LOCALLY ISOPERIMETRIC PARTITIONS

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Abstract. Locally isoperimetric N-partitions are partitions of the space $\mathbb{R}^d$ into N regions with prescribed, finite or infinite measure, which have minimal perimeter (which is the $(d-1)$-dimensional measure of the interfaces between the regions) among all variations with compact support preserving the total measure of each region. In the case when only one region has infinite measure, the problem reduces to the well known problem of isoperimetric clusters: in this case the minimal perimeter is finite, and variations are not required to have compact support.

In a recent paper by Alama, Bronsard and Vriend, the definition of isoperimetric partition was introduced, and an example, namely the *lens* partition, was shown to be locally isoperimetric in the plane. In the present paper we are able to give more examples of isoperimetric partitions: in any dimension $d \geq 2$ we have the *lens*, the *peanut* and the *Releaux* triangle. For $d \geq 3$ we also have a *thetrahedral* partition. To obtain these results we prove a *closure* theorem which enables us to state that the L^1_{loc}-limit of a sequence of isoperimetric clusters is an isoperimetric partition, if the limit partition is composed by *flat* interfaces outside a large ball. In this way we can make use of the known results about standard clusters.

In the planar case $d = 2$ we have a complete understanding of locally isoperimetric partitions: they exist if and only if the number of regions with infinite area is at most three. Moreover if the total number of regions is at most four then, up to isometries, there is a unique locally isoperimetric partition which is the lens, the peanut or the Releaux partition already mentioned.

Contents

1. Introduction

The isoperimetric problem is to find a set $E \subset \mathbb{R}^d$ which minimizes its perimeter $P(E)$ among all sets with fixed measure $|E| = m$. It is well known that the solution to this problem (unique up to translation) is the ball of volume m.

The isoperimetric problem can be extended to clusters, i.e. to N-uples of disjoint sets $E_1, \dots, E_N$ (regions) which have prescribed finite measure $|E_k| = m_k$ and which minimize the total surface area of the interfaces. Since the common boundary between two regions is only counted once, the regions in an isoperimetric cluster are indeed encouraged to share part of their boundary. This effectively forms a cluster of bubbles. The existence of isoperimetric clusters with given measures is guaranteed by the direct method of the calculus of variations [2]. The problem can be settled in the family of sets with finite perimeter, also called Caccioppoli sets. The perimeter of a Caccioppoli set is indeed lower semicontinuous with respect to L^1 convergence of sets. Even if there is no compactness (since $\mathbb{R}^d$ is unbounded) one can still use the tools of concentration compactness to obtain a minimizer (see, for example, [13, 17]). Isoperimetric clusters are partially regular in the sense that up to a closed *singular* set of zero $(d-1)$-dimensional measure the boundary of each region is a smooth hypersurface with constant mean curvature (see [2, 13]). In the planar case, $d = 2$, the boundaries between two regions are composed by a finite number of circular arcs or

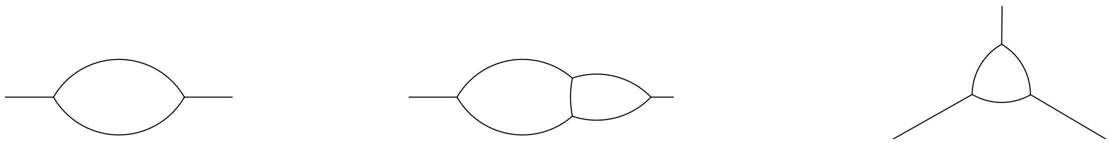

Figure 1. The lens, the peanut and the Releaux partition.

straight lines segments, the singular set is composed by a finite number of points (vertices), and in each vertex exactly three boundaries meet together with equal angles.

In the case $N \leq d+1$ for any given N-uple of positive measures, there is a natural cluster, the so called *standard* N-bubble (see Definition 3.1), which is conjectured to be the unique minimizer, up to isometries, for the isoperimetric problem.

For $N = 2$ (double bubbles) isoperimetric clusters have been proven to be standard by Foisy et al. [7] when $d = 2$, and by Hutchinson-Morgan-Ritoré-Ros [9] when $d \geq 2$. For $N = 3$ (triple bubbles) the result (isoperimetric clusters are standard) has been obtained by Wichiramala [24] in the case $d = 2$, and by the recent results of Milman and Neeman [14] when $d \geq 2$. For $N = 4$ the same result has been obtained again in [14] when $d \geq 4$. For $N = 5$, $d \geq 5$ in [15] the same authors prove that standard clusters are isoperimetric. They also prove, [14, Theorem 1.9], that any isoperimetric N-cluster of $\mathbb{R}^d$, has connected regions if $N \leq d$.

In the case $N \geq d+2$ there is no notion of standard cluster. But even in this case it is conjectured that all regions comprising an isoperimetric cluster should be connected. Even in the simplest case $N = 4$ and $d = 2$ the problem is open (but see [20, 19] for the case $d = 2$, $N = 4$ and equal areas).

Inspired by the paper [1] we are going to consider the case when some regions of the cluster may have infinite measure. If $E_1, \dots, E_N$ is a cluster in $\mathbb{R}^d$ we can add an *external* region E_0 of infinite measure to obtain a *partition* $E_0, E_1, \dots, E_N$ of the whole space $\mathbb{R}^d$. An N-cluster can thus be considered as an $(N+1)$-partition where one of the regions has infinite measure, and the others have prescribed finite measure. If we require two (or more) regions to have infinite measure then the partition cannot have finite perimeter and hence it makes no sense to minimize the perimeter of the whole partition. We are thus led to consider a different, local notion of minimizer as follows: a partition is said to be *locally isoperimetric* if for every bounded set Ω the perimeter of the partition in the interior of Ω is minimal among all partitions composed by regions with the same prescribed measures whose difference (set theoretic symmetric difference) with the corresponding region of the original partition is compactly contained in Ω (see Definition 2.2 below).

In [1] the authors consider the planar case $d = 2$ with $|E_0| = |E_1| = \infty > |E_2| > 0$, and introduce the concept of locally isoperimetric partitions. Moreover they show that the *lens* partition, whose boundary is composed by two symmetric arcs and two half lines joining with equal angles, (see Figure 1) is, in fact, locally isoperimetric.

In this paper we investigate the properties of locally isoperimetric partition. First of all we aim to prove a closure theorem. The idea is that isoperimetric clusters are a particular case of partitions with a single infinite region. But any partition, even with two or more infinite regions, can be seen as the limit (in the sense of L^1_{loc} topology) of a sequence of clusters. We are not currently able to show that the limit of any sequence of isoperimetric clusters is indeed a locally isoperimetric partition: the problem is that the local volume constraint of the finite regions which converge to an infinite region is lost in the limit. However we are able to prove a closure result (see Theorem 2.13) which guarantees that such a limit is isoperimetric with respect to variations which preserve the volumes not only of the finite regions but also of the infinite ones (since the variations have bounded support one can put a constraint on the volume in a large ball, taking into account that outside the ball the region coincides with its variation). Next we show that if the limit partition is *flat* at infinity (see Definition 2.15) then we are able to prove that the constraint on the infinite regions can be dropped because flat regions can be modified by a variation which allows us to change the volume by any large amount with a small change of the perimeter (see Theorem 2.16). These results allow us to find many examples of isoperimetric partitions in $\mathbb{R}^d$. The lens partition in $\mathbb{R}^d$ is the limit of double bubbles hence we can state that it is locally isoperimetric in any dimension. By taking the limit of a triple bubble we obtain the *peanut* partition, which is composed by two finite and two infinite regions, in any dimension. Again, by considering the limit of triple bubbles we obtain a partition with three infinite regions and only one finite region that we call *Releaux partition* since in dimension $d = 2$ the finite region is a Releaux triangle. See Figure 1. The other cases covered by the results of [14] are the partitions obtained as limits of quadruple bubbles in $\mathbb{R}^d$ for $d \geq 4$.

In the planar case we are able to find some more results. First of all we show that all locally isoperimetric partitions can have only one, two or three infinite regions. Then we prove that if we assign any number of areas for the finite regions, and we have from one to three infinite regions, a locally isoperimetric partition, with the given areas, always exists (Theorem 4.3). This is achieved by taking minimal clusters with the prescribed areas inside a ball whose radius is going to infinity. The difficult point in this approach is that of proving that no mass escapes to infinity: it is here that we need the assumption $d = 2$ and use some tools already developed for planar clusters. In the cases of the examples above, namely the lens (two infinite and one finite region), the peanut (two infinite and two finite regions), the Releaux triangle (three infinite and one finite region) we can show that these example, up to isometries, are in fact the unique locally isoperimetric partitions with their prescribed areas (Theorem 4.5).

In the Appendix we sketch how to adapt some statements on isoperimetric clusters, *e.g.* the so called Almgren's Lemma and the Infiltration Lemma, to limits of locally isoperimetric partitions, to get volume density estimates and a priori boundedness of their regions with finite volume (see also Theorem 2.4).

To summarize, in this paper we give a clear understanding of what locally isoperimetric partitions in the plane look like. We have also many examples in higher dimension, which rely on the corresponding results for clusters. These results are obtained by means of a closure theorem which is valid in any dimension while in the planar case we adapt to infinite regions the tools already developed in the study of planar clusters.

There are many questions left open. Of course any question which is open for the case of clusters is open in the case of partitions. For example we do not know, in general, if the bounded regions of an isoperimetric partition are connected. Also, in the case $d \geq 3$, when we prove that standard partitions are locally isoperimetric we are not able to prove that they are the unique example (up to isometries) with their volume.

A particular case of isoperimetric partitions is the case when all regions have infinite measure. In this case, since there are no volume constraints, isoperimetric partitions can be called *locally minimal partitions*. If $N = 2$ and $d \leq 7$ it is known that all locally minimal partitions are half-spaces (see [22, 8]) For $N = 2$ and $d \geq 8$ there are also minimal cones which are not hyperplanes, for example Simons cone (see [5]) and Lawson cones (see [12]). These are examples of locally isoperimetric partitions which are not standard. For $N = 2$ and $d \geq 9$ it is possible to find locally minimal partitions which are not cones (this is the Bernstein problem, see [5]). For any N in $d = 3$ all locally minimal conical partitions have been classified (see [23]): apart from the half-spaces there are only the dihedral angle with 120 degrees and the tetrahedral cone. Of course a cylinder over a locally minimal partition is also locally minimal, so we can produce many examples also in higher dimension, but a complete classification is still missing in the case $d > 3$. There are also a few more examples of minimal cones with $N > 2$ and $d > 3$: for $N = d + 1$ the cone over the skeleton of the symplex (see [10]) and for $N = 2d$ the cone over the skeleton of the hypercube (see [6]).

Notice that in $d \geq 8$ the Simons and Lawson cones are locally isoperimetric partitions which are not eventually flat (see Definition 2.15). However, it seems reasonable that the flatness condition that we have used in our examples could be replaced with the condition of having zero mean curvature, with some decay at infinity of the second fundamental form. In this respect we could expect to find a locally isoperimetric partition in $\mathbb{R}^8$ with prescribed volumes $(1, \infty, \infty)$ which is not eventually flat and hence is different from the lens partition with the same volumes.

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2. Notation and preliminary results

We shall denote by ω_d the volume of the unit ball in $\mathbb{R}^d$. A set $E \subset \mathbb{R}^d$ is said to be a *Caccioppoli set* or *set with locally finite perimeter* if E is measurable and the distributional derivative $D\mathbb{1}_E$ of its characteristic function is a Radon vector measure. We define $P(E, B)$, the perimeter of E in B by means of the total variation of the characteristic function: $\|D\mathbb{1}_E\|(B)$. We denote $P(E, \mathbb{R}^d)$ by $P(E)$.

We identify Caccioppoli sets which differ by a set of measure zero. In particular we can always choose a suitable representant of E so that the topological boundary coincides with the measure theoretic boundary (see for example [13, proposition 12.19]):

$$\partial E = \left\{x \in \mathbb{R}^d \colon \forall \rho > 0 \colon 0 < |E \cap B_\rho(x)| < |B_\rho|\right\}.$$

Notice that ∂E is a closed set. For a general measurable set E one can define a *reduced boundary* $\partial^* E \subset \partial E$ comprising the points where the approximate outer normal unit vector ν_E can be defined so that $D\mathbb{1}_E = \nu_E \cdot \|D\mathbb{1}_E\|$. When E is a Caccioppoli set one has $P(E, B) = \mathcal{H}^{d-1}(B \cup \partial^* E)$, for every borel set B.

Definition 2.1 (partition)**.** Let $\mathbf{E} = (E_1, \dots, E_N)$ be an N-uple of measurable subsets of $\mathbb{R}^d$, and Ω an open set. We say that $\mathbf{E}$ is an N-*partition* (of Ω), with *regions* $E_1, \dots, E_N$, if $|E_k \cap E_j \cap \Omega| = 0$ for every $k \neq j$ and $|\Omega \setminus \bigcup E_k| = 0$.

The *boundary* $\partial\mathbf{E}$ of a partition is the set of all interfaces between the regions:

$$\partial\mathbf{E} := \bigcup_{k=1}^{N} \partial E_k.$$

We define the *perimeter* of $\mathbf{E}$ on any borel set $B \subset \mathbb{R}^d$ as

$$P(\mathbf{E}, B) := \frac{1}{2}\sum_{k=1}^{N} P(E_k, B), \qquad P(\mathbf{E}) = P(\mathbf{E}, \mathbb{R}^d).$$

This quantity represents the $(d-1)$-dimensional surface area of the interfaces between the regions E_k inside B. In fact when $\mathbf{E}$ is sufficiently regular (namely when $\mathcal{H}^{d-1}(\partial E_k \setminus \partial^* E_k) = 0$ for all k) we have $P(\mathbf{E}, B) = \mathcal{H}^{d-1}(\partial\mathbf{E} \cap B)$.

Notice that we do not require the regions of a partition to have positive measure. If some of the regions have zero measure we say that the partition is *improper*, otherwise we say that it is *proper*. Improper partitions are useful to describe the limit of a sequence of proper partitions when the measure of some region goes to zero. Handling of improper regions is one of the difficulties we had to face in Theorem 2.8 and Theorem 2.13.

Definition 2.2 (locally isoperimetric partition)**.** We say that an N-partition $\mathbf{E} = (E_1, \dots, E_N)$ of $\Omega \subset \mathbb{R}^d$ is a *locally isoperimetric partition* if for every compact set $B \subset \Omega$ given any partition $\mathbf{F} = (F_1, \dots, F_N)$ such that for all k one has $|(E_k \triangle F_k) \setminus B| = 0$ and $|E_k| = |F_k|$ one has

$$P(\mathbf{E}, B) \leq P(\mathbf{F}_k, B).$$

If $\mathbf{E} = (E_0, E_1, \dots, E_N)$ is an $(N+1)$-partition of $\mathbb{R}^d$ such that $|E_k| < +\infty$ for all $k \neq 0$ we say that $\mathbf{E}$ is an N-*cluster* in $\mathbb{R}^d$. The sets E_k are called regions of the cluster $\mathbf{E}$. The region E_0 necessarily has infinite measure ad hence is always unbounded. It is usually called the *exterior region*. The cluster is determined by the N-uple $(E_1, \dots, E_N)$ of the finite regions (and this is the usual definition in the literature) because E_0 is uniquely determined as the complement of the union of the other regions. We choose to add E_0 to the definition so that N-clusters can be regarded as a particular case of $(N+1)$-partitions.

Notice that if a partition has at least two regions with infinite measure (we will say: *infinite* regions) then its perimeter is always $+\infty$. Clusters, instead, can have finite perimeter in the whole space, hence it is natural to give a *global* notion of minimizer for clusters.

Definition 2.3 (isoperimetric cluster)**.** We say that the N-cluster $\mathbf{E} = (E_0, \dots, E_N)$ is *isoperimetric* if for every other cluster $\mathbf{F} = (F_0, \dots, F_N)$ with $|F_k| = |E_k|$ for all $k \neq 0$ one has

$$P(\mathbf{E}) \leq P(\mathbf{F}).$$

Clearly every isoperimetric N-cluster $\mathbf{E}$ is also a locally isoperimetric $(N+1)$-partition. The converse statement is also true but requires a proof, see Proposition 2.11.

In the following theorem, we summarize some regularity results for locally isoperimetric partitions, that can be proven with minor modifications from the analogue statements for isoperimetric clusters (see the Appendix for a sketch of some of the proofs).

Theorem 2.4. *Let $\mathbf{E} = (E_1, \dots, E_N)$ be a locally isoperimetric N-partition in $\mathbb{R}^d$. Then $\partial\mathbf{E}$ is smooth outside a closed* singular *set of Hausdorff dimension at most $d-2$: each interface is an analytic hypersurface with locally constant mean curvature. The mean curvature is zero between two infinite regions.*

Moreover the following density estimates hold:

$$c_0 \omega_d r^d \leq |E_k \cap B(x,r)| \leq c_1 \omega_d r^d$$

for all $x \in \partial\mathbf{E}$ and $r < r_0$, with $c_0 = c_0(d,N)$, $c_1 = c_1(d,N)$ and $r_0 = r_0(\mathbf{E})$.

In particular, if $|E_k| < +\infty$ then E_k is bounded.

Lemma 2.5 (upper estimate of perimeter)**.** *Let $\mathbf{E} = (E_1, \dots, E_N)$ be a locally isoperimetric N-partition in $\mathbb{R}^d$. Then, for every $\mathbf{x} \in \mathbb{R}^d$ and any $R > 0$, one has*

$$P(\mathbf{E}, B_R(\mathbf{x})) \leq C_0 \cdot R^{d-1}$$

with $C_0 = C_0(d,N)$ a constant not depending on $\mathbf{E}$, $\mathbf{x}$ or R.

Proof. To simplify the notation suppose $\mathbf{x} = 0$. For every $\rho < R$ we can rearrange the regions of $\mathbf{E}$ inside the ball B_R into horizontal slices to construct a partition $\mathbf{F} = (F_1, \dots, F_N)$ such that for all k one has

(1) $F_k \setminus B_\rho = E_k \setminus B_\rho$,
(2) $|F_k \cap B_\rho| = |E_k \cap B_\rho|$,
(3) $\partial^*(F_k \cap B_\rho) \subset \partial B_\rho \cup \Pi_k \cup \Pi'_k$ where Π_k and Π'_k are parallel $(d-1)$-dimensional planes.

Since $\mathbf{E}$ is locally isoperimetric we have $P(\mathbf{E}, B_R) \leq P(\mathbf{F}, B_R)$ hence

$$\begin{aligned} 2P(\mathbf{E}, B_R) &\leq 2P(\mathbf{E}, B_R \setminus \overline{B_\rho}) + 2\mathcal{H}^{d-1}(\partial B_\rho) + 2\sum_{k=1}^{N} \mathcal{H}^{d-1}((\Pi_k \cup \Pi'_k) \cap B_\rho) \\ &\leq 2P(\mathbf{E}, B_R \setminus \overline{B_\rho}) + 2d\omega_d \rho^{d-1} + 4N\omega_{d-1}\rho^{d-1}. \end{aligned}$$

As $\rho \to R^-$ we have $P(\mathbf{E}, B_R \setminus \overline{B_\rho}) \to 0$ and the result follows. □

The following lemma uses the coarea formula to estimate the cost in perimeter when two Caccioppoli sets are joined together along the surface of a sphere.

Lemma 2.6 (glueing)**.** *Let E and F be Caccioppoli sets in $\mathbb{R}^d$. Define*

$$G_\rho = (E \cap B_\rho) \cup (F \setminus B_\rho).$$

Then for all $0 < r < R$ one has

$$\begin{aligned} \int_r^R P(G_\rho, \partial B_\rho)\, d\rho &= \int_r^R \left[P(G_\rho, B_R) - P(E, B_\rho) - P(F, B_R \setminus B_\rho)\right] d\rho \\ &= \left|(E \triangle F) \cap (B_R \setminus B_r)\right|, \end{aligned}$$

so that the set of $\rho \in (r,R)$ for which the estimate

$$P(G_\rho, \partial B_\rho) \leq \frac{|(E \triangle F) \cap (B_R \setminus B_r)|}{R - r}$$

holds, has positive measure.

Proof. For every ρ by additivity of measures one has $P(G_\rho, \partial B_\rho) = P(G_\rho, B_R) - P(E, B_\rho) - P(F, B_R \setminus \overline{B_\rho})$. Since E and F are Caccioppoli sets for almost all $\rho \in [r, R]$ we have

$$P(E, \partial B_\rho) = P(F, \partial B_\rho) = 0.$$

In particular $P(F, B_R \setminus \overline{B_\rho}) = P(F, B_R \setminus B_\rho)$. Moreover for all these ρ, denoting by E^1 and F^1 the points of density 1 of E and F respectively, we have (see for example [13, Chapter 15]):

$$\begin{aligned} P(G_\rho, \partial B_\rho) &= \mathcal{H}^{d-1}(\partial^* G_\rho \cap \partial B_\rho) \\ &= \mathcal{H}^{d-1}(((\partial^* E) \cap B_\rho \cap \partial B_\rho) \triangle ((\partial^* F) \setminus B_\rho \cap \partial B_\rho)) = \\ &= \mathcal{H}^{d-1}((\partial^*(E \cap B_\rho) \cap \partial B_\rho) \triangle (\partial^*(F \setminus B_\rho) \cap \partial B_\rho)) = \\ &= \mathcal{H}^{d-1}\left((E^1 \cap \partial B_\rho) \triangle (F^1 \cap \partial B_\rho)\right) \end{aligned}$$

Integrating in $d\rho$ on the interval $[r, R]$, and using coarea formula, we get the desired equality. □

Lemma 2.7 (perimeter estimate)**.** *Let $\mathbf{E} = (E_0, \dots, E_N)$ be a locally isoperimetric $(N+1)$-partition in $\mathbb{R}^d$ which is an N-cluster, i.e. $m = |E_1| + \dots + |E_N| < +\infty$. Then*

$$P(\mathbf{E}) \le C < +\infty$$

where $C = C(d, N, m)$ does not depend on $\mathbf{E}$.

Proof. Let us fix $\varepsilon > 0$. Since $|E_k| < +\infty$ for $k \neq 0$, we can find a radius $R \ge \frac{1}{\varepsilon}$ so large that $|E_k \setminus B_R| < \varepsilon/N$. Further enlarging R, we can also assume that B_R compactly contains N pairwise disjoint balls of volumes $|E_k|$, for $k = 1, \dots, N$. Since the partition has locally finite perimeter one has that $\mathcal{H}^{d-1}(\partial B_\rho \cap \partial^* E_k) = 0$ for all $k \ge 0$. and almost all $\rho > 0$. Among these $\rho > 0$, by Lemma 2.6 (applied with $\emptyset$ in place of E, with E_k in place of F, and $[R, R+1]$ in place of $[r, R]$) we can find $\rho = \rho(\varepsilon) \in (R, R+1)$ such that $P(E_k \setminus B_\rho, B_{R+1}) - P(E_k, B_{R+1} \setminus B_R) = P(E_k \setminus B_\rho, \partial B_\rho) < \varepsilon/N$, for all $k \ge 1$. Hence we can consider a new partition $\mathbf{F} = (F_0, \dots, F_N)$ such that:

(1) $F_k \setminus B_\rho = E_k \setminus B_\rho$, for all $k = 0, \dots, N$,
(2) $|F_k| = |E_k|$, for all $k = 0, \dots N$,
(3) for $k \neq 0$ then the intersections $F_k \cap B_\rho$ are pairwise disjoint balls compactly contained in B_ρ with measure $|F_k \cap B_\rho| = |E_k \cap B_\rho|$,
(4) $F_0 = \mathbb{R}^d \setminus \bigcup_{k=1}^{N} F_k$,

Note that since F_k are Caccioppoli sets, and ∂B_ρ can be viewed as the intersection of a decreasing sequence of open sets on which $P(\mathbf{F}, \cdot)$ is finite, one has

$$P(F_0, \partial B_\rho) = P\left(\bigcup_{k=1}^{N} F_k, \partial B_\rho\right) \le \sum_{k=1}^{N} P(F_k, \partial B_\rho).$$

Now, since $\mathbf{E}$ is locally isoperimetric, and taking in account that $\overline{B_R}$ is the intersection of a decreasing sequence of open sets on which $P(F_k, \cdot)$ is finite, we have

$$\begin{aligned} P(\mathbf{E}, \overline{B_\rho}) \le P(\mathbf{F}, \overline{B_\rho}) &= \frac{1}{2}\sum_{k=0}^{N} P(F_k, \overline{B_\rho}) \\ &= \frac{1}{2}\sum_{k=0}^{N} P(F_k, B_\rho) + \frac{1}{2}\sum_{k=0}^{N} P(F_k, \partial B_\rho) \\ &= C_d \sum_{k=1} |E_k \cap B_\rho|^{\frac{d-1}{d}} + \frac{1}{2}\sum_{k=0}^{N} P(F_k, \partial B_\rho) \le C_d N m^{\frac{d-1}{d}} + \varepsilon. \end{aligned}$$

Hence, since $\rho > R \ge \frac{1}{\varepsilon}$, letting $\varepsilon \to 0^+$, we get

$$P(\mathbf{E}) \le C_d N m^{\frac{d-1}{d}}.$$

□

Theorem 2.8 (volume fixing, possibly improper, variations)**.** *Let Ω be an open subset of $\mathbb{R}^d$. Let $\mathbf{F} = (F_1, \dots, F_N)$ be a (possibly improper) N-partition of Ω. For all $k = 1, \dots, N$ with $|F_k| > 0$, let $x_k \in \Omega$, and $\rho_k > 0$, be given so that the balls $B_{\rho_k}(x_k)$ are contained in Ω, are pairwise disjoint and $|F_k \cap B_r(x_k)| > \frac{1}{2}\omega_d r^d$ for all $r \le \rho_k$.*

Let $A = \bigcup_k B_{\rho_k}(x_k)$. Then for every $\mathbf{a} = (a_0, \dots, a_N)$ with $a_k \ge -|F_k \cap B_{\rho_k}(x_k)|$ when $|F_k| > 0$, $a_k \ge 0$ when $|F_k| = 0$, and $\sum_{k=1}^N a_k = 0$, there exists a partition $\mathbf{F}' = (F_1', \dots, F_N')$ such that for all $k = 1, \dots, N$:

1. $F_k' \triangle F_k \subset A$,
2. $|F_k' \cap A| = |F_k \cap A| + a_k$,
3. $P(F_k', A) \le P(F_k, A) + C_1 \cdot \displaystyle\sum_{j=1}^N |a_j|^{1-\frac{1}{d}}$,

with $C_1 = C_1(d, N)$ not depending on $\mathbf{F}$.

Remark 2.9. Notice that it is always possible to find such balls $B_{\rho_k}(x_k)$: since $|F_k| > 0$ just take any point x_k of full density in F_k, and ρ_k sufficiently small.

Remark 2.10. The previous theorem can be compared to the so called *Volume fixing variations* theorem leading to *Almgren's Lemma* (see Appendix 5.1, 5.2, [13, Theorem 29.14, Corollary 29.17]) with two important differences. First of all we do not require the regions to have positive measure. This enables us to make a *blow-down* with regions having measure going to zero (see the proof of Theorem 4.5). On the other hand our estimate (3) is weaker than in the usual volume fixing variations theorem, since in (3) we have an exponent $1 - \frac{1}{d}$ instead of 1. If $|F_k| = 0$ the exponent $1 - \frac{1}{d}$ is optimal in view of the isoperimetric inequality. Otherwise one could prove that the exponent $1 - \frac{1}{d}$ can be in fact replaced with 1 (this, however, requires a longer and more refined proof which we prefer to avoid here).

Proof of Theorem 2.8. Let $J = \{j \colon a_j < 0\}$. For all $j \in J$ take r_j such that $\left|F_j \cap B_{r_j}(x_j)\right| = -a_j$. By assumption $0 < -a_j \le \left|F_j \cap B_{\rho_j}(x_j)\right|$ and hence r_j exists and $0 < r_j \le \rho_j$. Coinsider the sets $F_k'' = F_k \setminus \bigcup_{j \in J} B_{r_j}(x_j)$. Clearly we have $|F_k'' \cap A| \le |F_k \cap A| + a_k$ for all $k = 1, \dots, N$ so that we are now required to add measure to each region. Since $\sum a_k = 0$ the total measure we need to add, which is $\sum_k |F_k \cap A| - \sum_k |F_k'' \cap A|$ is exactly equal to the total measure of the balls we have removed $\sum_{j \in J} \left|B_{r_j}(x_j)\right|$. This means that it is possible to find a partition $\mathbf{C} = (C_1, \dots, C_N)$ of $\bigcup_{j \in J} B_{r_j}(x_j)$ with $|C_k| = |F_k \cap A| - |F_k'' \cap A| + a_k \ge 0$. So we consider the partition $\mathbf{F}'$ with regions $F_k' = F_k'' \cup C_k$ to obtain the desired volumes:

$$|F_k' \cap A| = |F_k \cap A| + a_k.$$

By construction $F_k' \triangle F_k \subset A$ for all $k = 1, \dots, N$. To estimate the perimeter we observe that if we choose $\mathbf{C}$ by making slices of the balls with parallel planes for each $j \in J$, we are adding at most N slices, and also at most the perimeter of the ball. So the increase in perimeter is at most:

$$P(\mathbf{F}', \bar{A}) - P(\mathbf{F}, \bar{A}) \le \sum_{j \in J} \left[N\omega_{d-1} r_k^{d-1} + d \cdot \omega_d r_k^{d-1} \right].$$

Since, by assumption, $|a_j| = \left|F_j \cap B_{r_j}(x_j)\right| \ge \frac{1}{2}\omega_d r_j^d$ we have

$$r_j^{d-1} \le \frac{2}{\omega_d} |a_j|^{1-\frac{1}{d}}$$

hence, as desired,

$$P(\mathbf{F}', \bar{A}) - P(\mathbf{F}, \bar{A}) \le \sum_{j \in J} 2 \cdot \frac{N\omega_{d-1} + d \cdot \omega_d}{\omega_d} |a_j|^{1-\frac{1}{d}} \le C_1 \sum_{j=1}^N |a_j|^{1-\frac{1}{d}}.$$

□

Proposition 2.11 (equivalence of isoperimetric clusters and locally isoperimetric partitions)**.** *If* $\mathbf{E} = (E_0, E_1, \dots, E_N)$ *is an* N*-cluster in* $\mathbb{R}^d$ *then* $\mathbf{E}$ *is an isoperimetric* N*-cluster if and only if* $\mathbf{E}$ *is a locally isoperimetric* $(N+1)$*-partition.*

By using Almgren's Lemma (see Theorem 2.4, Lemma 5.2, Corollary 5.8) one could prove that the regions with finite measure of a locally isoperimetric partition are, in fact, bounded. This would make the following proof much easier, however are able to present a self contained proof which uses Theorem 2.8 instead of the classical one, adapted to partitions (see Appendix Theorem 5.1, and Lemma 5.2).

Proof of Proposition 2.11. Notice that any competitor in the definition of a locally isoperimetric partition is also a competitor in the definition of an isoperimetric cluster, where we drop the requirement of the variation to have compact support. Hence it is clear that an isoperimetric cluster is an isoperimetric partition.

On the other hand let $\mathbf{E} = (E_0, \dots, E_N)$ be a locally isoperimetric partition with $|E_k| < +\infty$ for $k \neq 0$ and let $\mathbf{F} = (F_0, \dots, F_N)$ be a *global* variation i.e. a partition such that $|F_k| = |E_k|$ for $k \neq 0$ (necessarily $|F_0| = |E_0| = +\infty$). To prove that $\mathbf{E}$ is an isoperimetric cluster it is enough to show that given any $\varepsilon > 0$ we have

$$P(\mathbf{E}) \leq P(\mathbf{F}) + 2\varepsilon.$$

By Lemma 2.7 we know that each E_k has finite perimeter. Suppose also $P(\mathbf{F}) < +\infty$ (otherwise there is nothing to prove).

Consider a large radius $\tilde{R}$ so that

$$\sum_{k=1}^{N} |F_k \setminus B_R| + |E_k \setminus B_R| < \varepsilon, \qquad \sum_{k=0}^{N} P(F_k \setminus B_R) + P(E_k \setminus B_R) < \varepsilon.$$

and define

$$F'_k = (F_k \cap B_R) \cup (E_k \setminus B_R).$$

Since for $k \neq 0$ the region E_k has finite measure and the complementary of E_0 has also finite measure, using Lemma 2.6 in an interval $[\tilde{R}, \tilde{R} + \delta]$ we choose $R > 0$ large enough we get

$$P(\mathbf{F}') \leq P(\mathbf{F}, B_R) + P(\mathbf{E}, \mathbb{R}^d \setminus B_R) + 2\varepsilon \leq P(\mathbf{F}) + 4\varepsilon.$$

Now applying Theorem 2.8 we can slightly modify $\mathbf{F}'$ inside B_R to obtain a partition $\mathbf{G} = (G_0, \dots, G_N)$ such that $|G_k| = |E_k|$ and $G_k \triangle E_k$ is bounded for all $k = 0, \dots, N$. Hence we can finally state that $P(\mathbf{E}) \leq P(\mathbf{G})$. Whence

$$P(\mathbf{E}) \leq P(\mathbf{G}) \leq P(\mathbf{F}') + \varepsilon \leq P(\mathbf{F}) + 4\varepsilon.$$

□

Definition 2.12 (isoperimetric partition with mixed constraint)**.** Let $J \subset \{1, \dots, N\}$ be a fixed set of indices. We say that an N-partition $\mathbf{E} = (E_1, \dots, E_N)$ of an open set Ω is locally J-isoperimetric, if, whenever we are given a compact set $B \subset \Omega$ and a partition $\mathbf{F} = (F_1, \dots, F_N)$ of Ω such that $F_i \triangle E_i \subset B$ for all $i = 1, \dots, N$, and $|F_j \cap B| = |E_j \cap B|$ for all $j \in J$, then, we have,

$$P(\mathbf{E}, B) \leq P(\mathbf{F}, B).$$

Theorem 2.13 (closure for J-isoperimetric partitions)**.** *Let* J *be a subset of* $\{1, \dots, N\}$ *and let* $\mathbf{E}^k = (E_1^k, \dots, E_N^k)$ *be a sequence of locally* J*-isoperimetric (possibly improper)* N*-partitions of* $\Omega_k \subset \mathbb{R}^d$ *where* Ω_k *is an increasing sequence of open sets such that* $\bigcup_k \Omega_k = \mathbb{R}^d$*. Suppose that there exists* $\mathbf{E} = (E_1, \dots, E_N)$*, a partition of* $\mathbb{R}^d$*, such that for all* $j = 1, \dots, N$*, we have* $E_j^k \to E_j$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *as* $k \to +\infty$*.*

Then $\mathbf{E} = (E_1, \dots, E_N)$ *is a locally* J*-isoperimetric partition of* $\mathbb{R}^d$*.*

Remark 2.14. Since given any $K \subset \mathbb{R}^d$, compact set, one has $\Omega_k \supset K$ for k large enough, the L^1_{loc} convergence makes sense in this setting.

Proof. Let $\mathbf{F} = (F_1, \dots, F_N)$ be a competitor to $\mathbf{E}$ i.e. an N-partition of $\mathbb{R}^d$ such that $F_i \triangle E_i \Subset \mathbb{R}^d$ for all $i = 1, \dots, N$ and $|F_j \setminus E_j| = |E_j \setminus F_j|$ for all $j \in J$. For all j such that $|F_j| > 0$ we can take any point $\mathbf{x}_j \in \mathbb{R}^d$ with full density in F_j. Then take ρ_j such that the hypotesis of Theorem 2.8 are satisfied. Define

$$\mu = \frac{1}{N} \cdot \min_{|F_j|>0} \left| F_j \cap B_{\rho_j}(\mathbf{x}_j) \right|.$$

Let A be the union of the balls $B_{\rho_j}(\mathbf{x}_j)$ given by the Theorem. Let $r > 0$ be large enough so that $A \Subset B_r$ and $E_i \triangle F_i \Subset B_r$ for all $i = 0, \dots, N$. By slightly enlarging r we can also assume that $P(\mathbf{E}, \partial B_r) = P(\mathbf{F}, \partial B_r) = 0$ and hence

$$P(\mathbf{E}, \bar{B}_r) = P(\mathbf{E}, B_r), \qquad P(\mathbf{F}, \bar{B}_r) = P(\mathbf{F}, B_r). \tag{1}$$

Let $\varepsilon > 0$ be given. By possibly decreasing ρ_j we can assume that

$$C_1 \cdot N^2 \cdot \mu^{1-\frac{1}{d}} \leq \varepsilon \tag{2}$$

where C_1 is the constant given by Theorem 2.8.

To conclude the proof it is enough to prove that

$$P(\mathbf{E}, B_r) \leq P(\mathbf{F}, B_r) + 4\varepsilon.$$

First choose $\delta > 0$, so that thanks (1)

$$P(\mathbf{E}, B_{r+\delta}) < P(\mathbf{E}, \bar{B}_r) + \varepsilon = P(\mathbf{E}, B_r) + \varepsilon. \tag{3}$$

Then we can choose $k \in \mathbb{N}$ sufficiently large so that $B_{r+\delta} \Subset \Omega_k$. Let

$$m_i = \left| (E_i^k \triangle E_i) \cap B_{r+\delta} \right|.$$

By the L^1_{loc} convergence of $\mathbf{E}^k$ to $\mathbf{E}$, by taking k sufficiently large, we might also assume that for all $i = 1, \dots, N$ one has

$$m_i \leq \mu, \qquad m_i \leq \frac{\delta \cdot \varepsilon}{N}. \tag{4}$$

Using the semicontinuity of perimeter we can finally also assume k so large that

$$P(\mathbf{E}, B_r) \leq P(\mathbf{E}^k, B_r) + \varepsilon. \tag{5}$$

Take now $\rho \in (r, r+\delta)$ and consider the N-partition $\mathbf{F}^k = (F_1^k, \dots, F_N^k)$ of Ω_k defined by

$$F_i^k = (F_i \cap B_\rho) \cup (E_i^k \setminus B_\rho), \qquad i = 1, \dots, N$$

so that F_i^k is a variation of E_i^k with compact support in $\bar{B}_\rho$.

By a suitable choice of ρ in the interval $(r, r+\delta)$, thanks to Lemma 2.6 and (4), we are not spending too much perimeter:

$$\begin{aligned} \left| P(\mathbf{F}^k, \bar{B}_\rho) - P(\mathbf{F}, \bar{B}_\rho) \right| = P(\mathbf{F}^k, \partial B_\rho) &\leq \frac{1}{\delta} \sum_{i=1}^N \left| (E_i \triangle E_i^k) \cap B_{r+\delta} \right| \\ &= \frac{1}{\delta} \sum_{i=1}^N m_i \leq \varepsilon. \end{aligned} \tag{6}$$

To have a competitor to the minimality of $\mathbf{E}^k$ we need to slightly modify the partition $\mathbf{F}^k$ in Ω_k to satisfy the mixed volume constraint. To achieve this we want to apply Theorem 2.8, modify the partition $\mathbf{F}$ inside of $A \Subset B_r \Subset B_\rho \Subset B_{r+\delta} \Subset \Omega_k$. Consider the sets of indices $K = \{j \notin J \colon |F_j| = 0\}$ and $L = \{j \notin J \colon |F_j| > 0\}$ so that $\{1, \dots, N\} = J \cup K \cup L$. Then define

$$a_j = \begin{cases} \left| E_j^k \cap B_\rho \right| - \left| F_j^k \cap B_\rho \right| & \text{if } j \in J \\ c & \text{if } j \in L \\ d & \text{if } j \in K \end{cases}$$

where c and d are defined by taking

$$S = \sum_{j\in J} a_j, \qquad c = \begin{cases} \text{irrelevant} & \text{if } L = \emptyset \\ -\frac{S}{\#L} & \text{if } L \neq \emptyset. \end{cases}, \qquad d = \begin{cases} \text{irrelevant} & \text{if } K = \emptyset \\ -\frac{S}{\#K} & \text{if } L = \emptyset \\ 0 & \text{if } L \neq \emptyset. \end{cases}$$

We claim that $\sum a_j = 0$, as required by Theorem 2.8. Notice that since both E^k_j and F^k_j cover the whole ball B_ρ we have

$$\sum_{j=1}^N \left|E^j_k \cap B_\rho\right| - \left|F^k_j \cap B_\rho\right| = 0.$$

So, if $K = \emptyset$ and $L = \emptyset$ the claim is verified. If $L \neq \emptyset$ then $d = 0$ and, by definition,

$$\sum_{j=1}^N a_j = S + \#L \cdot c + \#K \cdot d = S - S + 0 = 0.$$

Otherwise, if $L = \emptyset$ then

$$\sum_{j=1}^N a_j = S + \#K \cdot d = S - S = 0.$$

Now we want to prove that $a_j \geq -\left|F_j \cap B_{\rho_j}(\mathbf{x}_j)\right|$ when $|F_j| > 0$ while $a_j \geq 0$ when $|F_j| = 0$, as required by Theorem 2.8.

If $j \in J$ we have $F^k_j \cap B_\rho = F_j \cap B_\rho$ and $|F_j \cap B_\rho| = |E_j \cap B_\rho|$ hence $|a_j| = \left|\left|E^k_j \cap B_\rho\right| - |E_j \cap B_\rho|\right| \leq \left|(E^k_j \triangle E_j) \cap B_\rho\right| = m_j \leq \mu$, by (4). If $|F_j| > 0$ we have $\mu \leq \left|F_j \cap B_{\rho_j}(\mathbf{x}_j)\right|$, by definition, and hence $a_j \geq -\mu \geq -\left|F_j \cap B_{\rho_j}(\mathbf{x}_j)\right|$. If instead $|F_j| = 0$ just notice that $a_j = \left|E^k_j \cap B_\rho\right| \geq 0$ while $-\left|F_j \cap B_{\rho_j}(\mathbf{x}_j)\right| \leq 0$.

If $\ell \in L$ we have $|F_\ell| > 0$ and $a_\ell = c$. Hence

$$|a_\ell| = |c| = \frac{|S|}{\#L} \leq \sum_{j\in J} |a_j| \leq N \cdot \mu \leq |F_\ell \cap B_{\rho_\ell}(\mathbf{x}_\ell)|$$

by defintion of μ.

If $h \in K$ we have $a_h = d$ and $|F_h| = 0$. So to satisfy the hypothesis of Theorem 2.8 we have to prove that $a_h = d \geq 0$. If $L \neq \emptyset$ by definition we have $a_h = d = 0$ and the conclusion is trivial. If instead $L = \emptyset$, *i.e.* $|F_i| > 0 \Rightarrow i \in J$, by definition $a_h = d = -\frac{S}{\#K}$, and, since we are dealing with partitions, it follows:

$$\sum_{j\in J} \left|E^k_j \cap B_\rho\right| \leq \sum_{i=1}^N \left|E^k_i \cap B_\rho\right| = \sum_{i=1}^N |F_i \cap B_\rho| = \sum_{j\in J} |F_j \cap B_\rho|,$$

so that

$$-a_h \cdot \#K = S = \sum_{j\in J} \left|E^k_j \cap B_\rho\right| - |F_j \cap B_\rho| \leq 0.$$

Notice that in particular we have $|a_j| \leq N \cdot \mu$, for all $j = 1, \dots, N$.

We are now in the position to apply Theorem 2.8 to $\mathbf{F}^k$ in $A \Subset B_\rho$, getting a partition $\mathbf{G}^k$ of Ω_k, such that

$$G^k_j \triangle F_j = G^k_j \triangle F^k_j \subset A \Subset B_\rho, \qquad \text{for all } j = 1, \dots, N, \tag{7}$$

$$\left|G^k_j \cap B_\rho\right| = \left|E^k_j \cap B_\rho\right|, \qquad \text{for all } j \in J, \tag{8}$$

and, using (2),

$$P(\mathbf{G}^k, \bar{B}_\rho) - P(\mathbf{F}^k, \bar{B}_\rho) \leq C_1 \cdot \sum_{j=1}^N |a_j|^{1-\frac{1}{d}} \leq C_1 \cdot N^2 \cdot \mu^{1-\frac{1}{d}} \leq \varepsilon. \tag{9}$$

We eventually obtain that $\mathbf{G}^k$ is a competitor to $\mathbf{E}^k$ with the correct mixed volume constraint. Hence by local minimality of $\mathbf{E}^k$ one has:

$$P(\mathbf{E}^k, \bar{B}_\rho) \le P(\mathbf{G}^k, \bar{B}_\rho). \tag{10}$$

So the proof is concluded, using (1), and letting $\delta \to 0^+$ in the following inequality:

$$\begin{aligned}
P(\mathbf{E}, B_{r+\delta}) &\le P(\mathbf{E}, B_r) + \varepsilon && \text{by (3)}\\
&\le P(\mathbf{E}^k, B_r) + 2\varepsilon && \text{by (5)}\\
&\le P(\mathbf{E}^k, \bar{B}_\rho) + 2\varepsilon \\
&\le P(\mathbf{G}^k, \bar{B}_\rho) + 2\varepsilon && \text{by (10)}\\
&\le P(\mathbf{F}^k, \bar{B}_\rho) + 3\varepsilon && \text{by (9)}\\
&\le P(\mathbf{F}, \bar{B}_\rho) + 4\varepsilon && \text{by (6)}\\
&\le P(\mathbf{F}, B_{r+\delta}) + 4\varepsilon.
\end{aligned}$$

□

Definition 2.15 (eventually flat partitions)**.** We say that an N-partition $\mathbf{E} = (E_1, \dots, E_N)$ of $\mathbb{R}^d$ is *eventually flat* if for every pair $i \ne j$ of indices such that E_i and E_j have infinite measure there exists a $(d-1)$-dimensional half space contained in the interface $\partial E_i \cap \partial E_j$.

Theorem 2.16 (volume fixing of large volumes)**.** *Let $\mathbf{E} = (E_1, \dots, E_N)$ be an eventually flat partition of $\mathbb{R}^d$ and let $\varepsilon > 0$ and $\mathbf{a} = (a_1, \dots, a_N)$ be given. Suppose that $\sum a_k = 0$ and $a_k = 0$ if $|E_k|$ is finite. Then for every $r > 0$, there exists $R > r$ and a partition $\mathbf{F}$ of $\mathbb{R}^d$ such that for all $k = 1, \dots, N$ one has*

$$\begin{aligned}
&E_k \triangle F_k \Subset B_R \setminus \bar{B}_r\\
&|F_k \cap B_R| = |E_k \cap B_R| + a_k\\
&P(\mathbf{F}, B_R) \le P(\mathbf{E}, B_R) + \varepsilon.
\end{aligned}$$

Proof. By assumptions on a_k we only need to fix the volumes of the regions with infinite volume. Since the partition is assumed to be eventually flat, such regions have interfaces which contain arbitrarily large flat $(d-1)$-dimensional disks. To fix the volumes we can simply add or remove a cylinder of very large radius and very small height with basis on such disks. This enables us to obtain arbitrarily large changes in volumes with arbitrarily small change in perimeter. □

Theorem 2.17 (closure for locally isoperimetric partitions)**.** *Let $\mathbf{E}^k = (E^k_1, \dots, E^k_N)$ be a sequence of locally isoperimetric partitions. Suppose that there exists $\mathbf{E} = (E_1, \dots, E_N)$ a partition of $\mathbb{R}^d$ such that $E^k_i \to E_i$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\left|E^k_i\right| \to |E_i|$ whenever $|E_i| < +\infty$.*

If $\mathbf{E}$ is eventually flat, then $\mathbf{E}$ is itself a locally isoperimetric partition.

Proof. Let $J = \{j\colon |E_j| < +\infty\}$. For all $j \in J$, since $\left|E^k_j\right| \to |E_j|$, also $\left|E^k_j\right| < +\infty$ for k large enough. This means that $\mathbf{E}^k$ is in particular locally J-isoperimetric. By Theorem 2.13 we hence obtain that also $\mathbf{E}$ is locally J-isoperimetric. Let $\mathbf{F} = (F_0, \dots, F_N)$ be a competitor to $\mathbf{E}$ in the sense of local isoperimetricity. This means that $F_j \triangle E_j$ are bounded and that $|F_j| = |E_j|$ for all $j = 0, \dots, N$. In particular $\mathbf{F}$ is eventually flat as $\mathbf{E}$. Let $r > 0$ be so large that outside B_r the two partitions $\mathbf{E}$ and $\mathbf{F}$ coincide. Define, for all $j = 1, \dots, N$,

$$a_j = |E_j \cap B_r| - |F_j \cap B_r|.$$

If E_j has finite measure, since $E_j \triangle F_j \subset B_r$, we have $a_j = 0$. So for any $\varepsilon > 0$ we can apply Theorem 2.16 to obtain a partition $\mathbf{G}$ which differs from $\mathbf{F}$ only inside a larger ball B_R, which agrees with $\mathbf{F}$ inside B_r and such that

$$P(\mathbf{G}, B_R) \le P(\mathbf{F}, B_R) + \varepsilon.$$

Now we have $|G_j \cap B_R| = |E_j \cap B_R|$ for all $j = 1, \dots, N$ hence, by the J-isoperimetricity of $\mathbf{E}$ we conclude that

$$P(\mathbf{E}, B_R) \le P(\mathbf{G}, B_R) \le P(\mathbf{F}, B_R) + \varepsilon$$

and since $\mathbf{E}$ ane $\mathbf{F}$ agree outside B_r we conclude

$$P(\mathbf{E}, B_r) \le P(\mathbf{F}, B_r) + \varepsilon.$$

□

3. Standard isoperimetric partitions

Definition 3.1. (See [14]). We say that a partition $\mathbf{E} = (E_1, \dots, E_N)$ of $\mathbb{R}^d$ is *standard* if it can be obtained as *any stereographic projection* of an *equal-volume standard* $(N-1)$*-bubble* in $\mathbb{S}^d$, i.e. a partition of the sphere $\mathbb{S}^d$ which is the *Voronoi partition* corresponding to N equidistant points in $\mathbb{S}^d$ as a subset of $\mathbb{R}^{d+1}$.

If only one of the regions of a standard partition has infinite measure we say that the partition is a *standard cluster* or standard bubble.

We call standard N-partition, or $(N-1)$-bubble, of $\mathbb{S}^d$, any stereographic projection of an $(N-1)$-cluster of $\mathbb{R}^d$.

Remark 3.2. Standard N-partitions only exist for $N \le d+2$. For $N \le d+2$, $(N-1)$-standard clusters of $\mathbb{R}^d$ and N-standard partitions of $\mathbb{S}^d$, are unique, up to isometries, if the volumes of the regions have been fixed (see [3, 14]). It is conjectured that all isoperimetric $(N-1)$-clusters in $\mathbb{R}^d$ (recall that a $(N-1)$-cluster is an N-partition) are standard when $N \le d+2$. Each region of a standard partition shares a boundary with every other region.

Lemma 3.3 (approximation of a standard partition by standard clusters)**.** *Let* $\mathbf{E} = (E_1, \dots, E_N)$ *be a standard* N*-partition in* $\mathbb{R}^d$*. Then there exists a sequence* $\mathbf{E}^k = (E_0^k, \dots, E_N^k)$ *of standard* $(N-1)$*-clusters which converge in* $L^1_{\rm loc}$ *to* $\mathbf{E}$*, and* $\left|E_i^k\right| \to |E_i|$.

Proof. Let $\mathbf{F}$ be the Voronoi partition of $\mathbb{S}^d$ which corresponds to $\mathbf{E}$ by means of the stereographic projection. If $\mathbf{E}$ is itself a cluster then $\partial\mathbf{F}$ does not contain the north pole of $\mathbb{S}^d$. Otherwise with an arbitrarily small rotation of $\mathbf{F}$ on $\mathbb{S}^d$ we obtain a partition $\mathbf{F}'$ such that $\partial\mathbf{F}'$ does not contain the north pole, and it belongs to a fixed region F_j'. The corresponding stereographic projection $\mathbf{E}'$ will be a cluster in $\mathbb{R}^d$ and when the rotation converges to the identity we obtain L^1 convergence of the partitions $\mathbf{F}' \to \mathbf{F}$ on the sphere and $L^1_{\rm loc}$ convergence of their stereographic projections $\mathbf{E}' \to \mathbf{E}$ in $\mathbb{R}^d$. Clearly if ∂E_i does not contain the north pole we have L^1 convergence in a ball containing E_i hence $|E_i'| \to |E_i|$. Otherwise $|E_i| = +\infty$ and $|E_i'| \to +\infty$. □

Corollary 3.4 (examples of locally isoperimetric partitions)**.** *If* $\mathbf{E} = (E_1, \dots, E_N)$ *is a standard* N*-partition of* $\mathbb{R}^d$ *and if we know that all standard* $(N-1)$*-clusters of* $\mathbb{R}^d$ *are isoperimetric, then* $\mathbf{E}$ *is locally isoperimetric.*

Proof. If the partition $\mathbf{E}$ is a cluster then the result follows from Proposition 2.11. Otherwise notice that the partition is eventually flat, because it is the stereographic projection in $\mathbb{R}^d$ of a standard partition on the sphere $\mathbb{S}^d$ which has the north pole on the boundary. Each of the interfaces joining at the north pole are contained in maximal $(d-1)$-spheres in $\mathbb{S}^d$ so that their stereographic projection is contained in a $(d-1)$-dimensional plane in $\mathbb{R}^d$. Moreover, each region of a standard partition shares a boundary with every other region. So Definition 2.15 is satisfied. The conclusion follows from Lemma 3.3, Theorem 2.17, and Proposition 2.11. □

By the results on standard clusters already mentioned in the introduction, the above corollary assures that any standard N-partition of $\mathbb{R}^d$ is locally isoperimetric for $N \le \min\{5, d+1\}$ [14] or $N = 4$ and $d = 2$ [24]. This enables us to give a lot of examples of locally isoperimetric partitions.

For $N = 2$ we have that *half-spaces* are locally isoperimetric partitions in every $\mathbb{R}^d$. These can be obtained as the limit of a ball with volume going to infinity. It is well known that the ball solves the isoperimetric problem.

For $N = 3$ we have the *lens* partitions which is the partition of $\mathbb{R}^d$ composed by two half-spaces and a lens-shaped region between them. The lens is composed by two symmetrical $(d-1)$-dimensional sphere caps joining in a $(d-2)$-dimensional sphere lying in the plane containing the interface between the two unbounded regions. This partition can be obtained as the limit of double bubbles with a bubble converging to the lens and the other converging to an unbounded region.

The isoperimetricity of double bubbles has been proven in [7] for $d = 2$, in [9] for $d = 3$ and in [21] for all dimensions. This partition was already shown to be isoperimetric in [1].

Again for $N = 3$ and $d = 2$ we can have the *triple junction* partition of $\mathbb{R}^2$ composed by three unbounded regions whose boundary is the union of three half-lines joining with equal angles in a single point. If $d > 2$ we obtain a cylinder over the triple junction partition of $\mathbb{R}^2$. These partitions can be seen, again, as the limit of a double bubble in $\mathbb{R}^d$ by letting the two volumes go to infinity.

For $N = 4$ we have the *peanut* partition which is a partition with two bounded and two unbounded regions obtained by merging together two lens partitions with a common planar interface. The two lenses can have different volumes. This partition can be obtained as the limit of a triple bubble with two bubbles converging to the two bounded regions and the third bubble converging to an unbounded region. In the planar case, this partition has been described in [1] and in fact was conjectured to be locally isoperimetric.

Again for $N = 4$ we have the *Relaux* partition, composed by one bounded and three unbounded regions. It is obtained by adding three spherical slices to a triple junction partition. If $d = 2$ the bounded region has the shape of a Reuleaux triangle. In $d = 3$ it has the shape of a Brazil nut. This partition can be obtained as the limit of a triple bubble with one bubble converging to the bounded region and the other two (symmetrical) bubbles converging to unbounded regions.

For $N = 4$ and $d = 3$ we can have the *tetrahedral* partition which is a cone-like partition obtained by considering any regular tetrahedron and taking as regions each of the four cones with vertex at the center of the tetrahedron generated by the four faces of the tetrahedron itself. This partition, which is standard, is the blow up of a triple bubble in $\mathbb{R}^3$ centered in one of the two points in common to all the four regions. For $d > 3$ we obtain a cylinder over the tetrahedral partition of $\mathbb{R}^3$.

In the case $d \geq 3$ we don't know if these example are unique (up to isometries) with their prescribed volumes. In fact, also for $N \leq \min\{5, d + 1\}$, we cannot exclude that, there exists a locally isoperimetric partition $\mathbf{F}$ of $\mathbb{R}^d$ which is not the limit of standard clusters. In that case we would have two different locally isoperimetric partitions, the standard one and a non-standard one, with the same prescribed volumes. In the case $d = 2$ we have instead a uniqueness result, Theorem 4.5.

The main result of [14] also gives examples of 5-partitions in $\mathbb{R}^d$ with $d \geq 4$ which are locally isoperimetric, we do not try to describe their geometry.

4. The planar case

The following theorem resumes well known properties of minimizers. See for example [16, 13, 2].

Theorem 4.1 (regularity of planar local minimizers). *Let $\mathbf{E} = (E_1, \ldots, E_N)$ be a locally isoperimetric partition of an open set $\Omega \subset \mathbb{R}^2$ i.e. a partition such that for any other given partition $\mathbf{F}$ of Ω with $|F_k \cap \Omega| = |E_k \cap \Omega|$ and $F_k \triangle E_k \subset B$ for some open bounded $B \Subset \Omega$, one has*

$$P(\mathbf{E}, B) \leq P(\mathbf{F}, B).$$

Then the following properties hold:

1. *$\partial\mathbf{E}$ is a locally finite graph composed by straight segments or circular arcs meeting in triples with equal angles of 120 degrees;*
2. *the three signed curvatures of the arcs meeting in a vertex have zero sum;*
3. *it is possible to define a* pressure *p_i for each $i = 1, \ldots, N$ such that the curvature of an arc separating the regions E_i and E_j has curvature $p_i - p_j$ (the sign is chosen so that the curvature is positive when the arc has the concavity towards E_i)*

If $\mathbf{E}$ is any partition satisfying the above properties we say that $\mathbf{E}$ is stationary. *If $\mathbf{E}(t) = (E_1(t), \ldots, E_N(t))$ is a one-parameter curve of partitions of Ω such that $\mathbf{E}(t_0)$ is stationary, and $E_k(t) \setminus B = E_k(t_0) \setminus B$ for some open set $B \subset \Omega$, then one has*

$$\left[\frac{d}{dt} P(\mathbf{E}(t), B)\right]_{t=t_0} = \sum_{k=1}^{N} p_k \cdot \left[\frac{d}{dt} |E_k(t) \cap B|\right]_{t=t_0} \tag{11}$$

where $p_1, \ldots, p_N$ are the pressures of the regions of $\mathbf{E}(t_0)$.

Theorem 4.2. *Let $\mathbf{E} = (E_1, \dots, E_N)$ be any locally isoperimetric N-partition of the plane $\mathbb{R}^2$. Then $\partial\mathbf{E}$ is connected and the number of regions of $\mathbf{E}$ with infinite area is at least 1 and at most 3. If only one area is infinite then $\mathbf{E}$ is a bounded cluster. If two areas are infinite then $\partial\mathbf{E}$ coincides with a straight line outside a sufficiently large ball. If three areas are infinite then $\partial\mathbf{E}$ coincides, outside a sufficiently large ball, with three half-lines whose prolongations define angles of 120 degrees with each other (but not necessarily passing through a single point).*

Moreover the total number of all the connected components of all the regions is finite and if we consider the union D of all the bounded connected components of all the regions, then $\bar{D}$ is connected.

Proof. By Theorem 2.4, see Corollary 5.8, we know that the regions with finite area are bounded. By Theorem 4.1 we known that the boundary of the partition is a locally finite planar graph. Some of the arcs of this graph might be unbounded, in that case we imagine that the arc has one or two vertices of order 1 at infinity (which means that different unbounded arcs have different unbounded vertices at infinity). All other vertices have order 3 because the regularity of the boundary in the planar case states that exactly three edges can meet at a vertex point with equal angles of 120 degrees. Since the regions with finite measure are bounded we can find a large radius $R > 0$ such that all the bounded regions are compactly contained in B_R. Outside this ball the arcs of the graph $\partial\mathbf{E}$ have zero curvature because we do not have any local constraint on the area enclosed by infinite regions. So, outside B_R, the graph $\partial\mathbf{E}$ is composed by straight lines (with two end-points at infinity), lines segments (with two end-points in $\mathbb{R}^2$), or half-lines (with one end-point in $\mathbb{R}^2$ and one end-point at infinity).

We claim that every bounded closed (hence finite) loop contained in $\partial\mathbf{E}$ is contained in B_R. In fact take any bounded loop γ and suppose that there is an arc α not completely contained in B_R. The two regions separated by this arc have both infinite area, because the regions with finite measure are all contained in B_R. So that α has to be a straight line segment adjacent to two connected components, each of just one among the two infinite-area regions. One of the two infinite regions separated by α has a connected component C which is in the interior of the loop γ and is adjacent to the arc α. If we remove α, and reassign this component C to the other infinite-area region we strictly decrease the perimeter, while preserving the area constraints, because we are exchanging a finite area between two regions with infinite area. This is also a variation with compact support since γ is bounded. Hence we obtain a contradiction with the local isoperimetricity of $\mathbf{E}$.

Now we claim that the graph $\partial\mathbf{E}$ has a finite number of vertices (and hence a finite number of edges since every vertex has finite order). Recall that all vertices of the graph have order 3 apart from the vertices at infinity which have order 1 by convention. We will call *bounded vertices* the vertices which are not at infinity. The estimate $P(\mathbf{E}, B_\rho \setminus \bar{B}_R) \le P(\mathbf{E}, B_\rho) \le C_0(2,N) \cdot \rho$ (given by Lemma 2.5) implies that the number of vertices at infinity is not larger than C_0 because each vertex at infinity is the end-point of an half-line which asymptotically gives a contribution of ρ to the perimeter in the ball B_ρ. So the graph has a finite number of vertices of order one at infinity, in particular there is only a finite number of parallel entire straight lines.

Suppose now, by contradiction, that we have an infinite number of bounded vertices, which have order three. Since there are only a finite number of arcs that are entire straight lines enlarging R, we can suppose that outside B_R there are no entire straight lines. Since the graph is locally finite, we must have at least a sequence of bounded vertices going to infinity. Since the loops of $\partial\mathbf{E}$ are all contained in B_R and since the graph $\partial\mathbf{E}$ is locally finite, we have a finite number of closed loops. By removing a finite number of arcs in B_R we obtain a subgraph Γ without cycles, which is composed by a finite number of trees, each one touching B_R. So in Γ (and hence in $\partial\mathbf{E}$) it is possible to find a tree with infinitely many vertices of order three. This tree contains hence infinitely many disjoint paths each composed by infinitely many arcs: each such path must go to infinity because the graph is locally finite. But this, again, is in contradiction with the estimate $P(\mathbf{E}, B_\rho) \le C_0 \cdot \rho$.

Since the graph $\partial\mathbf{E}$ is finite, by further enlarging R we might suppose that B_R contains all the bounded vertices of the graph so that $\partial\mathbf{E} \setminus B_R$ is composed by a finite number, let say n,

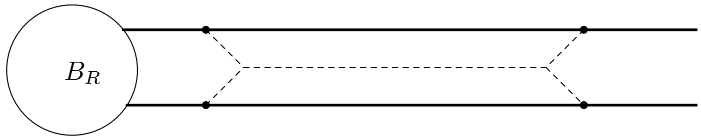

FIGURE 2. The modification performed in the proof of Theorem 4.2 when two rays have the same direction.

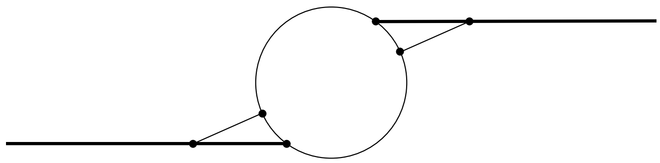

FIGURE 3. The rotation performed in the proof of Theorem 4.2. We can suppose that the two non collinear half-lines are emitted at diametrically opposite points of B_R.

$n \neq 1$, of disjoint half-lines emitted by B_R and going to infinity. These half lines, have all different directions towards infinity, because if we had two parallel half-lines with the same direction we could easily merge them into a single long segment and then split it again to obtain a partition with smaller perimeter (see Figure 2).

Now we can consider a *blow-down* of $\mathbf{E}$. By letting $\mathbf{E}^k = \frac{\mathbf{E}}{k}$ be a rescaled partition of $\mathbf{E}$ we easily notice that $\mathbf{E}^k$ converges in L^1_{loc} as $k \to +\infty$ to a partition $\mathbf{E}^\infty$ of $\mathbb{R}^2$ delimited by n half-lines with a common vertex at the origin, each line parallel to the n-half-lines of $\mathbf{E} \setminus B_R$. Since each $\mathbf{E}^k$ is a locally J-isoperimetric partition, with J being the set of indices of the finite regions of $\mathbf{E}$, by Theorem 2.13 we deduce that also $\mathbf{E}^\infty$ is a locally J-isoperimetric partition. However for $j \in J$ the regions $E^k_j = \frac{E_j}{k}$ converge to the empty set because E_j is bounded. So, we can remove the regions E^∞_j from $\mathbf{E}^\infty$ and obtain a locally isoperimetric partition of $\mathbb{R}^2$ composed by n angles with a common vertex in the origin. These angles cannot be smaller than 120 degrees (by the general regularity results or by simple geometric considerations) hence $n \leq 3$. In the case $n = 0$ the graph $\partial\mathbf{E}$ has no vertices at infinity and hence the partition $\mathbf{E}$ is a cluster. In the case $n = 3$ we have that $\partial\mathbf{E} \setminus \bar{B}_R$ is made of three half-lines going to infinity with relative angles of 120 degrees.

In the case $n = 2$ we have that $\partial\mathbf{E} \setminus \bar{B}_R$ is made of two half-lines going to infinity at opposite directions. In this case we claim that the two lines are collinear (i.e. are contained in the same straight line).

In fact if they were not collinear we could rotate the bounded cluster to which they are attached, creating two angles in the half-lines and strictly decreasing the perimeter (see Figure 3).

We now claim that $\partial\mathbf{E}$ is connected. Consider a connected component of $\partial\mathbf{E}$. If it is bounded, then it is the only connected component, otherwise we could move it until it touches another component and this would contrast with the local regularity (Theorem 4.1). Otherwise every connected component is unbounded. But it must have at least two half-lines going to infinity because otherwise the line would have the same region on both sides. Since we know that the total number of half-lines is at most 3 we conclude that there is a single connected component.

Let us now consider the set D which is the union of all the bounded connected components of all regions and let $D_1, \dots, D_m$ be the connected components of $\bar{D}$. The connected components are all contained in the large ball, hence they are a finite number by known regularity results (see Theorem 4.1). The unbounded connected components are all components of the infinite regions $E_1, \dots, E_n$ and, by the previous discussion, we have exactly one unbounded component for each infinite region and $n = 1, 2, 3$. If $n = 1$ we are in the case of a cluster and it is known, in this case, that $\bar{D}$ is connected (otherwise move D_1 and let it bump against the rest of the cluster). So we are left with the cases $n = 2, 3$. From now on suppose, by contradiction, that $m > 1$.

Consider an abstract planar graph Γ which has, as edges, the arcs of $\partial\mathbf{E}$ separating two unbounded connected components of two (infinite) regions. These arcs have zero curvature and can be half-lines (n half-lines going to infinity) or straight line segments ending either in one of the components D_k or in a triple point where three infinite regions meet. So the vertices of the graph are represented by the components $D_1, \dots, D_m$, by n vertices at infinity and by the triple points separating three infinite regions (only possible if $n = 3$). The vertices at infinity have order 1 by construction.

The graph Γ is connected because if not also $\partial\mathbf{E}$ would be disconnected. Now we claim that this graph Γ contains no cycles and hence is a tree. In fact, if we had a cycle in Γ, we could consider any edge α of this cycle and consider the infinite region E_k adjacent to α on the interior of the cycle. The connected component of E_k adjacent to α is bounded, being inside the cycle (notice that the cycle cannot pass through the points at infinity because they have order 1 by construction). This is not possible because the arcs of Γ separate two unbounded components by definition.

Let us show that the vertices of Γ represented by a component D_k must have order at least 3. If D_k had order 0 and since $m > 1$ we could move D_k until it touches some other arc of the partition, violating the regularity results of Theorem 4.1. If D_k has order 1, the single arc would have the same region on both sides, which is not possible. If D_k has order 2 then it is adjacent to two straight arcs. These two arcs must be parallel, otherwise we could decrease the local perimeter by translating D_k towards the interior of the convex angle formed by the two lines containing the arcs, so that both arcs become shorter. Moreover they are collinear, because otherwise we could rotate the component D_k (see Figure 3). Notice now that it cannot happen that both arcs joining D_k are going to infinity because either D_k is the only component (and hence $n = 2$ and $m = 1$ as we want to prove) or the graph Γ would be disconnected. So, if we move D_k along the bounded arc, the partition does not change perimeter and preserves the area constraints but, eventually, it would bump against another component or a triple point, violating the regularity properties of Theorem 4.1.

At this point we have proven that the graph Γ is a tree with exactly n vertices of order 1 (terminal points) and all other vertices of order at least 3. A simple combinatorial inductive argument (remove the terminal points one by one) shows that we cannot have more than one vertex of order at least 3, since each such vertex increases the number of terminal points of the tree. So we are either in the case $n = 2$ and $m = 0$, or in the case $n = 3$ and $m \leq 1$. □

Theorem 4.3 (existence)**.** *Let $m_k \in [0, +\infty]$ for $k = 1, \dots, N$ be a given N-uple of areas such that at least one and at most three of the m_k are infinite. Then there exists an isoperimetric partition $\mathbf{E} = (E_1, \dots, E_N)$ of $\mathbb{R}^2$ whose regions have the prescribed measures.*

If all the areas are finite or at least four of them are infinite then there are no isoperimetric partitions with the prescribed measures.

Proof. Theorem 4.2 guarantees that in a locally isoperimetric partition there are at least one and at most three infinite areas, so the second part of the statement has already been proved.

Let M be the number of infinite areas, $1 \leq M \leq 3$. Without loss of generality suppose that the infinite areas are the first M: m_1 if $M = 1$, m_1, m_2 if $M = 2$ and m_1, m_2, m_3 if $M = 3$ while $m_k < +\infty$ for $k = M + 1, \dots, N$.

If $M = 1$ there exists an isoperimetric $(N - 1)$-cluster $(E_2, \dots, E_N)$ with the prescribed finite measures $m_2, \dots, m_N$. By adding the external region $E_1 = \mathbb{R}^2 \setminus \bigcup_{k=2}^N E_k$ we obtain a locally isoperimetric partition with the given measures $|E_k| = m_k$ for $k = 1, \dots, N$.

We now consider the cases $M = 2$ and $M = 3$. If $M = 2$ we let $\mathbf{C} = (C_1, C_2, \emptyset, \dots, \emptyset)$ be an N-partition such that $\partial\mathbf{C}$ is a straight line passing through the origin. If $M = 3$ we let $\mathbf{C} = (C_1, C_2, C_3, \emptyset, \dots, \emptyset)$ be an N-partition such that $\partial\mathbf{C}$ is the union of three half-lines emitted by the origin with equal angles of 120 degrees.

In both cases we consider a radius $R>0$ so large that $|B_R| > m_{M+1}+\dots+m_N$. We then consider the family of all partitions $\mathbf{E}$ of $\mathbb{R}^2$ which coincide with $\mathbf{C}$ outside of $\overline{B_R}$ and such that $|E_k| = m_k$ for $k = M+1,\dots,N$. In this family we minimize $P(\mathbf{E},\overline{B_R})$ (thus taking into account also the length of $\partial B_R \cap \partial \mathbf{E}$). By compactness and semicontinuity we know that a minimizer $\mathbf{E}^R$ always exists. The minimizer, restricted to B_R, is a J-isoperimetric partition of B_R with $J=\{M+1,\dots,N\}$ (see Definition 2.12). The idea is now to let $R\to\infty$ and prove that the minimizers of the problem in B_R converge to a local minimizer in $\mathbb{R}^2$. The difficulty is to prove that the components with finite area are not going to infinity. The rest of the proof is devoted to this.

We are going to complete the proof in the case $M=3$. The case $M=2$ is similar but simpler so we don't treat it here.

Step 1: obtaining an estimate on the perimeter of the bounded components. Let R be fixed and let $\mathbf{E}=\mathbf{E}^R$ be a minimizer of the auxiliary problem stated above. Let us define D to be the union of all the bounded connected components of the regions $E_1,\dots,E_N$. Since the bounded components are all contained in $\bar B_R$ we have that $D\subset \bar B_R$ and since $E_4,\dots,E_N$ are bounded we have $D\supset E_4\cup\dots\cup E_N$. Moreover D will also contain the bounded connected components of E_1,E_2,E_3, if they exist. For $j=1,2,3$ let $E_j' = E_j\setminus D$ be the only unbounded connected component of E_j and consider the following partitions of $\mathbb{R}^2$:

$$\mathbf{F}^1 = (E_1'\cup D, E_2', E_3'),\qquad \mathbf{F}^2 = (E_1', E_2'\cup D, E_3'),\qquad \mathbf{F}^3 = (E_1', E_2', E_3'\cup D),$$

and

$$\mathbf{G} = (E_1', E_2', E_3', D).$$

Let $\partial\mathbf{C}\cap\partial B_R = \{p_1,p_2,p_3\}$ be the three fixed points on ∂B_R enumerated so that the half-line terminating in p_j is non contained in ∂E_j, for $j=1,2,3$. Notice that

$$P(\mathbf{F}^1,\bar B_R) = \mathcal{H}^1((\partial E_2'\cup\partial E_3')\cap \bar B_R).$$

Since E_2' is simply connected we know that $\partial E_2'\cap\bar B_R$ is a compact and connected set containing the two point p_1 and p_3. Analogously $\partial E_3'\cap\bar B_R$ is a compact connected set containing p_1 and p_2. Hence $(\partial E_2'\cup\partial E_3')\cap\bar B_R$ is a compact connected set containing $\{p_1,p_2,p_3\}$. We know that the shortest compact connected set containing $\{p_1,p_2,p_3\}$ is the classical Steiner tree on the three vertices, which is known to have length $3R$ (see, for example, [18]). The same reasoning can be applied to the other two partitions so we have

$$P(\mathbf{F}^1,\bar B_R)+P(\mathbf{F}^2,\bar B_R)+P(\mathbf{F}^3,\bar B_R)\ge 9R.$$

On the other hand we have

$$P(\mathbf{F}^1,\bar B_R) = \mathcal{H}^1(\partial E_1'\cap\partial E_2')+\mathcal{H}^1(\partial E_1'\cap\partial E_3')+\mathcal{H}^1(\partial E_2'\cap\partial E_3')+\mathcal{H}^1(\partial D)-\mathcal{H}^1(\partial E_1'\cap\partial D)$$

and analogously for $\mathbf{F}^2$ and $\mathbf{F}^3$. Summing up we obtain

$$\begin{aligned}9R &\le 3\left[\mathcal{H}^1(\partial E_1'\cap\partial E_2')+\mathcal{H}^1(\partial E_1'\cap\partial E_3')+\mathcal{H}^1(\partial E_2'\cap\partial E_3')\right]+2P(D)\\ &= 3P(\mathbf{G},\bar B_R)-2P(D)\end{aligned}$$

And since $P(\mathbf{G},\bar B_R)\le P(\mathbf{E},\bar B_R)$ we obtain

$$P(D)\le 3P(\mathbf{E},\bar B_R)-9R.$$

Now it is not diffucult to estimate $P(\mathbf{E},\bar B_R)$ by taking a competitor of $\mathbf{E}$ composed by the triple junction $\mathbf{C}$ with $N-M$ balls with given areas. This implies that

$$P(\mathbf{E},\bar B_R)\le 3R+2\pi\sum_{k=4}^N\sqrt{\frac{m_k}{\pi}}$$

In conclusion we have found

$$P(D)\le d := 6\sqrt{\pi}\sum_{k=4}^N\sqrt{m_k}.$$

Step 2: proving that there is a single "component". This is similar to what we did in the proof of the second part of Theorem 4.2. We consider the connected components $D_1,\dots,D_m$ of $\bar D$ and

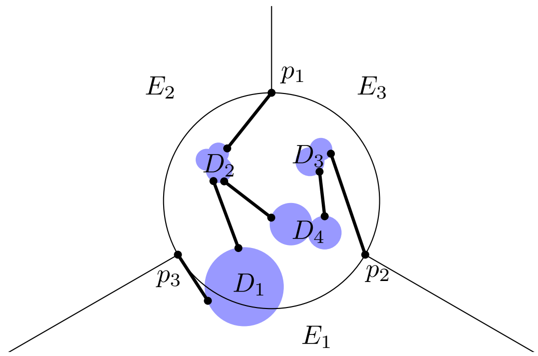

FIGURE 4. The components $D_1, \dots, D_4$ in the proof of Theorem 4.3 Step 2.

call them *components*. The arcs of $\partial\mathbf{E}$ which are not contained in $\bar{D}$ are separating two of the infinite regions. Having no constraint on the area of these regions we conclude that such arcs are straight line segments. The union of $\bar{D}$ with these segments is a compact connected set containing the three points p_1, p_2, p_3. It is connected because otherwise the three external components E_1', E_2' and E_3' would not be separated by $\partial\mathbf{E}$.

Now we would like to prove that the components of $\bar{D}$, if R is large enough, are not touching the boundary of B_R. To this aim, we consider another (auxiliary) minimization problem where we take any possible rigid motion of each connected component of $\bar{D}$ (in the whole plane) and any possible union of straight line segments so that the union of the components and the segments is a compact connected set containing the three points p_1, p_2, p_3 (see Figure 4). This problem has a solution since we only need to fix the position of a finite number of segments and the total length of the segments is continuous and coercitive with respect to the position of the end-points of the segments. The components of $\bar{D}$ are originally contained in B_R but now we allow to move them anywhere in the plane. We will see that, for R large enough, the original partition $\mathbf{E}$ is a minimizer also for this auxiliary problem.

Consider any minimizer of the auxiliary problem. By minimality, the segments have disjoint interiors and their end-points are either points of the boundary of the components, or the points p_1, p_2, p_3, or else triple junctions of three segments. The configuration is described by an abstract graph Γ whose vertices are the components of $\bar{D}$ and the three points p_1, p_2, p_3 and whose arcs are the segments joining them. If a component is touching one of the points p_j ($j = 1, 2, 3$), we pretend that there is a degenerate segment of length zero joining p_j to the component. The resulting graph Γ is connected because of the corresponding requirement on the union of the segments and the components. Also we can suppose that Γ has no cycles because otherwise we could remove a segment of the cycle and decrease the total length preserving the connectedness. Hence Γ is a tree. A vertex represented by a component of $\bar{D}$ has order at least two because otherwise it would have a single segment attached to it and we could move the component along the segment to decrease the total length until the component touches another component. When two components touch each other we can merge them into a single component. At the end of this process we obtain a graph where the only vertices or order one are among the three points p_1, p_2, p_3. A simple combinatorial argument shows that such a tree, having at most three terminal vertices, can have at most one vertex of order three while all other non terminal vertices have order two. See Figure 4.

We can now eliminate all vertices of order two. First we notice that we can suppose that the two segments emitted by a component of order two are collinear. In fact if they are not, we could translate the component together with the two end points on it, so that both the the segments become shorter. We can do this until the segments become collinear or until the component touches another component. In the latter case we can merge the two components as they touch and consider them as a single component repeating the argument from the beginning. Now, if the two segments emitted by a component of order two are collinear, we can translate the component along the common line containing them until it touches another component or a triple point. Iterating this procedure we can eliminate all vertices of order two.

Step 3. Equi-boundedness of the component. Let q be any point such that the component is contained in $\bar{B}_d(q)$. We claim that for R large enough we have $|q| \leq 3\sqrt{dR}$. Suppose this is not true and we can have a sequence of $R \to +\infty$ such that for such values of R one has $|q| \geq 3\sqrt{dR}$.

Clearly the total length of the three segments emitted by the component, is not smaller than $|p_1 - q| + |p_2 - q| + |p_3 - q| - 3d$. If we translate the component so that p goes into the origin, we obtain a competitor such that the total length of the three segments is smaller than $3R$. So we have

$$|p_1 - q| + |p_2 - q| + |p_3 - q| - 3d \leq 3R.$$

Consider the function

$$\ell(t) = \frac{1}{R} \inf_{|p| \leq tR} (|p - p_1| + |p - p_2| + |p - p_3|).$$

By definition we have

$$|p_1 - q| + |p_2 - q| + |p_3 - q| \geq R \cdot \ell\left(\frac{|q|}{R}\right).$$

Let $t = 3\sqrt{d/R}$ and notice that $t \to 0^+$ as $R \to +\infty$ since d does not depend on R. Since $|q|/R \geq t$ and ℓ is increasing, we have $R \cdot \ell(t) \leq 3R + 3d$. In Lemma 4.4 below, we prove that $\ell(t) \geq 3 + \frac{3}{4}t^2 + o(t^2)$ as $t \to 0$ so we have

$$3 + \frac{3}{4}t^2 + o(t^2) \leq 3 + 3\frac{d}{R} = 3 + \frac{3}{9}t^2$$

which gives the contradiction $\frac{3}{4}t^2 + o(t^2) \leq \frac{1}{3}t^2$.

Step 4. Letting $R \to +\infty$. For each sufficiently large R we consider the minimizer given by Step 3 and call it $\mathbf{E}^R$ to make explicit the dependence on R.

We know that the finite regions E_k^R, $k = 4, \ldots, N$ of $\mathbf{E}^R$ are all contained in a ball of radius d centered in some point q_R such that $|q_R| \leq 3\sqrt{dR}$. Hence the translated regions $F_k^R = E_k^R - q_R$, for $k > 3$, are all contained in the same ball B_d. So there is a sequence $R_n \to \infty$ and measurable sets $F_4, \ldots, F_N$ such that for all $k > 3$ one has $F_k^{R_n} \to F_k$ in L^1.

Moreover, since $|q_R| \leq 3\sqrt{dR} \ll R$, we notice that the translated balls $B_R(-q_R)$ invade the whole plane as $R \to +\infty$. So, up to a subsequence, each of the three rays emanating from the component of $\mathbf{E}^{R_n}$ must converge to a ray which has the same direction of the corresponding ray of the reference cone $\mathbf{C}$. These rays divide the complement of the union of the regions $F_4, \ldots, F_N$ into three regions F_1, F_2, F_3 which are thus the L^1_{loc} limit of the corresponding regions $F_1^{R_k}$, $F_2^{R_k}$, $F_3^{R_k}$ as $k \to +\infty$. So we have found a partition $\mathbf{F} = (F_1, \ldots, F_N)$ of $\mathbb{R}^2$ which is the L^1_{loc} limit of the partitions $\mathbf{F}^{R_k}$ of the balls B_{R^k}. Clearly $|F_1| = |F_2| = |F_3| = +\infty$ and $|F_k| = m_k$ for $k = M + 1, \ldots, N$.

The partitions $\mathbf{E}^{R_n}$ are all locally J-isoperimetric in $\Omega_n = B_{R_n} - q_{R_n}$ with $J = \{4, \ldots, N\}$ and hence, by Theorem 2.13, we obtain that $\mathbf{F}$ is locally J-isoperimetric in $\mathbb{R}^2$. But this is equivalent to say that $\mathbf{F}$ is locally isoperimetric in $\mathbb{R}^2$, which is our conclusion.

The proof for the case $M = 2$ is similar but much simpler. In that case, in Step 3 we have a single component D connected to two diametrically opposite points p_1 and p_2 by means of two collinear segments. Translating the component along the segments we can place the component D inside the ball B_d centered in the origin. So we have $q_R = 0$ and also Step 4 is simpler because we don't need any estimate on $|q_R|$. □

Lemma 4.4. *Let p_1, p_2, p_3 be the three vertices of a regular triangle inscribed in the unit circle ∂B_1 centered in the origin of the plane $\mathbb{R}^2$. Let q be any point and let $\Sigma_q = [q, p_1] \cup [q, p_2] \cup [q, p_3]$ be the union of the three segments joining q with the vertices of the triangle. Let*

$$\ell(t) = \inf \left\{ \mathcal{H}^1(\Sigma_q) \colon \ |q| \geq t \right\}.$$

Then the infimum is attained, ℓ is increasing, and, for $t \to 0^+$, one has

$$\ell(t) \geq 3 + \frac{3}{4}t^2 + o(t^2).$$

Proof. By assumption we have $|p_i| = 1$, $p_1 + p_2 + p_3 = 0$. Since the function $q \mapsto \mathcal{H}^1(\Sigma_q)$ is coercitive and convex on $\mathbb{R}^2$, the infimum defining $\ell(t)$ is attained in a point q such that $|q| = t$. The proof is concluded by the following computation:

$$\begin{aligned}
\mathcal{H}^1(\Sigma_q) &= \sum_{i=1}^{3} |q - p_i| = \sum_{i=1}^{3} \sqrt{1 + t^2 - 2\langle q, p_i\rangle} \\
&= \sqrt{1+t^2} \sum_{i=1}^{3} \sqrt{1 - 2\frac{\langle q, p_i\rangle}{1+t^2}} \\
&= (1 + t^2 + o(t^2)) \sum_{i=1}^{3} \left[1 - \frac{\langle q, p_i\rangle}{1+t^2} - \frac{1}{4}\left(\frac{\langle q, p_i\rangle}{1+t^2}\right)^2 + o(t^2)\right] \\
&= (1 + t^2 + o(t^2)) \left[3 - \frac{1}{4}\sum_{i=1}^{3}\left(\frac{\langle q, p_i\rangle}{1+t^2}\right)^2 + o(t^2)\right] \\
&\geq (1 + t^2 + o(t^2)) \left[3 - \frac{3}{4}t^2 + o(t^2)\right] \\
&= \big(1 + t^2 + o(t^2)\big) \cdot \big[3 - t + o(t^2)\big] = 3 + \frac{3}{4}t^2 + o(t^2).
\end{aligned}$$

□

Theorem 4.5 (uniqueness). *Let* $\mathbf{E}$ *be any locally isoperimetric* N*-partition of the plane* $\mathbb{R}^2$ *with* $N \leq 4$*. Then* $\mathbf{E}$ *is standard. This means that in the planar case the standard partitions, enumerated in section 3, are the only locally isoperimetric partitions with their given areas.*

Proof. For $N = 1$ there is only one (trivial) partition $\mathbf{E} = (\mathbb{R}^2)$.

For $N = 2$ the cluster cannot have triple points hence the boundary is either a circle or a single straight line: both are standard.

Consider the case $N = 3$. If we have a single region with infinite area then we have a standard double-bubble cluster.

If we have two regions with infinite area then, by Theorem 4.2 we know that outside a large ball the interface is contained in a single line separating the two unbounded regions while the region with finite measure is contained in the ball. By the second part of Theorem 4.2 if we remove the two half-lines, we obtain that the set $\bar{D}$, which is the closure of the union of all the bounded connected components of the regions, is connected. We claim that $\bar{D}$ coincides with (the closure of) the region of finite area E_3. Indeed, since every component is simply connected, a bounded connected component of a region of infinite area (say E_1) necessarily touches a component of the other region of infinite are (say E_2), contradicting the minimality by deleting the common edge. It follows that E_3 is also connected and simply connected. This implies that E_3 is a single two-sided component and the cluster is a lens cluster.

Suppose now that $N = 3$ and we have three regions with infinite area. By Theorem 4.2, there are three half-lines emanating from the component. The three half lines separate three unbounded connected components, say C_1, C_2 and C_3, of the three regions E_1, E_2 and E_3. We claim that in this case all the regions are connected. In fact suppose that a region, say E_1, is disconnected and take a bounded connected component C of E_1. If we give C to any other component of the partition we strictly decrease the perimeter without changing any prescribed area, since they are all infinite. Since all the regions are connected the component must be empty and what we get is a partition whose boundary is composed by three half lines joining with equal angles at a triple point: this is the triple junction.

Suppose now that $N = 4$. If we have only one region with infinite measure than the partition is a cluster and it is known that it must be a standard triple bubble. By Theorem 4.2 we cannot have four regions with infinite measure. If we have three regions with infinite area, Theorem 4.2 tells us that the boundary of the partition contains three half-lines emanating by the component D. The three half lines separate three unbounded connected components C_1, C_2 and C_3 of three regions, say respectively E_1, E_2 and E_3. As in the case $N = 3$ with two infinite regions, we can easily state that the external arcs of D must be adjacent to a component of the fourth region E_4 in the inside, otherwise by removing the arc we decrease the perimeter leaving the area of E_4 unchanged. So the set D contains a single component of E_4 which must be a triangular region. Since all angles are of 120 degrees and the three half lines also define angles of 120 degrees one with the other, we conclude that the only possibility is that E_4 is a Reuleaux triangle, and the partition is a Reuleaux partition.

The last case is $N = 4$ with only two regions with infinite measure. In this case Theorem 4.2 tells us that the boundary of the partition contains two collinear half lines emanating from a bounded component. The rest of the proof is devoted to this case, which is much harder than the previous ones.

We will use some of the ideas from [11]. Let $\mathbf{E} = \mathbf{E}(m_3, m_4)$ be a locally isoperimetric partition with measures $(+\infty, +\infty, m_3, m_4)$ and let $\mathbf{F} = \mathbf{F}(m_3, m_4)$ be the *peanut* partition with the same measures. If $\mathbf{E}(t)$ is a one-parameter family of partitions and $\mathbf{E} = \mathbf{E}(t_0)$ is a *stationary* partition we have

$$\left[\frac{d}{dt}P(\mathbf{E}(t), B_R)\right]_{t=t_0} = \sum_{k=1}^{N} p_k \left[\frac{d}{dt}|E_k|\right]_{t=t_0} \tag{12}$$

where p_k are the *pressures* of the regions E_k of $\mathbf{E}$.

By the considerations above, we know that there is a ball B_R such that $\partial\mathbf{E} \setminus B_R$ is contained in a straight line. By translating we can also assume that such a line is passing through the origin, which is the center of the ball. Do the same for $\mathbf{F}$. Since we know that $\mathbf{F}$ is locally isoperimetric (Theorem 3.4) we have $P(\mathbf{E}, B_R) = P(\mathbf{F}, B_R)$. Now define

$$\tilde{P}(\mathbf{E}) = P(\mathbf{E}, B_R) - 2R$$

and notice that this definition does not depend on R, and that (11) holds true for $\tilde{P}$ as well, with $B = B_R$, since $P(\mathbf{E}, B_R)$ differs from $\tilde{P}(\mathbf{E})$ by a constant depending only on R.

Claim 1: the function $m_3 \mapsto \tilde{P}(\mathbf{F}(m_3, m_4))$ is increasing in both variables. In fact by using formula 11 with $\mathbf{F}(t) = \mathbf{F}(t, m_4)$ we get

$$\frac{d}{dt}\tilde{P}(\mathbf{F}(t)) = p(t)$$

where $p(t)$ is the curvature of the external arcs of $F_3(t, m_4)$ which is clearly positive. Similarly, for the other variable.

Claim 2: the infinite regions E_1 and E_2 of the locally isoperimetric partition $\mathbf{E}$ are connected. Otherwise one of the two, say E_1, would have a connected component C. If we give C to a neighbouring region we would find a partition $\mathbf{E}'$ with strictly less perimeter than $\mathbf{E}$ and without decreasing any of the two prescribed area. This is impossible because the corresponding standard partition $\mathbf{F}'$ with the same areas of $\mathbf{F}$ would have strictly less perimeter that $\mathbf{F}$, which is not possible by the previous claim.

Claim 3: the regions of $\mathbf{E}$ have the same pressures of the regions of $\mathbf{F}$. Notice that both E_3 (and the same is true for E_4) has an edge in common with at least one of the two infinite regions E_1, E_2 (say E_1) because otherwise E_3 would be completely surrounded by E_4 and E_4 would not be simply connected. If α is an arc separating E_3 from E_1 we know by Theorem 4.1 that the curvature of α is $p_3 - p_1 = p_3$, since $p_1 = 0$. Consider a one family of clusters $\mathbf{E}(t)$ with $\mathbf{E}(0) = E_0$ and such that for t varying in a neighborhood of $t = 0$ the arc α is replaced with an arc with curvature $p_3 + t$ while the rest of the cluster is unchanged. Let $\mathbf{F}(t) = \mathbf{F}(|E_3(t)|, m_4)$ be the peanut partition with the same measures as $\mathbf{E}(t)$ so that $\mathbf{F}(0) = \mathbf{F}(m_3, m_4)$. Let q_3 and q_4 be the pressures of the regions F_3 and F_4 of $\mathbf{F}$. Since $|E_k(t)| = |F_k(t)|$ we have that

$$\left[\frac{d}{dt}\tilde{P}(\mathbf{F}(t))\right]_{t=0} = q_3\left[\frac{d}{dt}|F_3(t)|\right]_{t=0} + q_4\left[\frac{d}{dt}|F_4(t)|\right]_{t=0} = q_3\left[\frac{d}{dt}|E_3(t)|\right]_{t=0} \tag{13}$$

$$\left[\frac{d}{dt}\tilde{P}(\mathbf{E}(t))\right]_{t=0} = p_3\left[\frac{d}{dt}|E_3(t)|\right]. \tag{14}$$

Now notice that $\tilde{P}(\mathbf{E}(t)) \geq \tilde{P}(\mathbf{F}(t))$ for all t, since $\mathbf{F}(t)$ is locally isoperimetric. Moreover, $\tilde{P}(\mathbf{E}(0)) = \tilde{P}(\mathbf{F}(0))$ because $\mathbf{E}$ is also isoperimetric. Hence $\tilde{P}(\mathbf{E}(t)) - \tilde{P}(\mathbf{F}(t))$ has a local minimum at $t = 0$, and from (13) we deduce $q_3 = p_3$. Repeating the same argument with an external arc of E_4 we obtain also $q_4 = p_4$. So the pressures of $\mathbf{E}$ coincide with the pressures of $\mathbf{F}$: $p_i = q_i$ for $i = 1, 2, 3, 4$.

Now notice that two triangles with angles of 120 degrees and with the same curvatures of the three sides are congruent. This means that the triangular connected components of E_3 and E_4 are congruent with the triangular regions F_3 and F_4 and in particular they have the same area. This means that if E_3 (respectively E_4) has at least one triangular component then it has only that component and is congruent to F_3 (respectively F_4). But if we consider the triple point at the end of one of the two half-lines containing $\partial\mathbf{E}$, the three regions around this triple point are E_1, E_2 and a component C of E_3 or E_4. To fix the notation suppose C is a component of E_3. Since E_1 and E_2 are connected, C can have a single arc in common with E_1 and a single arc in common with E_2 (see [4, Theorem 6]). It means that C is triangular because it cannot have two consecutive arcs in common with E_4 (and it cannot have only two edges, otherwise the set $\bar{D}$ considered before would be disconnected). Then, by the previous claim, $C = E_3$ and it is congruent to F_3. But then the triple point on the other half line must be a triple point adjacent to E_4 and hence we can repeat the same reasoning with E_4 in place of E_3 to conclude that also E_4 is congruent to F_4. This means that $\mathbf{E}$ is congruent to $\mathbf{F}$ and hence $\mathbf{E}$ is standard. ☐

5. Appendix

We now give a path of statements leading to first regularity results for isoperimetric partitions, see Theroem 2.4, and for their limits. Their proofs can be obtained with minor modifications from the corresponding for isoperimetric clusters. For reader convenience we refer to the monograph [13, Ch. 29, 30], and give only some outline to link the argument with the proofs there exposed.

Theorem 5.1 (volume fixing variations)**.** *If $\mathbf{F} = (F_1, \dots, F_N)$, $N \geq 1$, is a partition of an open connected set $B \subseteq \mathbb{R}^d$, with $|F_i \cap B| > 0$ for all $i = 1, \dots, N$, for every suitably choosen family of interface points, doubly linking each region of the partition with a fixed one with infinite volume, there exist positive constants ε_1, C_1, ε_2, η, and an open bounded set $A \Subset B$ (a finite union of open balls centered in the choosen interface points of $\mathbf{E}$ with radius ε_1), with the following property: for every proper partition $\mathbf{F}' = (F'_1, \dots, F'_N)$ of B such that $\sum |(F_i \triangle F'_i) \cap A| < \varepsilon_2$ and every $\mathbf{a} \in V =: \{(a_1, \dots, a_N) \in \mathbb{R}^N : |a_i| < \eta, \sum_i a_i = 0\}$, there exists a C^1 function $\Phi\colon V \times B \to \mathbb{R}^d$ with $\Phi_{\mathbf{a}}\colon B \to B$ diffeomorphism of class C^1, such that*

1. *$\{x \in B : \Phi_{\mathbf{a}}(x) \neq x\} \Subset A$;*
2. *for all $i = 0, \dots, N$*
$$|\Phi_{\mathbf{a}}(F'_i) \cap A| = |F'_i \cap A| + a_i;$$
3. *given any $\mathcal{H}^{d-1}$-rectifiable set Σ one has*
$$\left|\mathcal{H}^{d-1}(\Phi_{\mathbf{a}}(\Sigma)) - \mathcal{H}^{d-1}(\Sigma)\right| \leq C_1 \mathcal{H}^{d-1}(\Sigma) \cdot \sum_{i=0}^{N} |a_i|,$$

so that for every open bounded set $\Omega \subseteq B$ containg $\overline{A}$ one has

$$|P(\Phi_{\mathbf{a}}(\mathbf{F}'), \Omega) - P(\mathbf{F}', \Omega)| \leq C_1 \cdot P(\mathbf{F}', A) \sum_{i=0}^{N} |a_i|$$

Proof. See [13, Lemma 29.13, Theorem 29.14]. ☐

Lemma 5.2. *(Almgren's Lemma) Let* $\mathbf{E}$ *a partition of* B *an open connected set* ($|B \cap E_j| > 0$, *for all* j). *Then there exist: a finite union of well separated disjoint open balls* A, *compactly contained in* B, *constants* C, $\varepsilon > 0$, $\eta > 0$ *depending on* $\mathbf{E}$ *and* B *such that: for each* $\triangle \subseteq B$ *open bounded disjoint from* A, *for each partition* $\mathbf{E}'$ *such that:* $\left|E'_j \cap B\right| > 0$, *for all* j, $\sum \left|E_j \triangle E'_j \cap A\right| < \varepsilon$, *and each partition* $\mathbf{F}$ *of* B, *such that is a variation of* $\mathbf{E}'$ *compactly contained in* $\triangle$ *and* $\sum ||E'_j \cap \triangle| - |F_j \cap \triangle|| < \eta$, *there exists a partition* $\mathbf{F}'$ *of* B, *such that*

(1) $F_j \triangle F'_j$ *is compactly contained in* A, *for all* j,
(2) *for any bounded open set* Ω *containing* A *it holds* $|F'_j \cap \Omega| = |E'_j \cap \Omega|$, *for all* j,
(3) *for any bounded open set* Ω *containing* A *it holds*
$$|P(\mathbf{F}', \Omega) - P(\mathbf{F}, \Omega)| \leq C \cdot P(\mathbf{E}', A)| \cdot \sum \left||F_j \cap \triangle| - \left|E'_j \cap \triangle\right|\right|.$$

Hence if $\mathbf{E}'$ *is an isoperimetric partition for any bounded open set* $\Omega \supseteq A$:

$$P(\mathbf{E}', \Omega) \leq P(\mathbf{F}, \Omega) + C \cdot P(\mathbf{E}', A) \cdot \sum \left||F_j \cap \triangle| - \left|E'_j \cap \triangle\right|\right|$$

Proof. Is exactly the same proof of Lemma 29.16, Corollary 29.17 in [13]. □

Remark 5.3. Following [13, Corollary 29.17], one choose two unions, A_1, A_2, with $dist(A_1, A_2) > 0$, of the well separated balls with centers two families of interface points of $\mathbf{E}$, as in Theorem 5.1. Put η be the minimum among the relative η_1, η_2. Then if $r_1 < \frac{dist(A_1,A_2)}{2}$, $\omega_d r_1^d < \eta$ (depending only on $\mathbf{E}$), one can choose as $\triangle$ any $B(x, r_1)$, $x \in \mathbb{R}^d$, and A one among A_1 and A_2.

With minor modifications Lemma 30.2 in [13] (Infiltration Lemma) still holds for locally isoperimtric partitions:

Lemma 5.4. *(Infiltration Lemma) For any partition* $\mathbf{E}$ *of* $\mathbb{R}^d$ *consider* A_1, A_2, $\varepsilon(\mathbf{E})$, $r_1(\mathbf{E})$ *as in Lemma 5.2 and Remark 5.3, and put* $O = A_1 \cup A_2$. *Then for* $d \geq 2$ *there is* $\varepsilon_0(d) < \omega_d$, *and for any* $K \geq 1$ *there exist* $0 < r_0(\mathbf{E}, K) < \min\{1, r_1\}$, *such that for every locally isoperimetric partition* $\mathbf{E}'$ *with*

$$P(\mathbf{E}', O) < K, \sum |E_j \triangle E'_j \cap O| < \varepsilon,$$

for each $x \in \mathbb{R}^d$, $r < r_0$, *and for each* $\Lambda \subseteq \{1, \ldots, N\}$, *if*

$$\sum_{j \in \Lambda} |E'_j \cap B(x, r)| < \varepsilon_0 r^d$$

then

$$\sum_{j \in \Lambda} \left|E'_j \cap B\left(x, \frac{r}{2}\right)\right| = 0$$

Proof. Let $x \in \mathbb{R}^d$. Modifing the proof of [13, Lemma 30.2], one applies Lemma 5.2 and Remark 5.3 to $\mathbf{E}'$ with pivot $\mathbf{E}$, finding: A among A_1, A_2, $C(\mathbf{E})$, $\varepsilon(\mathbf{E})$, $\eta(\mathbf{E})$, $r_0(\mathbf{E}) < \min\left\{\frac{dist(A_1,A_2)}{2}, 1, \left(\frac{\eta}{\omega_d}\right)^{\frac{1}{d}}\right\}$, so that, for open bounded $\Omega \supseteq A$, $r < r_0$, and for $\mathbf{F}$ partition with $F_h \triangle E'_h \Subset B_r(x)$, it holds good

$$P(\mathbf{E}', \Omega) \leq P(\mathbf{F}, \Omega) + C \cdot K \cdot \sum \left||F_j \cap B_r(x)| - \left|E'_j \cap B_r(x)\right|\right|.$$

Hence the proof is the same as reported in [13, Lemma 30.2] to choose the competitor $\mathbf{F}$ to cancel the infiltration; at the end, with the notation there used, to get the decay estimate (30.19) $m(s)^{1-\frac{1}{d}} \leq 6m'(s)$, one observes that suffices decreasing r_0 putting $r_0 < \frac{1}{8CK}$. □

Moreover using universal upper $(d-1)$-density estimate of the perimeter of a locally isoperimetric N-partition, for example given here in lemma 2.5, one gets the local perimeter bound K on any isoperimetric partions $\mathbf{E}'$ depending only on d, N, $\mathbf{E}$ and O, so that it holds:

Corollary 5.5. *For $d \geq 2$ there is $\varepsilon_0(d) < \omega_d$ such that if* $\mathbf{E}$ *is a partition that is L^1_{loc}-limit of a sequence $\mathbf{E}^k$ of locally isoperimetric partitions, then there is $r_0(\mathbf{E}) < 1$, such that for all $x \in \mathbb{R}$, $\Lambda \subseteq \{1, \ldots, N\}$, $r < r_0$, if*

$$\sum_{j \in \Lambda} |E_j \cap B(x, r)| \leq \varepsilon_0 r^d$$

then

$$\sum_{j \in \Lambda} \left| E_j \cap B\left(x, \frac{r}{2}\right) \right| = 0$$

Corollary 5.6. *Such a limit partion of locally isoperimetric ones can be considered with open regions.*

This allow to extend the regularity of minimizing clusters to locally isoperimetric partitions (Theorem 2.4, and Theorem 30.1 in [13]), and get also the following density estimates (same proof of Lemma 30.6 of [13]):

Lemma 5.7. *For $d \geq 2$ there exists* positive *constants $c_0 \leq c_1 < 1$ and c_2, such that if* $\mathbf{E}$ *is a locally isoperimetric partition there exists $r_0 > 0$ such that for every region E_j whenever $\rho < r$ and $x \in \partial E_j$*

(1) $c_0 \omega_d \rho^d \leq |E_j \cap B(x, \rho)| \leq c_1 \omega_d \rho^d$
(2) $c_2 \omega_{d-1} \rho^{d-1} \leq P(E_j, B(x, \rho))$.

Corollary 5.8. *If* $\mathbf{E}$ *is a locally isoperimetric partition then each regions with finite measure is bounded. Hence the component of the chambers with finite volume is bounded and with finite perimeter.*

Proof. Fix F a region of $\mathbf{E}$ with finite positive volume and $|F| > \varepsilon > 0$, take a large ball $B = B(\mathbf{0}, R)$ such that $0 < |F \setminus B| \leq \varepsilon$. On other hand if F were unbounded then it would be that for every $B' \Supset B$ concentric ball $|F \setminus B'| > 0$. Both $|F| < \infty$ and $|F \setminus B'| > 0$ yield $\partial F \setminus B' \neq \emptyset$. So that if r_0 is given as in the volume density estimate, and the radius of B' is greater than $R + r_0$, for $x \in \partial F \setminus B'$ one has $B(x, \rho) \subset \mathbb{R}^d \setminus B$ for each $\rho < r_0$. Summing up $c_0 \omega_d \rho^d \leq |F \cap B(x, \rho)| \leq |F \setminus B| \leq \varepsilon$, so $c_0 \omega_d r^d \leq \varepsilon$ that can not hold for every $\varepsilon > 0$.

□

Thanks to the infiltration lemma with pivot partition $\mathbf{E}$, 5.4, 5.5, one has to observe that the density estimate are rather uniform, so that it holds:

Lemma 5.9. *For $d \geq 2$ there are positive constants $c_0 \leq c_1 < 1$ and c_2, depending only on d, and for every* $\mathbf{E}$ *N-partition L^1_{loc} limits of locally isoperimetric partitions, there exists $0 < r_0(\mathbf{E}) < 1$ such that whenever $\rho < r_0$ and $x \in \partial E_j$, $1 \leq j \leq N$*
$c_0 \omega_d \rho^d \leq |E_j \cap B(x, \rho)| \leq c_1 \omega_d \rho^d$
$c_2 \omega_{d-1} \rho^{d-1} \leq P(E_j, B(x, \rho))$

Remark 5.10. Notice that if $\mathbf{E}^k \to \mathbf{E}$, in L^1_{loc}, are locally isoperimetric N-partitions, then with the same constants, whenever $\rho < r_0$, and k is large enough, for any $1 \leq j \leq N$ and $x \in \partial E^k_j$
$c_0 \omega_d \rho^d \leq |E^k_j \cap B(x, \rho)| \leq c_1 \omega_d \rho^d$
$c_2 \omega_{d-1} \rho^{d-1} \leq P(E^k_j, B(x, \rho))$

Similarly for partitions that are limit of isoperimetric partions we have that the regions with finite volume are bounded.

Corollary 5.11. *If* $\mathbf{E}$ *is a partition that is L^1_{loc} limit of isoperimetric partitions $\mathbf{E}^k$, then its region with finite volume are bounded.*

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