0^2 \frac{\cosh \sqrt{2}c(l - (x - x_0))}{\cosh \sqrt{2}cl}, \ \ x \in (x_0, x_0 + 2l),$$

is the solution of

$$\begin{cases} \ddot{\varphi} = 2c^2\varphi, \ \ x \in (x_0, x_0 + 2l), \\ \varphi(x_0) = \varphi(x_0 + 2l) = r_0^2. \end{cases}$$

From (2.5) and $\varphi_l(x) \leq 2r_0^2 e^{-\sqrt{2}c(x - x_0)}, \ \ x \in [x_0, x_0 + l]$ which holds for all $l > 0$, it follows

$$|u(x) - a_+| \leq \sqrt{2}r_0 e^{-\frac{c}{\sqrt{2}}(x - x_0)}, \quad \text{for} \ x \geq x_0.$$

The first estimate in (2.2), with $k = \frac{c}{\sqrt{2}}$ follows from this and from the fact that u is bounded. The estimate for $|u(x) - a_-|$ can be obtained in a similar way. □

In Mechanics the functional (1.4) is called the *Action* and Theorem 1.1 corresponds to the Hamilton principle of least action. This is equivalent to the Jacobi principle that concerns the minimization of the Jacobi functional $L : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$

$$L(u) = \int_{l_-^u}^{l_+^u} \sqrt{2W(u(x))}|\dot{u}(x)|dx.$$

We have indeed

Proposition 2.3.

$$\tilde{\sigma}_0 = \inf_{u \in \mathcal{A}} L(u) = \inf_{u \in \mathcal{A}} J(u) = \sigma_0.$$

Proof. The elementary inequality $a^2 + b^2 \geq 2ab$ implies

$$L(u) \leq J(u), \quad u \in \mathcal{A},$$

with equality if and only if (1.8) holds

$$\frac{1}{2}|\dot{u}|^2 = W(u), \quad a.e. \text{ on } (l_-^u, l_+^u). \tag{2.6}$$

This proves

$$\tilde{\sigma}_0 \leq \sigma_0.$$

To prove $\sigma_0 \leq \tilde{\sigma}_0$ we use the fact that $L(u)$ does not depend on the parametrization. Assume that $u : (l_-^u, l_+^u) \to \mathbb{R}^m$ is C^1 smooth and let $\phi : (l_-^v, l_+^v) \to (l_-^u, l_+^u)$ be a C^1 bijection with $\dot{\phi} > 0$ and inverse ψ. For the map $v : (l_-^v, l_+^v) \to \mathbb{R}^m$ defined by $v(s) = u(\phi(s))$ we have

$$L(v) = \int_{l_-^v}^{l_+^v} \sqrt{2W(u(\phi(s))}|\dot{u}(\phi(s))|\dot{\phi}(s)ds = L(u),$$

where we have made the substitution $s = \psi(x)$ to derive the last equality.

The idea is to show that each $u \in \mathcal{A}$ can be reparametrized into a $v \in \mathcal{A}$ that satisfies (2.6). This implies $\sigma_0 \leq \tilde{\sigma}_0$ via

$$\sigma_0 \leq J(v) = L(v) = L(u).$$

This program can not be realized in this simple way since we need to take care of the fact that $u \in \mathcal{A}$ may be not smooth and may have the set $\{\dot{u} = 0\}$ of positive measure.

We first show that we can assume $-\infty < l_-^u < l_+^u < +\infty$. If $l_+^u = \infty$, given $\delta > 0$ small, there are x_δ and $a \in A$ such that $0 < |u - a| \leq \delta$ for $x \geq x_\delta$. Set $u_\delta = \mathbb{1}_{(l_-^u, x_\delta]} u + \mathbb{1}_{(x_\delta, x_\delta + 1)} \tilde{u}$ where $\tilde{u} = (1 - x + x_\delta)u(x_\delta) + (x - x_\delta)a$, for $x \in (x_\delta, x_\delta + 1)$. We have that $L(\tilde{u}, (x_\delta, x_\delta + 1)) \leq \eta_\delta := \delta\sqrt{2\max_{|u-a|\leq\delta} W(u)} \to 0$ as $\delta \to 0$. Since we can proceed in a similar way if $l_-^u = -\infty$ we conclude that, given $u \in \mathcal{A}$, for each $\epsilon > 0$ small there is $u_\epsilon = u_{\delta_\epsilon}$, $u_\epsilon \in \mathcal{A}$ with $-\infty < l_-^{u_\epsilon} < l_+^{u_\epsilon} < +\infty$ that satisfies

$$L(u_\epsilon) \leq L(u) - \epsilon.$$

This proves the claim. If $[l_-^u, l_+^u]$ is bounded, $C^\infty([l_-^u, l_+^u], \mathbb{R}^m)$ is dense in $W^{1,2}([l_-^u, l_+^u], \mathbb{R}^m)$. This and the fact that L is continuous in $W^{1,2}([l_-^u, l_+^u], \mathbb{R}^m)$ imply that we can assume that u is smooth. Therefore in the remaining part of the proof we suppose that u is smooth and defined in a bounded set (l_-^u, l_+^u).

By arguing as before we choose l_-, l_+ with $l_-^u < l_- < l_+ < l_+^u$ and construct a map $u_\epsilon \in \mathcal{A}$ of the form

$$u_\epsilon = \mathbb{1}_{[l_-, l_+]} u + \mathbb{1}_{(l_- - 1, l_-) \cup (l_+, l_+ + 1)} \tilde{u} \tag{2.7}$$

and such that

$$J(\tilde{u}, (l_- - 1, l_-) \cup (l_+, l_+ + 1)) \leq \epsilon. \tag{2.8}$$

Consider the reparametrized map $v : (\lambda_-, \lambda_+) \to \mathbb{R}^m$ of $u : (l_-, l_+) \to \mathbb{R}^m$ defined by $x = \phi(s)$ where $\phi : (\lambda_-, \lambda_+) \to (l_-, l_+)$ is the inverse of the map $s = \psi(x)$ defined by

$$\psi(x) = \int_{\frac{l_-^u + l_+^u}{2}}^{x} \frac{\max\{|\dot{u}(t)|, \delta\}}{\sqrt{2W(u(t))}} dt, \quad x \in (l_-, l_+). \tag{2.9}$$

Note that ϕ satisfies $\phi(0) = \frac{l_-^u + l_+^u}{2}$ and the equation

$$\dot{\phi} = \frac{\sqrt{2W(u(\phi))}}{\max\{|\dot{u}(\phi)|, \delta\}}, \ \ s \in (\lambda_-, \lambda_+), \ \lambda_\pm = \psi(l_\pm), \tag{2.10}$$

which is approximately the condition one must impose to ϕ in order that v satisfies (2.6). In (2.9) and (2.10) we use the approximate expression $\max\{|\dot{u}|, \delta\}$ instead $|\dot{u}|$ to have well defined strictly increasing maps ψ and ϕ even when $\dot{u}$ vanishes in a set of positive measure. From (2.10) we obtain

$$\frac{1}{2}|\dot{u}(\phi)|^2\dot{\phi}^2 + W(u(\phi)) - \sqrt{2W(\phi)}|\dot{u}(\phi)|\dot{\phi} = \gamma_\delta, \ \ s \in (\lambda_-, \lambda_+), \tag{2.11}$$

where

$$\gamma_\delta = \begin{cases} 0, & \text{if } \ |\dot{u}| > \delta, \\ \frac{W}{\delta^2}(|\dot{u}| - \delta)^2, & \text{if } \ |\dot{u}| \leq \delta. \end{cases} \tag{2.12}$$

From (2.9) and (2.10) we obtain

$$|\{s \in (\lambda_-, \lambda_+) : |\dot{u}(\phi(s))| \leq \delta\}| = \int_{\{x \in (l_-, l_+) : |\dot{u}(x)| \leq \delta\}} \frac{\max\{|\dot{u}(x)|, \delta\}}{\sqrt{2W(u(x))}} dx \leq C\delta, \tag{2.13}$$

where $|S|$ denotes the measure of S and $C = \frac{l_+ - l_-}{\min_{x \in [l_-, l_+]} \sqrt{2W(u(x))}}$. Therefore, integrating (2.11) in (λ_-, λ_+) and using that $\gamma_\delta \leq 2\max_{x \in [l_-, l_+]} W(u(x))$ yields

$$J(v, (\lambda_-, \lambda_+)) - L(v, (\lambda_-, \lambda_+)) = J(v, (\lambda_-, \lambda_+)) - L(u, (l_-, l_+)) \leq C\delta, \tag{2.14}$$

with $C > 0$ independent of δ. Now extend $v = u \circ \phi$ from (λ_-, λ_+) to $(\lambda_- - 1, \lambda_+ + 1)$ by setting

$$v = \begin{cases} \tilde{u}(l_- + s - \lambda_-), & \text{for } \ s \in (\lambda_- - 1, \lambda_-], \\ \tilde{u}(l_+ + s - \lambda_-), & \text{for } \ s \in [\lambda_+, \lambda_+ + 1), \end{cases}$$

where $\tilde{u}$ is as in (2.7). The map v so extended belongs to $\mathcal{A}$. This and (2.8) imply

$$\sigma_0 \leq J(v, (\lambda_-, \lambda_+)) + \epsilon.$$

Therefore from (2.14) it follows, for $\delta > 0$ small,

$$\sigma_0 - 2\epsilon \leq L(u, (l_-, l_+)) \leq L(u, (l_-^u, l_+^u)).$$

The proof is complete. □

3 The proof of Theorem 1.1

The first observation is that J is translation invariant on $\mathcal{A}$ in the sense that

$$J(u^\lambda) = J(u), \quad \text{for } \ u \in \mathcal{A}, \ \lambda \in \mathbb{R},$$

where $u^\lambda = u(\cdot - \lambda) \in \mathcal{A}$. This generates a loss of compactness that manifests itself in the existence of minimizing sequences $\{u_j\} \in \mathcal{A}$ that converges in C^1_{loc} to a map u that fails to satisfy (1.9) in Theorem 1.1. For example this happens for $m = 1$ and $W = \frac{1}{2}(1 - u^2)^2$. In this case $u = \tanh x$ is a minimizer and $\{\tanh(\cdot - j)\}$ a minimizing sequence that converges to -1. We remove this pathology by an elementary observation. Since a_- is an isolated zero of W, for small fixed $r_0 > 0$, we have

$$\min_{a \in A, |u - a| = r_0} W(u) = W_0 > 0,$$

and any map $u \in \mathcal{A}$ has to satisfies $W(u(x_0)) = W_0$ for some $x_0 \in (l_-^u, l_+^u)$. Taking $x_0 = 0$ restricts the possible translations to a compact set and removes the obstruction of noncompactness. It follows that we can assume

$$W(u(0)) = W_0, \tag{3.1}$$

and restrict J to the subset of $\mathcal{A}$ where (3.1) holds.

Given $a_- \in A$ let $\bar{a} \in A$ be such that

$$|a_- - \bar{a}| = \min_{a \in A \setminus \{a_-\}} |a_- - a|,$$

and set

$$\tilde{u}(x) = (1 - (x + x_0))a_- + (x + x_0)\bar{a}, \ \ x \in (-x_0, 1 - x_0),$$

where $x_0 \in (0, 1)$ is chosen so that $W(\tilde{u}(0)) = W_0$. Then $\tilde{u} \in \mathcal{A}$, $l_-^{\tilde{u}} = -x_0$, $l_+^{\tilde{u}} = 1 - x_0$ and

$$J(\tilde{u}) = \sigma < +\infty.$$

In the following, when we wish to specify that the action functional is relative to some interval (x_1, x_2), we write $J(u, (x_1, x_2))$.

Next we show that there are constants $M > 0$ and $l_0 > 0$ such that each $u \in \mathcal{A}$ with

$$J(u) \leq \sigma, \tag{3.2}$$

satisfies

$$\begin{aligned} &\|u\|_{L^\infty((l_-^u, l_+^u);\mathbb{R}^n)} \leq M, \\ &l_-^u \leq -l_0 < l_0 \leq l_+^u. \end{aligned} \tag{3.3}$$

The L^∞ bound on u follows from (H). Indeed, if $|u(\bar{x})| = M$ for some $\bar{x} \in (l_-^u, l_+^u)$, we have

$$\sigma \geq J(u, (l_-^u, \bar{x})) \geq \int_{l_-^u}^{\bar{x}} \sqrt{2W(u(x))}|\dot{u}(x)|dx \geq \sqrt{2}\int_{r_0}^{M} \rho(s)ds.$$

If $l_-^u = -\infty, l_+^u = +\infty$ the existence of l_0 is obvious, if $l_-^u > -\infty$ and/or $l_+^u < +\infty$ it follows from

$$\frac{d_0^2}{|l_-^u|} \leq \int_{l_-^u}^{0} |\dot{u}(x)|^2 dx \leq 2\sigma, \qquad \frac{d_0^2}{l_+^u} \leq \int_{0}^{l_+^u} |\dot{u}(x)|^2 dx \leq 2\sigma,$$

where $d_0 = d(A, \{u : W(u) > W_0\})$.

Let $\{u_j\} \subset \mathcal{A}$ be a minimizing sequence

$$\lim_{j \to +\infty} J(u_j) = \inf_{u \in \mathcal{A}} J(u,) := \sigma_0 \leq \sigma. \tag{3.4}$$

We can assume that each u_j satisfies (3.2) and (3.3). By considering a subsequence, that we still denote by $\{u_j\}$, we can also assume that there exist l_-^∞, l_+^∞ with $-\infty \leq l_-^\infty \leq -l_0 < l_0 \leq l_+^\infty \leq +\infty$ and a continuous map $u^* : (l_-^\infty, l_+^\infty) \to \mathbb{R}^n$ such that

$$\begin{aligned} &\lim_{j \to +\infty} l_\pm^{u_j} = l_\pm^\infty, \\ &\lim_{j \to +\infty} u_j(x) = u^*(x), \ \ x \in (l_-^\infty, l_+^\infty), \end{aligned} \tag{3.5}$$

and in the last limit the convergence is uniform on bounded intervals. This follows from (3.3) which implies that the sequence $\{u_j\}$ is equi-bounded and from (3.2) which implies

$$|u_j(x_1) - u_j(x_2)| \leq \left| \int_{x_1}^{x_2} |\dot{u}_j(x)|dx \right| \leq \sqrt{\sigma}|x_1 - x_2|^{\frac{1}{2}}, \tag{3.6}$$

so that the sequence is also equi-continuous.

By passing to a further subsequence we can also assume that $u_j \rightharpoonup u^*$ in $W^{1,2}((l_1,l_2);\mathbb{R}^n)$ for each l_1, l_2 with $l_-^\infty < l_1 < l_2 < l_+^\infty$. This follows from (3.2), which implies

$$\frac{1}{2}\int_{l_-^{u_j}}^{l_+^{u_j}} |\dot{u}_j|^2 dx \leq J(u_j) \leq \sigma,$$

and from the fact that each u_j satisfies (3.3) and therefore is bounded in $L^2((l_1,l_2);\mathbb{R}^n)$.
We also have

$$J(u^*,(l_-^\infty,l_+^\infty)) \leq \sigma_0. \tag{3.7}$$

Indeed, from the lower semicontinuity of the norm, for each l_1, l_2 with $l_-^\infty < l_1 < l_2 < l_+^\infty$ we have

$$\int_{l_1}^{l_2} |\dot{u}^*|^2 dx \leq \liminf_{j\to+\infty} \int_{l_1}^{l_2} |\dot{u}_j|^2 dx.$$

This and the fact that u_j converges to u^* uniformly in $[l_1,l_2]$ imply

$$J(u^*,(l_1,l_2)) \leq \liminf_{j\to+\infty} J(u_j,(l_1,l_2)) \leq \liminf_{j\to+\infty} J(u_j,(l_-^{u_j},l_+^{u_j})) = \sigma_0.$$

Since this is valid for each $l_-^\infty < l_1 < l_2 < l_+^\infty$ the claim (3.7) follows.

Lemma 3.1. *Define $l_-^\infty \leq l_- \leq -l_0 < l_0 \leq l_+ \leq l_+^\infty$ by setting*

$$\begin{aligned} l_- &= \inf\{x \in (l_-^\infty, 0] : u^*((x,0]) \subset \mathbb{R}^m \setminus A\} \\ l_+ &= \sup\{t \in (0, l_+^\infty) : u^*([0,x)) \subset \mathbb{R}^m \setminus A\}. \end{aligned}$$

Then u^* with $l_\pm^{u^*} = l_\pm$ belongs to $\mathcal{A}$ and is a minimizer. That is

$$J(u^*) = \sigma_0. \tag{3.8}$$

Proof. If $l_+ < +\infty$ the existence of

$$a_+ = \lim_{x\to l_+} u^*(x) \tag{3.9}$$

follows from (3.6) which implies that u^* is a $C^{0,\frac{1}{2}}$ map. The limit a_+ belongs to A. Indeed, $a_+ \notin A$ would imply the existence of $\lambda > 0$ such that, for j large enough,

$$d(u_j([l_+, l_+ + \lambda], A) \geq \frac{1}{2} d(a_+, A),$$

in contradiction with the definition of l_+. If $l_+ = +\infty$ and (3.9) does not hold there is $\delta > 0$ and a diverging sequence $\{x_j\}$ such that

$$d(u^*(x_j), A) \geq \delta.$$

Set $W_\delta = \min_{d(u,A)=\delta} W(u) > 0$. From the uniform continuity of W in $\{|u| \leq M\}$ (M as in (3.3)) it follows that there is $\lambda > 0$ such that

$$|W(u_1) - W(u_2)| \leq \frac{1}{2} W_\delta, \quad \text{for } |u_1 - u_2| \leq \lambda,\ u_1, u_2 \in \{|u| \leq M\}.$$

This and $u^* \in C^{0,\frac{1}{2}}$ imply

$$W(u^*(x)) \geq \frac{1}{2} W_\delta,\ \ x \in I_j = \Big(x_j - \frac{l^2}{\sigma}, x_j + \frac{\lambda^2}{\sigma}\Big),$$

and, by passing to a subsequence, we can assume that the intervals I_j are disjoint. Therefore for each $L > 0$ we have

$$\sum_{x_j \leq L} \frac{\lambda^2 W_\delta}{\sigma} \leq \int_0^L W(u^*(x))dx \leq \sigma_0,$$

which is impossible for L large. This proves that, also when $l_+ = +\infty$ there is $A \ni a_+ = \lim_{x\to+\infty} u^*(x)$. To show that $a_+ \neq a_-$ we observe that $a_+ = a_-$ implies the existence of a sequence $\{x_j\} \subset [l_0, l_+]$ that satisfies

$$\begin{aligned} &\lim_{j\to+\infty} x_j = l_+, \\ &\lim_{j\to+\infty} u_j(x_j) = a_-. \end{aligned} \tag{3.10}$$

Since $W(u_j(0)) = W_0$ from the uniform continuity of W in $\{|u| \leq M\}$ and (3.6) it follows

$$W(u_j(x)) \geq \frac{1}{2} W_0, \quad \text{for } \ x \in (-\delta, \delta),$$

for some $\delta > 0$. Therefore, for j large, we have

$$J(u_j, (l_-^{u_j}, x_j) \geq \delta W_0.$$

On the other hand from $(3.10)_2$ we have

$$J(u_j, (x_j, l_+^{u_j})) \geq \sigma_0 - \epsilon_j,$$

where $\epsilon_j \to 0$ as $j \to +\infty$. These inequalities contradict the minimizing character of the sequence $\{u_j\}$ and prove $a_+ \neq a_-$. We have seen that u^* with $l_\pm^{u^*} = l_\pm$ satisfies all the properties required for membership in $\mathcal{A}$. Therefore we have $J(u^*) \geq \sigma_0$ that together with (3.7) show that $u^* \in \mathcal{A}$ is indeed a minimizer. The proof of the lemma is complete. □

Remark. It is actually possible that $l_+ < l_+^\infty$ and/or $l_- > l_-^\infty$. Assume $W = \frac{\pi^2}{8}(1-u^2)$ for $u \in (-1, 1)$. Then the solution of (2.1) that satifies $u(0) = 0$ is $u = \sin(\frac{\pi}{2}x)$, $x \in (-1, 1)$ and $J(u) = \frac{\pi^2}{4}$. Consider the sequence $\{u_j\}$ defined by

$$u_j(x) = \begin{cases} \sin(\frac{\pi}{2}x), & x \in (-1, 1-\epsilon_j), \\ \sin(\frac{\pi}{2}(1-\epsilon_j)), & x \in (1-\epsilon_j, x_j), \\ \sin(\frac{\pi}{2}(1-\epsilon_j + x - x_j)), & x \in (x_j, x_j+\epsilon_j), \end{cases}$$

where $\epsilon_j \to 0^+$ and $x_j \to +\infty$. We have $J(u_j) = \frac{\pi^2}{4} + \frac{\pi^2}{8}(x_j - 1 + \epsilon_j)\cos^2(\frac{\pi}{2}(1-\epsilon_j))$ and we can choose the sequence $\{x_j\}$ in such a way that $J(u_j) \to \frac{\pi^2}{4}$. Then $\{u_j\}$ is a minimizing sequence and it results $1 = l_+ < l_+^\infty = +\infty$.

Lemma 3.2. *The map u^* satisfies (1.8) in (l_-, l_+).*

Proof. Given x_0, x_1 with $l_- < x_0 < x_1 < l_+$, let $\phi : [x_0, x_1 + \xi] \to [x_0, x_1]$ be linear, with $|\xi|$ small, and with $\phi(x_0) = x_0, \phi(x_0 + \xi) = x_1$. Let $\psi : [x_0, x_1] \to [x_0, x_1 + \xi]$ be the inverse of ϕ. Define $u_\xi : [l_-, l_+ + \xi] \to \mathbb{R}^n$ by setting

$$u_\xi(x) = \begin{cases} u^*(x), & x \in [l_-, x_0], \\ u^*(\phi(x)), & x \in [x_0, x_1+\xi], \\ u^*(x-\xi), & x \in (x_1+\xi, l_+ + \xi] \end{cases} \tag{3.11}$$

Note that $u_\xi \in \mathcal{A}$ with $l_-^{u_\xi} = l_-$ and $l_+^{u_\xi} = l_+$ if $l_+ = +\infty$ and $l_+^{u_\xi} = l_+ + \xi$ if $l_+ < +\infty$. Since u^* is a minimizer we have

$$\frac{d}{d\xi} J(u_\xi, (l_-^{u_\xi}, l_+^{u_\xi}))|_{\xi=0} = 0. \tag{3.12}$$

From (3.11), using also the change of variables $x = \psi(s)$, it follows

$$\begin{aligned}
& J(u_\xi, (l_-^{u_\xi}, l_+^{u_\xi})) - J(u^*, (l_-, l_+)) \\
&= \int_{x_0}^{x_1+\xi} \Big(\frac{\dot{\phi}^2(x)}{2} |\dot{u}^*(\phi(x))|^2 + W(u^*(\phi(x))) \Big) dx - \int_{x_0}^{x_1} \Big(\frac{1}{2} |\dot{u}^*(x)|^2 + W(u^*(x)) \Big) dx \\
&= \int_{x_0}^{x_1} \Big(\frac{1 - \dot{\psi}(x)}{2\dot{\psi}(x)} |\dot{u}^*(x)|^2 + (\dot{\psi}(x) - 1) W(u^*(x)) \Big) dx \\
&= \int_{x_0}^{x_1} \Big(\frac{-\frac{\xi}{x_1 - x_0}}{2(1 + \frac{\xi}{x_1 - x_0})} |\dot{u}^*(x)|^2 + \frac{\xi}{x_1 - x_0} W(u^*(x)) \Big) dx \\
&= -\frac{\xi}{x_1 - x_0} \int_{x_0}^{x_1} \Big(\frac{|\dot{u}^*(x)|^2}{2(1 + \frac{\xi}{x_1 - x_0})} - W(u^*(x)) \Big) dx.
\end{aligned}$$

This and (3.12) imply

$$\int_{x_0}^{x_1} \Big(\frac{1}{2} |\dot{u}^*(x)|^2 - W(u^*(x)) \Big) dx = 0. \tag{3.13}$$

Since this holds for all x_0, x_1, with $l_- < x_0 < x_1 < l_+$, then (1.8) follows. □

On the basis of Lemmas 3.1 and 3.2 $u^* : (l_-, l_+) \to \mathbb{R}^m$ can be identified with the map u in Theorem 1.1. To complete the proof of Theorem 1.1 it remain to show that if W is of class C^1 in $\mathbb{R}^m \setminus A$, then u^* is a classical solution of (1.1). Since u^* is a minimizer, if $w : (l_1, l_2) \to \mathbb{R}^m$ is a smooth map that satisfies $w(l_i) = 0$, $i = 1, 2$ we have

$$0 = \frac{d}{d\lambda} J(u^* + \lambda w)|_{\lambda=0} = \int_{l_1}^{l_2} (\dot{u}^* \cdot \dot{w} + W_u(u^*) \cdot w) dx = \int_{l_1}^{l_2} (\dot{u}^* - \int_{l_1}^{x} W_u(u^*(s)) ds) \cdot \dot{w} dx. \tag{3.14}$$

Since this is valid for all $l_- < l_1 < l_2 < l_+$ and $\dot{w} : (l_1, l_2) \to \mathbb{R}^m$ is an arbitrary smooth map with zero average (3.14) implies

$$\dot{u}^* = \int_{l_1}^{x} W_u(u^*(s)) ds + const.$$

The continuity of u^* and of W_u implies that the right hand side of this equation is a map of class C^1. It follows that we can differentiate and obtain

$$\ddot{u}^* = W_u(u^*), \quad x \in (l_-, l_+).$$

The proof of Theorem 1.1 is complete.

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