A second order gradient flow of p-elastic planar networks

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Abstract

We consider a second order gradient flow of the p-elastic energy for a planar theta-network of three curves with fixed lengths. We construct a weak solution of the flow by means of an implicit variational scheme. We show long-time existence of the evolution and convergence to a critical point of the energy.

Keywords: elastic flow of networks, minimizing movements, long-time existence
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1 Introduction

In this paper we consider a network composed of three inextensible planar curves. Each curve $\gamma_i = \gamma_i(s)$ of fixed length $L_i > 0$, $i = 1, 2, 3$, is parametrized by arc-length s over the domain $\bar{I}_i = [0, L_i]$. Without loss of generality we may assume that

$$0 < L_3 \leq \min\{L_2, L_1\}. \tag{1.1}$$

Let $T^i = T^i(s) = \gamma_i'(s)$ denote the unit tangent of the curve γ_i. It is well known that a planar curve is uniquely determined by its tangent indicatrix T^i, up to rotation and translation. Omitting for simplicity the indices of the curves, we recall the formulas $T' = \vec{\kappa} = \kappa N$, $N' = -\kappa T$, as well as $\theta'(s) = \kappa(s)$, where $T = (\cos\theta, \sin\theta)$. The map $\theta : I \to \mathbb{R}$ is called the indicatrix of the curve γ.

We shall consider a *theta-network* $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, where the three curves satisfy

$$\gamma_1(0) = \gamma_2(0) = \gamma_3(0),$$
$$\gamma_1(L_1) = \gamma_2(L_2) = \gamma_3(L_3).$$

Without loss of generality we shall assume that the first triple point is placed at the origin, that is, $\gamma_i(0) = 0$, for $i = 1, 2, 3$. From the concurrency conditions above it follows immediately that

$$\int_{I_1} T^1(s)ds = \int_{I_2} T^2(s)ds = \int_{I_3} T^3(s)ds. \tag{1.2}$$

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Letting $p \in (1, +\infty)$, the p-elastic energy of the network is defined as

$$E_p(\Gamma) = \sum_{i=1}^{3} E_p(\gamma_i),$$

where

$$E_p(\gamma_i) := \frac{1}{p}\int_{I_i} |\vec{\kappa}_i|^p ds = \frac{1}{p}\int_{I_i} |\partial_s T^i|^p ds =: F_p(T^i).$$

Minimizers for the elastic energy (i.e. with $p = 2$) plus an additional term penalizing the growth of the length of the curves have been investigated in [9], where an angle condition at the triple junctions has been imposed in order to avoid the collapse of a minimizing sequence to a point. Here the situation is different, because the length of each curve is fixed. In particular it is not necessary to impose the angle condition at the triple junctions.

Here we consider the evolution of the network Γ via a *second order* gradient flow first introduced by Y. Wen in [33] (see also [19, 31]). More precisely we will consider the L^2-gradient flow of the energy

$$F_p(\Gamma) := \sum_{i=1}^{3} F_p(T^i),$$

when expressed in terms of the angles corresponding to the tangent vectors. This gives rise to a second order parabolic system. We shall express the energy $F_p(\Gamma)$ and the corresponding gradient flow by means of the three scalar maps $\theta^i : I_i \to \mathbb{R}$ such that $T^i = (\cos\theta^i, \sin\theta^i)$.

Let us now state our main existence results. We let

$$H := \Big\{ \boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3) \in W^{1,p}(0, L_1) \times W^{1,p}(0, L_2) \times W^{1,p}(0, L_3) \,|\, \int_{I_1} (\cos\theta^1, \sin\theta^1) ds = \int_{I_2} (\cos\theta^2, \sin\theta^2) ds = \int_{I_3} (\cos\theta^3, \sin\theta^3) ds \Big\}.$$

Theorem 1.1. *Let $\boldsymbol{\theta}_0 \in H$ and let $T > 0$. Assume that the lengths of the three curves are such that*

$$L_3 < \min\{L_1, L_2\}. \tag{1.3}$$

Then, there exist functions $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$, with $\theta^j \in L^\infty(0, T; W^{1,p}(I_j)) \cap H^1(0, T; L^2(I_j))$, and Lagrange multipliers $\lambda^1, \lambda^2, \mu^1, \mu^2 \in L^2(0, T)$ such that the following properties hold:

(i) for any $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$ with $\varphi^j \in L^\infty(0, T; W^{1,p}(I_j))$, $j = 1, 2, 3$, there holds

$$\begin{aligned} 0 = &\sum_{j=1}^{3} \int_0^T \int_{I_j} \partial_t \theta^j \, \varphi^j ds dt + \sum_{j=1}^{3} \int_0^T \int_{I_j} |\theta^j_s|^{p-2} \theta^j_s \cdot \varphi_s \, ds dt \\ &- \int_0^T (\lambda^1 - \mu^1) \int_{I_1} \sin(\theta^1) \, \varphi^1 ds dt + \int_0^T (\lambda^2 - \mu^2) \int_{I_1} \cos(\theta^1) \, \varphi^1 ds dt \\ &+ \int_0^T \lambda^1 \int_{I_2} \sin(\theta^2) \, \varphi^2 ds dt - \int_0^T \lambda^2 \int_{I_2} \cos(\theta^2) \, \varphi^2 ds dt \\ &- \int_0^T \mu^1 \int_{I_3} \sin(\theta^3) \, \varphi^3 ds dt + \int_0^T \mu^2 \int_{I_3} \cos(\theta^3) \, \varphi^3 ds dt \,; \end{aligned} \tag{1.4}$$

(ii) the maps $|\partial_s\theta^j|^{p-2}\partial_s\theta^j$ *belong to* $L^\infty(0,T;L^{\frac{p}{p-1}}(I_j))\cap L^2(0,T;H^1(I_j))$, $j=1,2,3$, *and satisfy*

$$(|\partial_s\theta^1|^{p-2}\partial_s\theta^1)_s = \theta^1_t - (\lambda^1-\mu^1)\sin\theta^1 + (\lambda^2-\mu^2)\cos\theta^1, \tag{1.5}$$

$$(|\partial_s\theta^2|^{p-2}\partial_s\theta^2)_s = \theta^2_t + \lambda^1\sin\theta^2 - \lambda^2\cos\theta^2, \tag{1.6}$$

$$(|\partial_s\theta^3|^{p-2}\partial_s\theta^3)_s = \theta^3_t - \mu^1\sin\theta^3 + \mu^2\cos\theta^3, \tag{1.7}$$

$$\theta^j_s(0,t) = \theta^j_s(L_j,t) = 0, \ \textit{for } j=1,2,3 \textit{ and for a.e. } t\in(0,T); \tag{1.8}$$

(iii) for all $t\in[0,T]$, *there holds*

$$\int_{I_1}(\cos\theta^1,\sin\theta^1)ds = \int_{I_2}(\cos\theta^2,\sin\theta^2)ds = \int_{I_3}(\cos\theta^3,\sin\theta^3)ds. \tag{1.9}$$

Notice that the time $T>0$ can be chosen arbitrarily, so that the weak solutions $\boldsymbol{\theta}$ and the Lagrange multipliers $\vec{\lambda}=(\lambda^1,\lambda^2)$, $\vec{\mu}=(\mu^1,\mu^2)$ can be defined globally on the whole of $(0,+\infty)$, and Theorem 1.1 provides long-time existence of the evolution.

Concerning the behavior of the solutions as $t\to+\infty$, we will show that they converge, on a suitable sequence of times, to a critical point of the energy $F_p(\Gamma)$.

Theorem 1.2. *Assume* (1.3) *and let* $\boldsymbol{\theta}_0\in H$. *Let* $\boldsymbol{\theta}=(\theta^1,\theta^2,\theta^3)$, *with* $\theta^j\in L^\infty_{loc}(0,\infty;W^{1,p}(I_j))\cap H^1_{\rm loc}(0,\infty;L^2(I_j))$, *and* $\lambda^1,\lambda^2,\mu^1,\mu^2\in L^\infty(0,\infty)$ *be the solutions given by Theorem* 1.1. *Then there exist a sequence of times* $t_n\to\infty$, *Lagrange multipliers* $\lambda^1_\infty,\lambda^2_\infty,\mu^1_\infty,\mu^2_\infty\in\mathbb{R}$, *and limit functions* $\boldsymbol{\theta}_\infty=(\theta^1_\infty,\theta^2_\infty,\theta^3_\infty)$, *with* $\theta^j_\infty\in W^{1,p}(I_j)$, *such that the following system holds:*

$$\begin{aligned}
&(|\partial_s\theta^1_\infty|^{p-2}\partial_s\theta^1_\infty)_s = -(\lambda^1-\mu^1)\sin\theta^1_\infty + (\lambda^2-\mu^2)\cos\theta^1_\infty && \textit{in } I_1,\\
&(|\partial_s\theta^2_\infty|^{p-2}\partial_s\theta^2_\infty)_s = \lambda^1\sin\theta^2_\infty - \lambda^2\cos\theta^2_\infty && \textit{in } I_2,\\
&(|\partial_s\theta^3_\infty|^{p-2}\partial_s\theta^3_\infty)_s = -\mu^1\sin\theta^3_\infty + \mu^2\cos\theta^3_\infty && \textit{in } I_3,
\end{aligned} \tag{1.10}$$

together with the boundary conditions

$$\partial_s\theta^j_\infty(0) = \partial_s\theta^j_\infty(L_j) = 0 \qquad \textit{for } j=1,2,3. \tag{1.11}$$

Observe that Theorem 1.2 together with Remark 2.2 below yields the existence of configurations of planar theta-networks that are critical with respect to the elastic energy E_2 ($p=2$) and are subject to natural boundary conditions and given fixed lengths. This result is relevant for the investigations undertaken in [8, 14].

We notice that, by direct method of the Calculus of Variations (see for instance [9]), the energy E_p always admits a global minimizers among theta-networks with curves of fixed length, moreover such a minimizer is regular and satisfies the natural boundary conditions at the triple junctions, so that it is also a critical point of E_p. However, uniqueness of minimizers is not clear, and there might exist critical points of E_p which are not global minimizers.

If we do not assume (1.3) we are not able to show long-time existence, due to a technical difficulty in estimating the Lagrange multipliers $\lambda^1,\lambda^2,\mu^1,\mu^2$. However, if we assume that at least two initial curves are not flat, the same method yields the following short-time existence result.

Theorem 1.3. *Let* $\boldsymbol{\theta}_0\in H$ *be such that*

$$\min\Big(osc_{\bar I_{j_1}}\,\theta^{j_1}_0(t), osc_{\bar I_{j_2}}\,\theta^{j_2}_0(t)\Big)\geq c>0 \qquad \textit{for some } j_1,j_2\in\{1,2,3\}, \tag{1.12}$$

where $osc_{\bar I_j}\,\theta_0^j$ *denotes the oscillation of* θ_0^j *on the interval* $\bar I_j$*. Then there exist* $T = T(\boldsymbol{\theta}_0) > 0$ *and functions* $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$*, with* $\theta^j \in L^\infty(0,T; W^{1,p}(I_j)) \cap H^1(0,T; L^2(I_j))$*,* $\lambda^1, \lambda^2, \mu^1, \mu^2 \in L^2(0,T)$ *such that properties (i), (ii), (iii) of Theorem* 1.1 *hold. Moreover, letting* T_{max} *be the maximal existence time of the evolution, if* $T_{max} < +\infty$ *there holds*

$$\liminf_{t\to T_{max}^-} \max\Big(osc_{\bar I_{j_1}}\,\theta^{j_1}(t), osc_{\bar I_{j_2}}\,\theta^{j_2}(t)\Big) = 0 \qquad \text{for some } j_1, j_2 \in \{1,2,3\}. \tag{1.13}$$

In order to show existence of weak solutions in Theorems 1.1 and 1.3 we apply an implicit variational scheme to the energy $F_p(\Gamma)$ expressed in terms of the functions θ^j. Such time-discrete schemes have been used in the study of geometric evolutions starting from the pivotal works by Almgren, Taylor and Wang [2] and by Luckhaus and Sturzenhecker [20] in the case of the mean curvature flow. An extension of these techniques to multiple-phase systems can be found in [4], while an adaptation to the L^2-gradient flow for the elastic energy ($p = 2$) of an open curve, which gives rise to a fourth order flow, has been recently proposed in [13, 3].

Starting from the work by Polden [32], the fourth order evolution of elastic curves has been extensively studied in the literature, under various constraints and boundary conditions, we refer for instance to [18, 10, 25, 26, 7, 6, 11, 32, 16, 12, 17, 34, 29, 30, 28, 27, 35] and references therein. On the other hand, not many works treat the second order evolution that we consider here (see [33, 19, 31]).

The geometric evolution of planar networks is more complicated, since a network is intrinsically singular, due to the presence of the multiple junctions, and the evolution is typically described by a system instead of a single equation. However, the evolution by curvature of a network has been studied in many papers, starting from the work [5] where the authors first establish the short-time existence of solutions. We refer to [24, 21, 22, 23] for a discussion of the long-time existence in some particular cases, and the formation of singularities.

Finally, for the fourth order evolution of elastic networks we refer to [15] for the short-time existence of smooth solutions and to [8, 14] for the long-time existence, under the assumption that the tangent vectors of the three concurring curves are not collinear at a triple junction. With our approach we don't need such a condition, even if our notion of solution is considerably weaker than the one considered in [8, 14].

We conclude by observing that the result in Theorem 1.3 can be extended without significant modifications to the case of a network of three curves with a single triple junction and three fixed endpoints, which is the situation considered in [8, 15, 14]: this fact is briefly discussed in Remark 3.12.

The article is organized as follows: in Section 2 we derive and motivate the system (1.5)–(1.8) and discuss the well-posedness of the Lagrange multipliers. In Section 3 we investigate the construction of a weak solution via minimizing movements and provide proofs of our main results. For the reader's convenience some proofs are collected in the Appendix.

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2 First variation and preliminary results

Let us compute the first variation of the energy $F_p(\Gamma) = \sum_{i=1}^3 F_p(T^i)$. We consider variations $T^i_\epsilon = \frac{T^i+\epsilon\varphi^i}{|T^i+\epsilon\varphi^i|}$, for ϵ small enough and $\varphi^i \in C^\infty(\bar{I}_i, \mathbb{R}^2)$, $i = 1, 2, 3$. Then

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} T^i_\epsilon = \varphi^i - (\varphi^i \cdot T^i)T^i =: \varphi^{i\perp}.$$

Since we want to include the constraint (1.2) we compute

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \left[\left(\sum_{i=1}^3 F_p(T^i_\epsilon) \right) + \vec{\lambda} \cdot \left(\int_{I_1} T^1_\epsilon ds - \int_{I_2} T^2_\epsilon ds \right) + \vec{\mu} \cdot \left(\int_{I_3} T^3_\epsilon ds - \int_{I_1} T^1_\epsilon ds \right) \right] = 0,$$

where $\vec{\lambda} = (\lambda^1, \lambda^2)$, $\vec{\mu} = (\mu^1, \mu^2) \in \mathbb{R}^2$ are Lagrange multipliers. A direct computation gives

$$\begin{aligned}
0 &= \sum_{i=1}^3 \int_{I_i} |\partial_s T^i|^{p-2} \partial_s T^i \cdot \partial_s(\varphi^{i\perp}) ds \\
&\quad + \vec{\lambda} \cdot \left(\int_{I_1} \varphi^{1\perp} ds - \int_{I_2} \varphi^{2\perp} ds \right) + \vec{\mu} \cdot \left(\int_{I_3} \varphi^{3\perp} ds - \int_{I_1} \varphi^{1\perp} ds \right) \\
&= \int_{I_1} [-\partial_s(|\partial_s T^1|^{p-2}\partial_s T^1) + (\vec{\lambda} - \vec{\mu})] \cdot \varphi^{1\perp} ds + \int_{I_2} [-\partial_s(|\partial_s T^2|^{p-2}\partial_s T^2) - \vec{\lambda}] \cdot \varphi^{2\perp} ds \\
&\quad + \int_{I_3} [-\partial_s(|\partial_s T^3|^{p-2}\partial_s T^3) + \vec{\mu}] \cdot \varphi^{3\perp} ds \\
&= \int_{I_1} [-\nabla_s(|\partial_s T^1|^{p-2}\partial_s T^1) + ((\vec{\lambda} - \vec{\mu}) \cdot N^1) N^1] \cdot \varphi^1 \, ds \\
&\quad - \int_{I_2} [\nabla_s(|\partial_s T^2|^{p-2}\partial_s T^2) + (\vec{\lambda} \cdot N^2) N^2] \cdot \varphi^2 \, ds \\
&\quad + \int_{I_3} [-\nabla_s(|\partial_s T^3|^{p-2}\partial_s T^3) + (\vec{\mu} \cdot N^3) N^3] \cdot \varphi^3 \, ds,
\end{aligned}$$

where $\nabla_s \varphi = \partial_s \varphi - (\partial_s \varphi \cdot T)T$ denotes the normal component of the derivative $\partial_s \varphi$ and where we have used the fact that $\partial_s T^i$ vanishes at the boundary.

This motivates the study of the *second-oder* problem

$$\partial_t T^1 = \nabla_s(|\partial_s T^1|^{p-2}\partial_s T^1) - ((\vec{\lambda} - \vec{\mu}) \cdot N^1) N^1 \qquad \text{in } I_1 \times (0, t_*) \tag{2.1}$$

$$\partial_t T^2 = \nabla_s(|\partial_s T^2|^{p-2}\partial_s T^2) + (\vec{\lambda} \cdot N^2) N^2 \qquad \text{in } I_2 \times (0, t_*) \tag{2.2}$$

$$\partial_t T^3 = \nabla_s(|\partial_s T^3|^{p-2}\partial_s T^3) - (\vec{\mu} \cdot N^3) N^3 \qquad \text{in } I_3 \times (0, t_*) \tag{2.3}$$

$$\partial_s T^i = 0 \quad \text{on} \quad \partial I_i \times (0, t_*), \quad i = 1, 2, 3, \tag{2.4}$$

$$T^i(\cdot, 0) = T^i_0, \quad i = 1, 2, 3, \tag{2.5}$$

for some $t_* > 0$, and for smooth initial data T^i_0 satisfying (1.2) and

$$\vec{\kappa}^i_0(s) = \partial_s T^i_0(s) = 0 \quad \text{for } s \in \{0, L_i\}, \ i = 1, 2, 3. \tag{2.6}$$

Here $\vec{\lambda} = \vec{\lambda}(t) = (\lambda^1(t), \lambda^2(t))$, $\vec{\mu} = \vec{\mu}(t) = (\mu^1(t), \mu^2(t))$ are such that

$$\int_{I_1} |\partial_s T^1|^p T^1 ds - (\vec{\lambda} - \vec{\mu}) \cdot A^1 = \int_{I_2} |\partial_s T^2|^p T^2 ds + \vec{\lambda} \cdot A^2 \tag{2.7}$$

$$\int_{I_1} |\partial_s T^1|^p T^1 ds - (\vec{\lambda} - \vec{\mu}) \cdot A^1 = \int_{I_3} |\partial_s T^3|^p T^3 ds - \vec{\mu} \cdot A^3 \tag{2.8}$$

hold, where

$$A^i = A^i(t) = \int_{I_i} N^i \otimes N^i ds, \qquad i = 1, 2, 3, \tag{2.9}$$

are 2×2 time-dependent matrices. Note that if $\det A^i = 0$ then the Lagrange multipliers might not be well defined. We will comment on the well-posedness of the Lagrange multipliers below.

Under the assumption that such Lagrange multipliers exist, we observe that as long as the flow is well defined and smooth the constraint (1.2) is satisfied. Indeed, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{I_1} T^1 ds - \int_{I_2} T^2 ds\right) = \int_{I_1} T^1_t ds - \int_{I_2} T^2_t ds \\
&= \int_{I_1} \nabla_s(|\partial_s T^1|^{p-2}\partial_s T^1) - ((\vec{\lambda} - \vec{\mu}) \cdot N^1)N^1 ds - \int_{I_2} \nabla_s(|\partial_s T^2|^{p-2}\partial_s T) - (\vec{\lambda} \cdot N^2)N^2 ds \\
&= \int_{I_1} \partial_s(|\partial_s T^1|^{p-2}\partial_s T^1) ds - \int_{I_1} (\partial_s(|\partial_s T^1|^{p-2}\partial_s T^1) \cdot T^1)T^1 ds - (\vec{\lambda} - \vec{\mu}) \cdot A^1 \\
&\quad - \left(\int_{I_2} \partial_s(|\partial_s T^2|^{p-2}\partial_s T^2) ds - \int_{I_2} (\partial_s(|\partial_s T^2|^{p-2}\partial_s T^2) \cdot T^2)T^2 ds + \vec{\lambda} \cdot A^2\right) = 0,
\end{aligned}$$

due to the boundary conditions (2.4) (with the convention that $|0|^{p-2}0 = 0$), the fact that $\partial_{ss} T \cdot T = -|\partial_s T|^2$, $T \cdot \partial_s T = 0$, and (2.7). Similarly, using now (2.8), one verifies that

$$\frac{d}{dt}\left(\int_{I_3} T^3 ds - \int_{I_1} T^1 ds\right) = 0.$$

In other words the constraint (1.2) is satisfied along the flow.

Note also that the energy $F_p(\Gamma)$ decreases along the flow. Indeed, using the computation above, the fact that T^i_t are normal vector fields and $\partial_t(\int_{I_1} T^1 ds - \int_{I_2} T^2 ds) = 0 = \partial_t(\int_{I_3} T^3 ds - \int_{I_1} T^1 ds)$, we find

$$\begin{aligned}
\frac{d}{dt} F_p(\Gamma) &= \sum_{i=1}^3 \int_{I_i} [-\nabla_s(|\partial_s T^i|^{p-2}\partial_s T^i)] \cdot T^i_t ds \\
&= \sum_{i=1}^3 \int_{I_i} [-\nabla_s(|\partial_s T^i|^{p-2}\partial_s T^i)] \cdot T^i_t ds + \vec{\lambda} \cdot \int_{I_1} T^1_t ds - \vec{\lambda} \cdot \int_{I_2} T^2_t ds \\
&\quad + \vec{\mu} \cdot \int_{I_3} T^3_t ds - \vec{\mu} \cdot \int_{I_1} T^1_t ds \\
&= \int_{I_1} [-\nabla_s(|\partial_s T^1|^{p-2}\partial_s T^1)] \cdot T^1_t + ((\vec{\lambda} - \vec{\mu}) \cdot N^1)N^1 \cdot T^1_t ds \\
&\quad + \int_{I_2} [-\nabla_s(|\partial_s T^2|^{p-2}\partial_s T^2)] \cdot T^2_t - (\vec{\lambda} \cdot N^2)N^2 \cdot T^2_t ds \\
&\quad + \int_{I_3} [-\nabla_s(|\partial_s T^3|^{p-2}\partial_s T^3)] \cdot T^3_t + (\vec{\mu} \cdot N^3)N^3 \cdot T^3_t ds \\
&= -\sum_{i=1}^3 \int_{I_i} |\partial_t T^i|^2 ds \leq 0.
\end{aligned}$$

The system (2.1)– (2.5) can be converted into a system for the scalar maps $\theta^i : I_i \times (0,t_*) \to \mathbb{R}$ satisfying

$$T^i(s,t) = (\cos\theta^i(s,t), \sin\theta^i(s,t)), \qquad i = 1,2,3. \tag{2.10}$$

Indeed, since we have

$$\begin{aligned}\nabla_s(|\partial_s T^i|^{p-2}\partial_s T^i) &= \partial_s(|\partial_s T^i|^{p-2})\partial_s T^i + |\partial_s T^i|^{p-2}(\partial_s^2 T^i - (\partial_s^2 T^i \cdot T^i)T^i)\\ &= (|\theta^i_s|^{p-2})_s \theta^i_s N^i + |\theta^i_s|^{p-2}\theta^i_{ss} N^i,\end{aligned}$$

we obtain the system

$$\theta^1_t = (|\theta^1_s|^{p-2}\theta^1_s)_s + (\lambda^1(t) - \mu^1(t))\sin\theta^1 - (\lambda^2(t) - \mu^2(t))\cos\theta^1 \qquad \text{in } I_1 \times (0,t_*) \tag{2.11}$$

$$\theta^2_t = (|\theta^2_s|^{p-2}\theta^2_s)_s - \lambda^1(t)\sin\theta^2 + \lambda^2(t)\cos\theta^2 \qquad \text{in } I_2 \times (0,t_*) \tag{2.12}$$

$$\theta^3_t = (|\theta^3_s|^{p-2}\theta^3_s)_s + \mu^1(t)\sin\theta^3 - \mu^2(t)\cos\theta^3 \qquad \text{in } I_3 \times (0,t_*) \tag{2.13}$$

$$\theta^i_s(0,t) = \theta^i_s(L_i,t) = 0 \qquad t \in (0,t_*),\ i = 1,2,3, \tag{2.14}$$

$$\theta^i(\cdot,0) = \theta^i_0(\cdot) \qquad i = 1,2,3. \tag{2.15}$$

Regarding the Lagrange multipliers, recall that the matrix A^i is given by

$$A^i = A^i(t) = \begin{pmatrix} \int_{I_i}\sin^2\theta^i ds & -\int_{I_i}\sin\theta^i\cos\theta^i ds \\ -\int_{I_i}\sin\theta^i\cos\theta^i ds & \int_{I_i}\cos^2\theta^i ds \end{pmatrix} =: A^i(\theta^i), \tag{2.16}$$

so that by (2.7), (2.8), we can write

$$\int_{I_1}|\partial_s\theta^1|^p(\cos\theta^1,\sin\theta^1)ds - (\vec\lambda - \vec\mu)\cdot A^1(\theta^1) = \int_{I_2}|\partial_s\theta^2|^p(\cos\theta^2,\sin\theta^2)ds + \vec\lambda\cdot A^2(\theta^2) \tag{2.17}$$

$$\int_{I_1}|\partial_s\theta^1|^p(\cos\theta^1,\sin\theta^1)ds - (\vec\lambda - \vec\mu)\cdot A^1(\theta^1) = \int_{I_3}|\partial_s\theta^3|^p(\cos\theta^3,\sin\theta^3)ds - \vec\mu\cdot A^3(\theta^3). \tag{2.18}$$

Letting, for $i = 1,2,3$,

$$G^i = G^i(\theta^i) := \int_{I_i}|\partial_s\theta^i|^p(\cos\theta^i,\sin\theta^i)ds, \tag{2.19}$$

the above system reads as

$$G^1 - (\vec\lambda - \vec\mu)\cdot A^1 = G^2 + \vec\lambda\cdot A^2 \tag{2.20}$$

$$G^1 - (\vec\lambda - \vec\mu)\cdot A^1 = G^3 - \vec\mu\cdot A^3, \tag{2.21}$$

that is, recalling that $G^2 + \vec\lambda\cdot A^2 = G^3 - \vec\mu\cdot A^3$,

$$\vec\lambda\cdot A^2 + \vec\mu\cdot A^3 = G^3 - G^2 \tag{2.22}$$

$$-\vec\lambda\cdot(A^2 + A^1) + \vec\mu\cdot A^1 = G^2 - G^1. \tag{2.23}$$

Assuming that A^1, A^2 are invertible, we then get

$$\begin{aligned}\vec\lambda &= (G^3 - G^2 - \vec\mu\cdot A^3)\cdot(A^2)^{-1}\\ \vec\mu &= (G^2 - G^1 + \vec\lambda(A^2 + A^1))\cdot(A^1)^{-1},\end{aligned}$$

which yields

$$\vec{\mu}\,(I + A^3((A^2)^{-1} + (A^1)^{-1})) = (G^2 - G^1)(A^1)^{-1} + (G^3 - G^2)((A^1)^{-1} + (A^2)^{-1}).$$

Observe that if $\det(A^i) > 0$ for $i = 1, 2, 3$, then we can solve for $\vec{\mu}$ and $\vec{\lambda}$ and the Lagrange multipliers are well defined (simply write $(I + A^3((A^2)^{-1} + (A^1)^{-1})) = A^3(\sum_{i=1}^3 (A^i)^{-1})$ and use that A^i, $i = 1, 2, 3$ are symmetric real (hence diagonalisable) and positive definite matrices (by Sylvester criterion), and that the sum of positive definite matrices is again positive definite and hence invertible). Note also that, by Cauchy-Schwarz inequality we have $\det A^i \geq 0$ $i = 1, 2, 3$. A strict bound from below on the determinant is shown in [19, Lemma 1] (see Lemma 2.3 below), provided the considered curve is not a straight line (i.e. we need some oscillation of θ). In other words provided none of the curves is a straight line, then the system is well-posed.

The system for the Lagrange multipliers can be solved in a slightly more general situation. Indeed, if the matrices A^i are such that $\det(A^i) > 0$ for $i = 1, 2$, while $\det(A^3) = 0$, that is, $\theta^3 \equiv \theta^*$ for some constant θ^*, we deduce that:

(i) $(A^i)^{-1}$, $i = 1, 2$ exist, are positive definite and symmetric;

(ii) the matrix $M = (m_{ij})_{i,j=1,2} := (A^2)^{-1} + (A^1)^{-1}$ is symmetric and positive definite, and by writing it down explicitly we infer that m_{11} and m_{22} are nonnegative. Moreover, since $\det M > 0$, we have that

$$\sqrt{m_{11}}\sqrt{m_{22}} > |m_{21}| = |m_{12}|;$$

(iii) the symmetric matrix $A^3 = (a_{ij})_{i,j=1,2}$ is given by $A^3 = L_3 \begin{pmatrix} \sin^2\theta^* & -\sin\theta^*\cos\theta^* \\ -\sin\theta^*\cos\theta^* & \cos^2\theta^* \end{pmatrix} = L_3 P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$ where $P = \begin{pmatrix} \cos\theta^* & -\sin\theta^* \\ \sin\theta^* & \cos\theta^* \end{pmatrix}$ and $P^{-1} = P^t$. In particular, note that a_{11} and a_{22} are nonnegative and $\sqrt{a_{11}}\sqrt{a_{22}} \geq |a_{21}| = |a_{12}|$ holds;

(iv) $\det(A^3 M) = \det A^3 \det M = 0$. Moreover (ii) and (iii) yield

$$\begin{aligned} tr(A^3 M) &= a_{11}m_{11} + a_{12}m_{21} + a_{21}m_{12} + a_{22}m_{22} \\ &\geq a_{11}m_{11} + a_{22}m_{22} - 2|a_{12}||m_{12}| \\ &= (\sqrt{a_{11}}\sqrt{m_{11}} - \sqrt{a_{22}}\sqrt{m_{22}})^2 + 2\sqrt{a_{11}}\sqrt{m_{11}}\sqrt{a_{22}}\sqrt{m_{22}} - 2|a_{12}||m_{12}| \geq 0. \end{aligned}$$

This implies that the matrix $A^3 M$ has eigenvalues $\omega_1 = 0$ and $\omega_2 = tr(A^3 M) \geq 0$, and can be diagonalized. Hence there exists an invertible matrix Q such that

$$Q(A^3 M)Q^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_2 \end{pmatrix}, \quad \text{where } 0 \leq \omega_2 \leq C\left(L_3, L_2, L_1, \frac{1}{\det(A^1)}, \frac{1}{\det(A^2)}\right).$$

By writing

$$I + A^3((A^2)^{-1} + (A^1)^{-1}) = I + A^3 M = Q^{-1}(I + Q(A^3 M)Q^{-1})Q = Q^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1+\omega_2 \end{pmatrix} Q$$

we infer that such matrix is invertible and we can solve the system for $\vec{\mu}$ and $\vec{\lambda}$. Moreover, the above analysis yields that

$$|\lambda| + |\mu| \leq C\left(\sum_{i=1}^3 \int_{I_i} |\partial_s \theta^i|^p ds\right) \quad \text{with } C = C\left(L_1, L_2, L_3, \frac{1}{\det A^1}, \frac{1}{\det A^2}\right). \tag{2.24}$$

This is a bound that is important for the analysis that follows.

Summing up and recalling (1.1), we have that the Lagrange multipliers are well defined in the case $L_3 < \min\{L_1, L_2\}$, which corresponds to a situation where the curve γ_3 might be a straight line, whereas γ_1 and γ_2 necessarily have regions with non vanishing curvature. In fact, for the previous estimates on the Lagrange multipliers to hold, it is enough to assume that at most one of the three curves is flat. This is the case we will mostly concentrate on, the remaining cases are briefly discussed in the following remark.

Remark 2.1. Let us first consider the case where $L_3 = L_2 < L_1$. As shown above the Lagrange multipliers are well defined as long as none of the curve is a straight line. If the curve γ_3 becomes straight, the same must happen for γ_2. More precisely, due to the theta-network configuration, we have that $\gamma_2 = \gamma_3$, $A^3 = A^2$ with A^3 as in case (iii) discussed above, and $G^2 = G^3 = 0$. Summing up equations (2.20) and (2.21) yields

$$(\vec{\lambda} - \vec{\mu}) \cdot (2A^1 + A^3) = 2G^1$$
$$(\vec{\lambda} + \vec{\mu}) \cdot A^3 = 0$$

Since $(2A^1 + A^3)$ is positive definite and invertible, we get

$$\begin{aligned}(\vec{\lambda} - \vec{\mu}) &= 2G^1 \cdot (2A^1 + A^3)^{-1} \\ 0 = (\vec{\lambda} + \vec{\mu}) \cdot P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= (\vec{\lambda} + \vec{\mu}) \cdot \begin{pmatrix} 0 & -\sin\theta^* \\ 0 & \cos\theta^* \end{pmatrix} \\ &= (0, -(\lambda^1 + \mu^1)\sin\theta^* + (\lambda^2 + \mu^2)\cos\theta^*).\end{aligned}$$

In this case, a solution of the system (2.20), (2.21) is given by

$$\vec{\lambda} = -\vec{\mu} = G^1 \cdot (2A^1 + A^3)^{-1}.$$

However, the solution is not unique. Moreover, even if we can pick up a solution for which (2.24) holds, we have no means to control the constant C in (2.24) when two curves simultaneously become flat.

In the case $L_1 = L_2 = L_3$, the three curves can become straight necessarily at the same time. When this happens, the energy is minimal and equal to zero, and the trivial solution of three coinciding segments is attained. In this case $G^1 = G^2 = G^3 = 0$, $A^1 = A^2 = A^3$ and $\vec{\lambda} = \vec{\mu} = 0$ is a solution of the system (2.20), (2.21).

Remark 2.2 (Relation between classical formulation and θ-formulation)**.** For simplicity we first consider a smooth evolution of a single curve satisfying

$$\theta_t = (|\theta_s|^{p-2}\theta_s)_s + \mu^1(t)\sin\theta - \mu^2(t)\cos\theta.$$

For a stationary point this implies

$$0 = (|\theta_s|^{p-2}\theta_s)_s + \mu^1\sin\theta - \mu^2\cos\theta. \tag{2.25}$$

After mutiplying by θ_s we obtain

$$0 = \frac{d}{ds}\left(\frac{p-1}{p}|\theta_s|^p - \mu^1\cos\theta - \mu^2\sin\theta\right)$$

which gives

$$\frac{p-1}{p}|\theta_s|^p - (\mu^1\cos\theta + \mu^2\sin\theta) =: \tilde{\mu} \in \mathbb{R}.$$

On the other hand, by differentiating (2.25) we get

$$\begin{aligned}
0 &= (|\theta_s|^{p-2}\theta_s)_{ss} + \theta_s(\mu^1\cos\theta + \mu^2\sin\theta) \\
&= (|\theta_s|^{p-2}\theta_s)_{ss} + \theta_s\left(\frac{p-1}{p}|\theta_s|^p - \tilde{\mu}\right) = (|\theta_s|^{p-2}\theta_s)_{ss} + \frac{p-1}{p}|\theta_s|^p\theta_s - \tilde{\mu}\theta_s \\
&= (|\kappa|^{p-2}\kappa)_{ss} + \frac{p-1}{p}|\kappa|^p\kappa - \tilde{\mu}\kappa.
\end{aligned}$$

Now let us consider a network satisfying the system (2.11), (2.12),(2.13), (2.14). Reasoning as in the case of a single curve, we get that a stationary network solves

$$\begin{aligned}
(|\kappa^1|^{p-2}\kappa^1)_{ss} + \frac{p-1}{p}|\kappa^1|^p\kappa^1 - \tilde{\xi}\kappa^1 = 0, &\qquad \text{in } I_1 \\
(|\kappa^2|^{p-2}\kappa^2)_{ss} + \frac{p-1}{p}|\kappa^2|^p\kappa^2 - \tilde{\lambda}\kappa^2 = 0, &\qquad \text{in } I_2 \\
(|\kappa^3|^{p-2}\kappa^3)_{ss} + \frac{p-1}{p}|\kappa^3|^p\kappa^3 - \tilde{\mu}\kappa^3 = 0, &\qquad \text{in } I_3, \\
\kappa^i = 0 \quad \text{on } \partial I_i, &\quad i = 1,2,3,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mu} &= \frac{p-1}{p}|\theta_s^3|^p - (\mu^1\cos\theta^3 + \mu^2\sin\theta^3), \\
\tilde{\lambda} &= \frac{p-1}{p}|\theta_s^2|^p + (\lambda^1\cos\theta^2 + \lambda^2\sin\theta^2), \\
\tilde{\xi} &= \frac{p-1}{p}|\theta_s^1|^p - ((\lambda^1-\mu^1)\cos\theta^1 + (\lambda^2-\mu^2)\sin\theta^1).
\end{aligned}$$

Using the expressions above, the fact that at a triple junction $\kappa^i = \theta_s^i = 0$, and the equations (2.11), (2.12), (2.13) evaluated at a junction when the velocities $\theta_t^i = 0$, one verifies that at a triple junction there holds

$$\sum_{i=1}^{3}(|\theta_s^i|^{p-2}\theta_s^i)_s N^i = \tilde{\xi}T^1 + \tilde{\lambda}T^2 + \tilde{\mu}T^3. \tag{2.26}$$

If $p = 2$, noting that $\theta_{ss}^i N^i = (\partial_s\kappa^i)N^i = \nabla_s\vec{\kappa}^i$ we see that a stationary network satisfies the natural boundary conditions at the triple junctions derived for the L^2-gradient flow of elastic networks in [8, 15, 14], so that it is a critical network for the elastic energy $E_2(\Gamma)$.

We now collect some important estimates which will be useful in the sequel.

Lemma 2.3 ([19], Lemma 1)**.** *Let $I = (0, L)$ and $\varphi : \bar{I} \to \mathbb{R}$ be a continuous function with positive oscillation d_0, i.e.*

$$osc_{\bar{I}}\,\varphi \geq d_0 > 0.$$

Suppose $\omega : [0, +\infty] \to [0, +\infty]$ is a continuous monotonic function which is a modulus of continuity of φ, i.e., $\omega(0) = 0$ and

$$|\varphi(s) - \varphi(\sigma)| \leq \omega(|s - \sigma|) \qquad \forall s, \sigma \in \bar{I}.$$

Then we have the following estimates:

(i) There exists a positive constant

$$C = \sin^2(\delta_0/4) \cdot \min\{\omega^{-1}(\delta_0/4), L/2\},$$

where $\delta_0 = \min\{d_0, \pi\}$*, such that*

$$C \leq \int_I \sin^2(\varphi(s) + \varphi_*)ds, \qquad C \leq \int_I \cos^2(\varphi(s) + \varphi_*)ds,$$

for any arbitrary constant φ_**.*

(ii) There holds

$$\det \begin{pmatrix} \int_I \sin^2\theta ds & -\int_I \sin\theta\cos\theta ds \\ -\int_I \sin\theta\cos\theta ds & \int_I \cos^2\theta ds \end{pmatrix} \geq \frac{L}{2}C.$$

Proof. The proof given in [19, Lemma 1] relies on the fact that the determinant can be written as a double integral as follows

$$\det \begin{pmatrix} \int_I \sin^2\theta ds & -\int_I \sin\theta\cos\theta ds \\ -\int_I \sin\theta\cos\theta ds & \int_I \cos^2\theta ds \end{pmatrix} = \frac{1}{2}\int_I\int_I \sin^2(\theta(s) - \theta(\sigma))ds\, d\sigma.$$

□

If we consider a theta-network for which at most one curve can become a line, then the angles of the remaining two curves have always positive oscillation.

Corollary 2.4. *Suppose* $L_3 < \min\{L_1, L_2\}$*. There there exists a constant* $C > 0$ *such that*

$$\det(A^2) \geq C, \qquad \det(A^1) \geq C,$$

where the matrices A^i*,* $i = 1, 2$*, are defined as in* (2.16)*. The constant* C *depends on* L_1, L_2 *and the oscillation of the angle functions* θ^2 *and* θ^1*.*

Lemma 2.5. *Suppose* $L_3 < \min\{L_1, L_2\}$*. Then for the Lagrange multipliers* $\vec{\lambda}$*,* $\vec{\mu}$ *(unique solution of the system* (2.20)*,* (2.21)*) we have the bound*

$$|\vec{\lambda}| + |\vec{\mu}| \leq C\left(\sum_{i=1}^{3}\int_{I_i} |\partial_s\theta^i|^p ds\right)$$

where C *depends on* L_1, L_2*,* L_3 *and the oscillation of the angle functions* θ^2 *and* θ^1*.*

Proof. This follows directly from (2.24) and the Corollary 2.4. □

3 Existence of solutions

From now on we shall assume that condition (1.3) holds.

3.1 The discretization procedure

The discrete scheme

Let $\boldsymbol{\theta}_0 \in H$ and $T > 0$, $n \in \mathbb{N}$, $\tau_n = \frac{T}{n}$. We define a family of maps $\{\boldsymbol{\theta}_{i,n}\}_{i=0}^n \in H$, $\boldsymbol{\theta}_{i,n} = (\theta^1_{i,n}, \theta^2_{i,n}, \theta^3_{i,n})$, inductively by making use of a minimization problem. Set $\boldsymbol{\theta}_{0,n} = \boldsymbol{\theta}_0$. For each $i \in \{1, \dots, n\}$ consider the following variation problem:

$$(M_{i,n}) \qquad \min\{E_{i,n}(\boldsymbol{\theta}) \,|\, \boldsymbol{\theta} \in H\}$$

where

$$E_{i,n}(\boldsymbol{\theta}) := \sum_{j=1}^{3} \left(\frac{1}{p} \int_{I_j} |\partial_s \theta^j|^p ds + \frac{1}{2\tau_n} \int_{I_j} |\theta^j - \theta^j_{i-1,n}|^2 ds \right). \tag{3.1}$$

Existence of a minimizers $\boldsymbol{\theta} \in H$ follows by standard methods in the calculus of variations taking into account that $(H, \|\cdot\|_H)$ with $\|\boldsymbol{\theta}\|_H := \sum_{i=1}^3 \|\theta^j\|_{W^{1,p}(I_j)}$ is a Banach space (see [31, Thm 3.1] for similar arguments).

Discrete Lagrange multipliers

Let $\boldsymbol{\theta}\,(= \boldsymbol{\theta}_{i,n}) \in H$ denote a solution for $(M_{i,n})$. Moreover let

$$\boldsymbol{\psi} = (\psi^1, \psi^2, \psi^3) \in W^{1,p}(0, L_1) \times W^{1,p}(0, L_2) \times W^{1,p}(0, L_3) =: \boldsymbol{W}^{1,p}$$

and define

$$\begin{aligned}
C_1(\boldsymbol{\theta}) &= \int_{I_1} \cos\theta^1 ds - \int_{I_2} \cos\theta^2 ds, \\
C_2(\boldsymbol{\theta}) &= \int_{I_1} \sin\theta^1 ds - \int_{I_2} \sin\theta^2 ds, \\
C_3(\boldsymbol{\theta}) &= \int_{I_3} \cos\theta^3 ds - \int_{I_1} \cos\theta^1 ds, \\
C_4(\boldsymbol{\theta}) &= \int_{I_3} \sin\theta^3 ds - \int_{I_1} \sin\theta^1 ds.
\end{aligned}$$

To show the existence of Lagrange multipliers $\vec{\lambda}_{i,n} = (\lambda^1_{i,n}, \lambda^2_{i,n})$, $\vec{\mu}_{i,n} = (\mu^1_{i,n}, \mu^2_{i,n}) \in \mathbb{R}^2$ such that

$$\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\psi} + \lambda^1_{i,n}\,\delta C_1(\boldsymbol{\theta})\boldsymbol{\psi} + \lambda^2_{i,n}\,\delta C_2(\boldsymbol{\theta})\boldsymbol{\psi} + \mu^1_{i,n}\,\delta C_3(\boldsymbol{\theta})\boldsymbol{\psi} + \mu^2_{i,n}\,\delta C_4(\boldsymbol{\theta})\boldsymbol{\psi} = 0 \quad \forall \boldsymbol{\psi} \in \boldsymbol{W}^{1,p} \tag{3.2}$$

we consider the map

$$\mathbb{R}^5 \ni (\epsilon, \boldsymbol{t}) = (\epsilon, t_1, t_2, t_3, t_4) \mapsto \boldsymbol{C}(\epsilon, \boldsymbol{t}) = \begin{pmatrix} C_1(\boldsymbol{\theta} + \epsilon\boldsymbol{\psi} + \sum_{r=1}^4 t_r \boldsymbol{\varphi}_r) \\ C_2(\boldsymbol{\theta} + \epsilon\boldsymbol{\psi} + \sum_{r=1}^4 t_r \boldsymbol{\varphi}_r) \\ C_3(\boldsymbol{\theta} + \epsilon\boldsymbol{\psi} + \sum_{r=1}^4 t_r \boldsymbol{\varphi}_r) \\ C_4(\boldsymbol{\theta} + \epsilon\boldsymbol{\psi} + \sum_{r=1}^4 t_r \boldsymbol{\varphi}_r) \end{pmatrix}$$

for

$$\boldsymbol{\varphi}_r = (\varphi^1_r, \varphi^2_r, \varphi^3_r) \in \boldsymbol{W}^{1,p}, \quad r = 1, 2, 3, 4.$$

Note that $\boldsymbol{C}(0,\mathbf{0})=\mathbf{0}$ since $\boldsymbol{\theta}\in H$. If the maps $\boldsymbol{\varphi}_r$ can be chosen such that the matrix

$$\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})=\begin{pmatrix}\frac{\partial}{\partial t_1}C_1 & \dots & \frac{\partial}{\partial t_4}C_1\\ \vdots & & \vdots\\ \frac{\partial}{\partial t_1}C_4 & \dots & \frac{\partial}{\partial t_4}C_4\end{pmatrix}(0,\mathbf{0})=(\delta C_i(\boldsymbol{\theta})(\boldsymbol{\varphi}_j))_{i,j=1\dots4}$$

$$=\begin{pmatrix}-\int_{I_1}\sin\theta^1\varphi_1^1+\int_{I_2}\sin\theta^2\varphi_1^2 & -\int_{I_1}\sin\theta^1\varphi_2^1+\int_{I_2}\sin\theta^2\varphi_2^2 & -\int_{I_1}\sin\theta^1\varphi_3^1+\int_{I_2}\sin\theta^2\varphi_3^2 & -\int_{I_1}\sin\theta^1\varphi_4^1+\int_{I_2}\sin\theta^2\varphi_4^2\\ \int_{I_1}\cos\theta^1\varphi_1^1-\int_{I_2}\cos\theta^2\varphi_1^2 & \int_{I_1}\cos\theta^1\varphi_2^1-\int_{I_2}\cos\theta^2\varphi_2^2 & \int_{I_1}\cos\theta^1\varphi_3^1-\int_{I_2}\cos\theta^2\varphi_3^2 & \int_{I_1}\cos\theta^1\varphi_4^1-\int_{I_2}\cos\theta^2\varphi_4^2\\ -\int_{I_3}\sin\theta^3\varphi_1^3+\int_{I_1}\sin\theta^1\varphi_1^1 & -\int_{I_3}\sin\theta^3\varphi_2^3+\int_{I_1}\sin\theta^1\varphi_2^1 & -\int_{I_3}\sin\theta^3\varphi_3^3+\int_{I_1}\sin\theta^1\varphi_3^1 & -\int_{I_3}\sin\theta^3\varphi_4^3+\int_{I_1}\sin\theta^1\varphi_4^1\\ \int_{I_3}\cos\theta^3\varphi_1^3-\int_{I_1}\cos\theta^1\varphi_1^1 & \int_{I_3}\cos\theta^3\varphi_2^3-\int_{I_1}\cos\theta^1\varphi_2^1 & \int_{I_3}\cos\theta^3\varphi_3^3-\int_{I_1}\cos\theta^1\varphi_3^1 & \int_{I_3}\cos\theta^3\varphi_4^3-\int_{I_1}\cos\theta^1\varphi_4^1\end{pmatrix}$$

has maximal rank, then by the implicit function theorem we have that there exist C^1-maps σ_r, $r=1,2,3,4$, defined in a neighborhood of zero, such that $\sigma_r(0)=0$ for $r=1,2,3,4$, and

$$\boldsymbol{C}(\epsilon,\sigma_1(\epsilon),\dots,\sigma_4(\epsilon))=\mathbf{0}\qquad\text{for }\epsilon\in(-\epsilon_0,\epsilon_0),$$

i.e., $\boldsymbol{\theta}+\epsilon\boldsymbol{\psi}+\sum_{r=1}^4\sigma_r(\epsilon)\boldsymbol{\varphi}_r\in H$ for $\epsilon\in(-\epsilon_0,\epsilon_0)$. Differentiation in ϵ of the above equation gives

$$\begin{pmatrix}\sigma_1'(0)\\ \vdots\\ \sigma_4'(0)\end{pmatrix}=-\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)^{-1}\begin{pmatrix}\delta C_1(\boldsymbol{\theta})\boldsymbol{\psi}\\ \vdots\\ \delta C_4(\boldsymbol{\theta})\boldsymbol{\psi}\end{pmatrix}$$

so that, from the minimality of $\boldsymbol{\theta}$ we infer

$$\begin{aligned}0&=\frac{d}{d\epsilon}\Big|_{\epsilon=0}E_{i,n}\left(\boldsymbol{\theta}+\epsilon\boldsymbol{\psi}+\sum_{r=1}^4\sigma_r(\epsilon)\boldsymbol{\varphi}_r\right)=\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\psi}+\sum_{r=1}^4\sigma_r'(0)\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_r\\&=\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\psi}-\sum_{l=1}^4\left(\sum_{r=1}^4\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)^{-1}_{rl}\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_r\right)\delta C_l(\boldsymbol{\theta})\boldsymbol{\psi}.\end{aligned}$$

It follows that (3.2) holds with

$$\begin{aligned}\lambda^1_{i,n}&=-\sum_{r=1}^4\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)^{-1}_{r1}\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_r\\ \lambda^2_{i,n}&=-\sum_{r=1}^4\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)^{-1}_{r2}\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_r\\ \mu^1_{i,n}&=-\sum_{r=1}^4\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)^{-1}_{r3}\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_r\\ \mu^2_{i,n}&=-\sum_{r=1}^4\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)^{-1}_{r4}\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_r\end{aligned}$$

or equivalently

$$(\lambda^1_{i,n},\lambda^2_{i,n},\mu^1_{i,n},\mu^2_{i,n})\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)=-(\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_1,\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_2,\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_3,\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_4).$$

By letting

$$\boldsymbol{\varphi}_1:=(0,\sin\theta^2,-\sin\theta^3),\qquad\boldsymbol{\varphi}_3:=(\sin\theta^1,-\sin\theta^2,0)$$

$$\boldsymbol{\varphi}_2 := (0, -\cos\theta^2, \cos\theta^3), \qquad \boldsymbol{\varphi}_4 := (-\cos\theta^1, \cos\theta^2, 0)$$

we obtain that

$$\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right) = \left(\begin{array}{c|c} A^2 & -(A^1+A^2) \\ \hline A^3 & A^1 \end{array}\right)$$

with $A^i \in \mathbb{R}^{2\times 2}$ as in (2.16). Moreover, we compute

$$\begin{aligned}(\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_1, \delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_2) &= G^2 - G^3 \\ &+ \frac{1}{\tau_n}\int_{I_2}(\theta^2 - \theta^2_{i-1,n})(\sin\theta^2, -\cos\theta^2)ds - \frac{1}{\tau_n}\int_{I_3}(\theta^3 - \theta^3_{i-1,n})(\sin\theta^3, -\cos\theta^3)ds, \\ (\delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_3, \delta E_{i,n}(\boldsymbol{\theta})\boldsymbol{\varphi}_4) &= G^1 - G^2 \\ &- \frac{1}{\tau_n}\int_{I_2}(\theta^2 - \theta^2_{i-1,n})(\sin\theta^2, -\cos\theta^2)ds + \frac{1}{\tau_n}\int_{I_1}(\theta^1 - \theta^1_{i-1,n})(\sin\theta^1, -\cos\theta^1)ds\end{aligned}$$

where G^i is as in (2.19). Therefore the Lagrange multipliers solve

$$(\lambda^1_{i,n}, \lambda^2_{i,n})\cdot A^2 + (\mu^1_{i,n}, \mu^2_{i,n})\cdot A^3 = G^3 - G^2 + R^{23}_{i,n} \tag{3.3}$$

$$-(\lambda^1_{i,n}, \lambda^2_{i,n})\cdot (A^1+A^2) + (\mu^1_{i,n}, \mu^2_{i,n})\cdot A^1 = G^2 - G^1 + R^{21}_{i,n}, \tag{3.4}$$

where we set

$$\begin{aligned}-R^{23}_{i,n} &:= \frac{1}{\tau_n}\int_{I_2}(\theta^2 - \theta^2_{i-1,n})(\sin\theta^2, -\cos\theta^2)ds - \frac{1}{\tau_n}\int_{I_3}(\theta^3 - \theta^3_{i-1,n})(\sin\theta^3, -\cos\theta^3)ds, \\ -R^{21}_{i,n} &:= -\frac{1}{\tau_n}\int_{I_2}(\theta^2 - \theta^2_{i-1,n})(\sin\theta^2, -\cos\theta^2)ds + \frac{1}{\tau_n}\int_{I_1}(\theta^1 - \theta^1_{i-1,n})(\sin\theta^1, -\cos\theta^1)ds.\end{aligned}$$

Recalling the system (2.22), (2.23) and the subsequent discussion concerning its solvability, we can conclude that (under assumption (1.3)) the above system is solvable, that is, the matrix $\left(\frac{\partial}{\partial \boldsymbol{t}}\boldsymbol{C}(0,\mathbf{0})\right)$ has maximal rank. Moreover, similarly to Lemma 2.5 we infer that

$$|\vec{\lambda}_{i,n}| + |\vec{\mu}_{i,n}| \le C\left(\sum_{j=1}^3 \int_{I_j} |\partial_s\theta^j|^p\right) + \frac{C}{\tau_n}\sum_{j=1}^3\int_{I_j}|\theta^j - \theta^j_{i-1,n}|ds, \tag{3.5}$$

where C has the same dependencies given in Lemma 2.5, and $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3) = \boldsymbol{\theta}_{i,n}$ is solution to $(M_{i,n})$.

Regularity

Let $\boldsymbol{\theta}_{i,n} \in H$ be a solution to $(M_{i,n})$. Since (3.2) and (3.5) hold for $\boldsymbol{\theta} = \boldsymbol{\theta}_{i,n}$ it follows that the map $|\partial_s\theta^j|^{p-2}\partial_s\theta^j \in L^{\frac{p}{p-1}}(L_j)$ admits weak derivative in $L^1(L_j)$ with

$$|(|\partial_s\theta^j|^{p-2}\partial_s\theta^j)_s| \le C\left|\frac{\theta^j - \theta^j_{i-1,n}}{\tau_n}\right| + |\lambda^1_{i,n}| + |\lambda^2_{i,n}| + |\mu^1_{i,n}| + |\mu^1_{i,n}|. \tag{3.6}$$

Moreover, the natural boundary conditions

$$\partial_s\theta^j(s) = 0 \quad \text{for } s \in \{0, L_j\} \tag{3.7}$$

hold for $j = 1, 2, 3$.

Definition of approximating functions

First of all let us introduce some notation. We denote by $\boldsymbol{V}_{i,n} = (V^1_{i,n}, V^2_{i,n}, V^3_{i,n})$ the discrete velocity

$$\boldsymbol{V}_{i,n} := \frac{\boldsymbol{\theta}_{i,n} - \boldsymbol{\theta}_{i-1,n}}{\tau_n}$$

We will need maps that interpolate the three components of our maps $\{\boldsymbol{\theta}_{i,n}\}_{i=0,\dots,n}$ linearly in time:

Definition 3.1. *Let $\boldsymbol{\theta}_n : I_1 \times I_2 \times I_3 \times [0,T] \to \mathbb{R}^3$ be defined by*

$$\boldsymbol{\theta}_n(\boldsymbol{s}, t) := \boldsymbol{\theta}_{i,n-1}(\boldsymbol{s}) + (t - (i-1)\tau_n)\boldsymbol{V}_{i,n}(\boldsymbol{s})$$

if $(\boldsymbol{s}, t) = (s_1, s_2, s_3, t) \in I_1 \times I_2 \times I_3 \times [(i-1)\tau_n, \tau_n]$ for $i = 1, \dots, n$.

We will need also piecewise constant interpolations, that is,

Definition 3.2. *Let $\bar{\boldsymbol{\theta}}_n, \underline{\boldsymbol{\theta}}_n, \boldsymbol{V}_n : I_1 \times I_2 \times I_3 \times [0,T] \to \mathbb{R}^3$ be defined by*

$$\begin{aligned}
\underline{\boldsymbol{\theta}}_n(\boldsymbol{s}, t) &:= \boldsymbol{\theta}_{i-1,n}(\boldsymbol{s}),\\
\bar{\boldsymbol{\theta}}_n(\boldsymbol{s}, t) &:= \boldsymbol{\theta}_{i,n}(\boldsymbol{s}),\\
\boldsymbol{V}_n(\boldsymbol{s}, t) &:= \boldsymbol{V}_{i,n}(\boldsymbol{s})
\end{aligned}$$

if $(\boldsymbol{s}, t) = (s_1, s_2, s_3, t) \in I_1 \times I_2 \times I_3 \times [(i-1)\tau_n, \tau_n]$ for $i = 1, \dots, n$.

Similarly for the discrete Lagrange multipliers (recall (3.2)) we define

Definition 3.3. *Let $\vec{\boldsymbol{\lambda}}_n, \vec{\boldsymbol{\mu}}_n : [0,T] \to \mathbb{R}^2$ be defined by*

$$\begin{aligned}
\vec{\boldsymbol{\lambda}}_n(t) = (\lambda^1_n(t), \lambda^2_n(t)) &:= \vec{\lambda}_{i,n},\\
\vec{\boldsymbol{\mu}}_n(t) = (\mu^1_n(t), \mu^2_n(t)) &:= \vec{\mu}_{i,n}
\end{aligned}$$

if $t \in [(i-1)\tau_n, \tau_n]$ for $i = 1, \dots, n$.

To keep the notation as simple as possible we adopt from now on following conventions. For $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$ in an appropriate function space and $q \in [1, \infty)$ we write

$$\int_I |\boldsymbol{\theta}|^q ds := \sum_{j=1}^3 \int_{I_j} |\theta^j|^q ds, \qquad \int_I |\partial_s \boldsymbol{\theta}|^q ds := \sum_{j=1}^3 \int_{I_j} |\partial_s \theta^j|^q ds,$$

$$\int_0^T \int_I |\boldsymbol{\theta}|^q ds dt := \sum_{j=1}^3 \int_0^T \int_{I_j} |\theta^j|^q ds dt.$$

Uniform bounds for the approximating functions

We now derive some uniform bounds for solutions of $(M_{i,n})$.

Theorem 3.4. *Assume* (1.3). *Let* $\boldsymbol{\theta}_0 \in H$ *and* $T > 0$ *be given. Let* $\boldsymbol{\theta}_{i,n} \in H$ *be the solution of* $(M_{i,n})$ *and let* $\vec{\lambda}_{i,n}$, $\vec{\mu}_{i,n} \in \mathbb{R}^2$ *be the Lagrange multipliers fulfilling* (3.2). *Upon recalling the definitions and convention given above, write*

$$D(\boldsymbol{\theta}_{i,n}) := \frac{1}{p}\int_I |\partial_s \boldsymbol{\theta}_{i,n}|^p ds.$$

Then we have that

$$\begin{aligned} D(\boldsymbol{\theta}_{i,n}) \le D(\boldsymbol{\theta}_{i-1,n}) \le D(\boldsymbol{\theta}_0) \qquad & \text{for all } i = 1, \dots, n, \\ \frac{1}{2}\int_0^T \int_I |\boldsymbol{V_n}|^2 ds\, dt \le D(\boldsymbol{\theta}_0), & \\ \int_0^T |\vec{\boldsymbol{\lambda}}_n|^2(t) dt + \int_0^T |\vec{\boldsymbol{\mu}}_n|^2(t) dt \le C\big(TD(\boldsymbol{\theta}_0) + 1\big) D(\boldsymbol{\theta}_0), & \\ \int_I |\boldsymbol{\theta}_{i,n}|^2 ds \le C\int_I |\boldsymbol{\theta}_0|^2 ds + CTD(\boldsymbol{\theta}_0), & \end{aligned}$$

where C *has the same dependencies of the constant appearing in Lemma* 2.5.

Proof. We let

$$P_{i,n}(\boldsymbol{\theta}) := \sum_{j=1}^{3} \left(\frac{1}{2\tau_n} \int_{I_j} |\theta^j - \theta^j_{i-1,n}|^2 ds \right)$$

so that $E_{i,n}(\boldsymbol{\theta}) = D(\boldsymbol{\theta}) + P_{i,n}(\boldsymbol{\theta})$. The proof of the first statement follows by an induction argument. Fix $i \in \{1, \dots, n\}$ and assume that $D(\boldsymbol{\theta}_{j,n}) \le D(\boldsymbol{\theta}_0)$ for all $j = 1, \dots, i-1$. Then it follows from the minimality of $\boldsymbol{\theta}_{i,n}$ that

$$D(\boldsymbol{\theta}_{i,n}) \le D(\boldsymbol{\theta}_{i,n}) + P_{i,n}(\boldsymbol{\theta}_{i,n}) = E_{i,n}(\boldsymbol{\theta}_{i,n}) \le E_{i,n}(\boldsymbol{\theta}_{i-1,n}) = D(\boldsymbol{\theta}_{i-1,n}).$$

This gives the first statement. Next observe that from

$$P_{i,n}(\boldsymbol{\theta}_{i,n}) \le D(\boldsymbol{\theta}_{i-1,n}) - D(\boldsymbol{\theta}_{i,n}) \tag{3.8}$$

we obtain

$$\begin{aligned} \int_0^T \int_I |\boldsymbol{V_n}|^2 ds dt = \sum_{i=1}^n \tau_n \int_I |\boldsymbol{V_{i,n}}|^2 ds = 2\sum_{i=1}^n P_{i,n}(\boldsymbol{\theta}_{i,n}) & \\ \le 2\sum_{i=1}^n (D(\boldsymbol{\theta}_{i-1,n}) - D(\boldsymbol{\theta}_{i,n})) \le 2D(\boldsymbol{\theta}_0) & \end{aligned}$$

and the second statement follows. From (3.5) we infer that

$$|\vec{\lambda}_{i,n}| + |\vec{\mu}_{i,n}| \le CD(\boldsymbol{\theta}_{i,n}) + C\int_I |\boldsymbol{V}_{i,n}| ds \le CD(\boldsymbol{\theta}_0) + C\left(\int_I |\boldsymbol{V}_{i,n}|^2 ds\right)^{1/2}, \tag{3.9}$$

which gives the third statement after squaring and integrating in time. Finally, observe that for $j = 1, 2, 3$ we can write

$$\|\theta^j_{i,n}\|_{L^2(I_j)} \le \|\theta^j_{0,n}\|_{L^2(I_j)} + \sum_{r=1}^i \|\theta^j_{r,n} - \theta^j_{r-1,n}\|_{L^2(I_j)} = \|\theta^j_{0,n}\|_{L^2(I_j)} + \sum_{r=1}^i \sqrt{\tau_n} \left\| \frac{\theta^j_{r,n} - \theta^j_{r-1,n}}{\sqrt{\tau_n}} \right\|_{L^2(I_j)}$$

$$\leq \|\theta^j_{0,n}\|_{L^2(I_j)} + \sqrt{i\tau_n}\left(\sum_{r=1}^{i}\int_{I_j}\frac{|\theta^j_{r,n}-\theta^j_{r-1,n}|^2}{\tau_n}ds\right)^{1/2}$$

$$\leq \|\theta^j_{0,n}\|_{L^2(I_j)} + \sqrt{2T}\left(\sum_{r=1}^{i}P_{r,n}(\boldsymbol{\theta}_{r,n})\right)^{1/2} \leq \|\theta^j_{0,n}\|_{L^2(I_j)} + \sqrt{2TD(\boldsymbol{\theta}_0)},$$

where we have used again (3.8). The last statement follows. □

3.2 Convergence of the scheme

Having achieved some uniform bounds for the approximating maps, it is possible to pass to the limit as $n \to \infty$. The following three Lemmas are similar to the ones obtained in [31, Lemma 3.11, Lemma 3.12, Lemma 3.13]. For the reader's convenience, we include the proofs in the Appendix. We point out that condition (1.3) is not needed to prove these results, since the Lagrange multipliers are not involved.

Lemma 3.5. *Let $\boldsymbol{\theta}_0 \in H$ and $T > 0$ be as in Theorem 3.4. Let $\boldsymbol{\theta}_n = (\theta^1_n, \theta^2_n, \theta^3_n)$ be the piecewise linear interpolation of $\{\boldsymbol{\theta}_{i,n}\}$ given in Definition 3.1. Then, for $j = 1, 2, 3$, there exists a map*

$$\theta^j \in L^\infty(0,T;W^{1,p}(I_j)) \cap H^1(0,T;L^2(I_j))$$

such that

$$\frac{1}{2}\int_0^T\int_{I_j}|\partial_t\theta^j(s,t)|^2\,dsdt \leq D(\boldsymbol{\theta}_0), \tag{3.10}$$

$$\sup_{(0,T)}\|\partial_s\theta^j\|_{L^p(I_j)} \leq C = C(D(\boldsymbol{\theta}_0),p), \tag{3.11}$$

$$\sup_{(0,T)}\|\theta^j\|_{W^{1,p}(I_j)} \leq C = C(p,L_j,T,D(\boldsymbol{\theta}_0),\|\theta^j_0\|_{L^2(I_j)}), \tag{3.12}$$

and, for a subsequence which we still denote by θ^j_n,

$$\begin{cases} \theta^j_n \rightharpoonup \theta^j & \text{weakly}* \text{ in } \quad L^\infty(0,T;W^{1,p}(I_j)), \\ \theta^j_n \rightharpoonup \theta^j & \text{weakly in } \quad H^1(0,T;L^2(I_j)), \end{cases} \quad \text{as} \quad n \to \infty. \tag{3.13}$$

Moreover, for $\alpha = \min\{\frac{1}{4}, \frac{p-1}{2p}\}$ we have that

$$\theta^j_n \to \theta^j \quad \text{in} \quad C^{0,\alpha}([0,T]\times I_j). \tag{3.14}$$

In particular, $\theta^j(\cdot,t) \to \theta^j_0(\cdot)$ in C^0 as $t \downarrow 0$.

A direct consequence of equation (3.14) of Lemma 3.5 and Lemma A.1 is the following

Corollary 3.6. *Under the assumptions of Lemma 3.5, for all $j = 1, 2, 3$ and $n \in \mathbb{N}$ there holds*

$$osc_{\bar{I}_j}\,\theta^j_n,\ osc_{\bar{I}_j}\,\theta^j \in C^\alpha([0,T]).$$

In particular, using Definition 3.1, we can assert that the oscillations of the maps $\theta^j_{i,n}$ are close to the oscillation of $\theta^j_{0,n}$ if T is chosen sufficiently small. This fact will be used in the proof of Theorem 1.3.

Lemma 3.7. *Let $\boldsymbol{\theta}_0 \in H$ and $T > 0$ be as in Theorem 3.4. Let $\bar{\boldsymbol{\theta}}_n = (\bar\theta^1, \bar\theta^2, \bar\theta^3)$, $\underline{\boldsymbol{\theta}}_n = (\underline\theta^1, \underline\theta^2, \underline\theta^3)$ be the piecewise constant interpolations of $\{\boldsymbol{\theta}_{i,n}\}$ as given in Definition 3.2. Then we have*

$$\bar\theta^j_n \to \theta^j \quad \text{and} \quad \underline\theta^j_n \to \theta^j \text{ in } C^0([0,T]\times I_j), \quad j=1,2,3, \tag{3.15}$$

where θ^j, $j=1,2,3$, denote the maps obtained in Lemma 3.5. Moreover, it holds that

$$\partial_s\bar\theta^j_n \rightharpoonup \partial_s\theta^j \quad \text{and} \quad \partial_s\underline\theta^j_n \rightharpoonup \partial_s\theta^j \quad \text{weakly in} \quad L^p(0,T;L^p(I_j)) \quad \text{as} \quad n\to\infty. \tag{3.16}$$

Lemma 3.8. *Let $\bar{\boldsymbol{\theta}}_n = (\bar\theta^1_n, \bar\theta^2_n, \bar\theta^3_n)$ be the piecewise constant interpolation of $\{\boldsymbol{\theta}_{i,n}\}$ given in Definition 3.2. and let the assumptions of Lemma 3.5 hold. Then, for $j=1,2,3$, it holds that*

$$\int_0^T\int_{I_j} |(\bar\theta^j_n)_s|^{p-2}(\bar\theta^j_n)_s\cdot\varphi_s\,dsdt \to \int_0^T\int_{I_j}|\theta^j_s|^{p-2}\theta^j_s\cdot\varphi_s\,dsdt \quad \text{as} \quad n\to\infty$$

for any $\varphi\in L^\infty(0,T;W^{1,p}(I_j))$.

We can now prove our main existence result.

Proof of Theorem 1.1. (i) Equation (3.2) yields that for any $\boldsymbol{\varphi}=(\varphi^1,\varphi^2,\varphi^3)$ with $\varphi^j\in L^\infty(0,T;W^{1,p}(I_j))$, $j=1,2,3$, and (for almost every) $t\in((i-1)\tau_n, i\tau_n]$, $i=1,\dots,n$ we have

$$\begin{aligned}
0 = &\sum_{j=1}^3\int_{I_j} V^j_n(s,t)\varphi^j(s,t)ds + \sum_{j=1}^3\int_{I_j}|(\bar\theta^j_n)_s|^{p-2}(\bar\theta^j_n)_s\,(\varphi^j)_s\,ds\\
&-(\lambda^1_n(t)-\mu^1_n(t))\int_{I_1}\sin(\bar\theta^1_n)\,\varphi^1 ds + (\lambda^2_n(t)-\mu^2_n(t))\int_{I_1}\cos(\bar\theta^1_n)\,\varphi^1 ds\\
&+\lambda^1_n(t)\int_{I_2}\sin(\bar\theta^2_n)\,\varphi^2 ds - \lambda^2_n(t)\int_{I_2}\cos(\bar\theta^2_n)\,\varphi^2 ds\\
&-\mu^1_n(t)\int_{I_3}\sin(\bar\theta^3_n)\,\varphi^3 ds + \mu^2_n(t)\int_{I_3}\cos(\bar\theta^3_n)\,\varphi^3 ds
\end{aligned}$$

so that integration in time yields

$$\begin{aligned}
0 = &\sum_{j=1}^3\int_0^T\int_{I_j} V^j_n(s,t)\varphi^j(s,t)dsdt + \sum_{j=1}^3\int_0^T\int_{I_j}|(\bar\theta^j_n)_s|^{p-2}(\bar\theta^j_n)_s\,(\varphi^j)_s\,dsdt\\
&-\int_0^T(\lambda^1_n(t)-\mu^1_n(t))\int_{I_1}\sin(\bar\theta^1_n)\,\varphi^1 dsdt + \int_0^T(\lambda^2_n(t)-\mu^2_n(t))\int_{I_1}\cos(\bar\theta^1_n)\,\varphi^1 dsdt\\
&+\int_0^T\lambda^1_n(t)\int_{I_2}\sin(\bar\theta^2_n)\,\varphi^2 dsdt - \int_0^T\lambda^2_n(t)\int_{I_2}\cos(\bar\theta^2_n)\,\varphi^2 dsdt\\
&-\int_0^T\mu^1_n(t)\int_{I_3}\sin(\bar\theta^3_n)\,\varphi^3 dsdt + \int_0^T\mu^2_n(t)\int_{I_3}\cos(\bar\theta^3_n)\,\varphi^3 dsdt
\end{aligned}$$

for any $\boldsymbol{\varphi} \in L^\infty(0,T;\boldsymbol{W}^{1,2})$. We now let $n \to \infty$. The first two integrals are dealt with in Lemma 3.5 and Lemma 3.8. By the uniform bound given Theorem 3.4 we have that there exist $\lambda^1, \lambda^2, \mu^1, \mu^2 \in L^2(0,T)$ such that

$$\lambda_n^j \rightharpoonup \lambda^j \text{ weakly in } L^2(0,T), \qquad \mu_n^j \rightharpoonup \mu^j \text{ weakly in } L^2(0,T) \tag{3.17}$$

for $j = 1, 2$. Since $v_n(t) := \int_{I_1} \sin(\bar\theta_n^1)\varphi^1(s,t)ds \to \int_{I_1} \sin(\theta^1)\varphi^1(s,t)ds =: v(t)$ by Lemma 3.7, and $|v_n| \le C(\varphi^1)$, then also $v_n \to v$ in $L^2(0,T)$ and we infer that

$$\int_0^T \lambda_n^1(t) \int_{I_1} \sin(\bar\theta_n^1)\, \varphi^1 ds dt \to \int_0^T \lambda^1(t) \int_{I_1} \sin(\theta^1)\, \varphi^1 ds dt$$

for $n \to \infty$. The other integrals with the Lagrange multipliers are treated in a similar way and the first statement follows.

(ii) Equations (1.5), (1.6), (1.7), and the natural boundary conditions (1.8) follow directly from (1.4) by choosing test functions of the form $\varphi^j(s,t) = \tilde\varphi(t)\psi^j(s)$ with $\psi^j \in W^{1,p}(I_j)$ and $\tilde\varphi \in C_0^\infty(0,T)$. Also we exploit the fact that given any map $f \in L^1(I)$ with $f_s \in L^2(I)$ and $I \subset \mathbb{R}$ bounded interval, it follows from embedding theory that $f \in H^1(I)$.

(iii) By construction we have that $\boldsymbol{\theta}_{i,n} \in H$, so that

$$\int_{I_1} (\cos\bar\theta_n^1, \sin\bar\theta_n^1)ds = \int_{I_2} (\cos\bar\theta_n^2, \sin\bar\theta_n^2)ds = \int_{I_3} (\cos\bar\theta_n^3, \sin\bar\theta_n^3)ds$$

for all $t \in ((i-1)\tau_n, i\tau_n]$, $i = 1, \dots, n$. Passing to the limit as $n \to \infty$ and using (3.15) we obtain (1.9). □

We now show that the Lagrange multipliers in Theorem 1.1 are uniformly bounded in time.

Proposition 3.9. *Let $\boldsymbol{\theta}_0 \in H$, $T > 0$, $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$, $\vec\lambda$ and $\vec\mu$ be as in Theorem* 1.1. *Then we have that the system* (2.22), (2.23) *holds for almost every time and*

$$\|\vec\lambda\|_{L^\infty(0,T)} + \|\vec\mu\|_{L^\infty(0,T)} \le C(D(\boldsymbol{\theta}_0), p). \tag{3.18}$$

The constant C has the same dependencies as in Lemma 2.5.

Proof. Testing the weak formulation (1.4) with $\boldsymbol{\varphi}(\boldsymbol{s},t) = (-\tilde\varphi \sin\theta^1, 0, 0)$ and $\boldsymbol{\varphi}(\boldsymbol{s},t) = (\tilde\varphi\cos\theta^1, 0, 0)$, where $\tilde\varphi \in C_0^\infty(0,T)$, yields that for almost every time there holds

$$\frac{d}{dt}\int_{I_1} (\cos\theta^1, \sin\theta^1)ds = \int_{I_1} \partial_t\theta^1(-\sin\theta^1, \cos\theta^1)ds = G^1 - (\vec\lambda - \vec\mu)\cdot A^1$$

where we use the notation employed in (2.19), (2.16). Similarly testing with $\boldsymbol{\varphi}(\boldsymbol{s},t) = (0, -\tilde\varphi\sin\theta^2, 0)$ and $\boldsymbol{\varphi}(\boldsymbol{s},t) = (0, \tilde\varphi\cos\theta^2, 0)$, respectively $\boldsymbol{\varphi}(\boldsymbol{s},t) = (0,0,-\tilde\varphi\sin\theta^3)$ and $\boldsymbol{\varphi}(\boldsymbol{s},t) = (0,0,\tilde\varphi\cos\theta^3)$ where $\tilde\varphi \in C_0^\infty(0,T)$ yields that for almost every time

$$\frac{d}{dt}\int_{I_2} (\cos\theta^2, \sin\theta^2)ds = \int_{I_2} \partial_t\theta^2(-\sin\theta^2, \cos\theta^2)ds = G^2 + \vec\lambda\cdot A^2,$$
$$\frac{d}{dt}\int_{I_3} (\cos\theta^3, \sin\theta^3)ds = \int_{I_3} \partial_t\theta^3(-\sin\theta^3, \cos\theta^3)ds = G^3 - \vec\mu\cdot A^3.$$

Using (1.9) we infer that for almost every time the system (2.22), (2.23) holds for $\boldsymbol{\theta}$. Inequality (3.18) now follows from (2.24) and (3.11). □

Remark 3.10. Notice that Theorem 1.1 does not yield uniqueness of solutions. To that end a deeper analysis would be needed (see for instance [31, Lemma 3.20] for a similar issue in the case of a single evolving curve).

Remark 3.11. The method of proof of Theorem 1.1 slightly differs from the one presented in [31] since in this paper we treat the Lagrange multipliers implicitly. This has the advantage that no restriction on the time T is necessary to show existence, and that the decrease of the energy follows directly. In particular, there is no need to analyze higher regularity properties of solutions as in [31]. With the techniques presented here [31, Thm 1.1] can be generalized in the following sense: under the hypothesis of [31, Thm 1.1] then a weak solution to (P) can be defined for any time $T \in (0, +\infty)$.

So far we assumed (1.3). However, as noticed above, the estimates on the Lagrange multipliers given in Theorem 3.4 hold as long as we assume that two of the three curves have positive total curvature, that is, if the corresponding angle functions have positive oscillation. By this observation and by Corollary 3.6, we provide a partial extension of Theorem 1.1.

Proof of Theorem 1.3. Recalling Corollary 3.6, it follows from (1.12) that there exists $T > 0$ such that

$$\min\left(\mathrm{osc}_{\bar{I}_{j_1}} \bar{\theta}_n^{j_1}(t), \mathrm{osc}_{\bar{I}_{j_2}} \bar{\theta}_n^{j_2}(t)\right) \geq \frac{c}{2},$$

for all $t \in [0, T]$ and $n \in \mathbb{N}$. As a consequence the estimates on the Lagrange multipliers given in Theorem 3.4 still hold, and we can proceed exactly as in the proof of Theorem 1.1.

To show the final assertion, it is enough to observe that, if $T_{max} < +\infty$ and the oscillation of θ^{j_1} and of θ^{j_2} are uniformly bounded below by $\delta > 0$ on $[0, T_{max}]$, then we can extend the solution on a time interval $[0, T']$ with $T' = T'(\delta) > T_{max}$. □

Remark 3.12. Note that the result in Theorem 1.3 can be extended to the case of a network of three curves with a single triple junction and three fixed endpoints. We notice that for such network the second order evolution, expressed in terms of the functions θ^j, is again given by the equations (1.5)–(1.7). Moreover, the natural boundary conditions are still given by (1.8), whereas the condition (1.9) becomes

$$\int_{I_1} (\cos\theta^1, \sin\theta^1) ds - P_1 = \int_{I_2} (\cos\theta^2, \sin\theta^2) ds - P_2 = \int_{I_3} (\cos\theta^3, \sin\theta^3) ds - P_3\,,$$

where P_1, P_2, P_3 are the fixed endpoints.

3.3 Long-time behavior

We now show that the weak solutions given by Theorem 1.1 converge, on a suitable sequence of times, to a critical point of the energy.

Proof of Theorem 1.2. From (3.10) we know that, for $j = 1, 2, 3$ we have

$$\frac{1}{2}\int_0^\infty \int_{I_j} |\partial_t \theta^j|^2 ds dt \leq D(\boldsymbol{\theta}_0). \tag{3.19}$$

Together with (3.18) this yields the existence of a sequence of times $(t_n)_{n\in\mathbb{N}}$, and vectors $\vec{\lambda}, \vec{\mu} \in \mathbb{R}^2$ such that $t_n \to \infty$ and

$$\vec{\lambda}(t_n) \to \vec{\lambda}, \qquad \vec{\mu}(t_n) \to \vec{\mu}, \qquad \|\partial_t \theta^j(t_n)\|_{L^2(I_j)} \to 0, \tag{3.20}$$

for $j=1,2,3$, as $n\to\infty$. From (3.11) we infer that for $j=1,2,3$ we have

$$\|\partial_s\theta^j(t_n)\|_{L^p(I_j)}\le C(D(\boldsymbol{\theta}_0),p).$$

Moreover, from

$$|\theta^j(s,t_n)-\theta^j(0,t_n)|\le (L_j)^{\frac{p-1}{p}}\|\partial_s\theta^j(t_n)\|_{L^p(I_j)}\le C$$

for any $s\in[0,L_j]$, we obtain that the sequence $\tilde\theta^j(\cdot,t_n):=\theta^j(\cdot,t_n)-2\pi z_n$, with $z_n\in\mathbb{Z}$ chosen in such a way that $|\theta^j(0,t_n)-2\pi z_n|\le 2\pi$, satisfies in addition the uniform bound

$$\|\tilde\theta^j(t_n)\|_{W^{1,p}(I_j)}\le C(D(\boldsymbol{\theta}_0),p,L_j).$$

Therefore, by Arzelà-Ascoli Theorem, possibly extracting a further subsequence we have that $\tilde\theta^j(t_n)\to\theta^j_\infty$ uniformly as $n\to\infty$. Notice also that $\tilde\theta^j(t_n)\rightharpoonup\theta^j_\infty$ weakly in $W^{1,p}(I_j)$, and that from the uniform bounds

$$\||\partial_s\tilde\theta^j(t_n)|^{p-2}\partial_s\tilde\theta^j(t_n)\|_{L^{\frac{p}{p-1}}(I_j)}\le C\quad\text{ and }\quad\|(|\partial_s\tilde\theta^j(t_n)|^{p-2}\partial_s\tilde\theta^j(t_n))_s\|_{L^2(I_j)}\le C$$

(which follows from (3.20), (1.5), (1.6), (1.7)) we infer a uniform bound in the H^1-norm giving that $|\partial_s\tilde\theta^j(t_n)|^{p-2}\partial_s\tilde\theta^j(t_n)\rightharpoonup|\partial_s\theta^j_\infty|^{p-2}\partial_s\theta^j_\infty$ weakly in $H^1(I_j)$ and uniformly on I_j. Since, if $\boldsymbol{\theta}$ solves (1.5), (1.6), (1.7), then so does $\boldsymbol{\theta}-(2\pi z^1,2\pi z^2,2\pi z^3)$ with $z^i\in\mathbb{Z}$ and with no change in the Lagrange multipliers, we can pass to the limit as $n\to\infty$ and obtain that $\boldsymbol{\theta}_\infty:=(\theta^1_\infty,\theta^2_\infty,\theta^3_\infty)$ satisfies the constraint (1.9). Moreover, passing to the limit in (1.4) we infer that $\boldsymbol{\theta}_\infty$ also fulfills the system (1.10), together with the boundary conditions (1.11). □

A Appendix

Proof of Lemma 3.5. We adapt to the present setting the arguments presented in [31, Lemma 3.11]. Let $j\in\{1,2,3\}$, and let $q:=p/(p-1)$. First of all notice that by Theorem 3.4 we have that $\theta^j_n(\cdot,t)\in W^{1,p}(I_j)$ for any $t\in[0,T]$, and

$$\text{(A1)}\qquad \sup_{t\in[\,0,T\,]}\|\theta^j_n(\cdot,t)\|_{L^2(I_j)}\le C(T,D(\boldsymbol{\theta}_0),\|\theta^j_0\|_{L^2(I_j)}),\qquad \sup_{t\in[\,0,T\,]}\|\partial_s\theta^j_n(\cdot,t)\|^p_{L^p(I_j)}\le p\,D(\boldsymbol{\theta}_0).$$

Therefore there exists a map $\theta^j\in L^\infty(0,T;W^{1,p}(I_j))$ such that, up to a subsequence,

$$\theta^j_n\rightharpoonup\theta^j\quad\text{weakly}*\text{ in }L^\infty(0,T;W^{1,p}(I_j))\qquad\text{as}\quad n\to\infty.$$

Next note that, since $\theta^j_n(s,\cdot)$ is absolutely continuous in $[\,0,T\,]$, we infer from Hölder's inequality that

$$\begin{aligned}\|\theta^j_n(\cdot,t_2)-\theta^j_n(\cdot,t_1)\|_{L^2(I_j)}&=\left(\int_{I_j}\left|\int_{t_1}^{t_2}\frac{\partial\theta^j_n}{\partial t}(s,\tau)\,d\tau\right|^2 ds\right)^{\frac12}\\&\le\left(\int_{t_1}^{t_2}\int_{I_j}\left|\frac{\partial\theta^j_n}{\partial t}(s,\tau)\right|^2 ds d\tau\right)^{\frac12}(t_2-t_1)^{\frac12},\end{aligned}$$

for any t_1, $t_2 \in [\,0,T\,]$ with $t_1 < t_2$. Using Theorem 3.4, we find that

$$\int_{t_1}^{t_2}\int_{I_j}\left|\frac{\partial\theta_n^j}{\partial t}(s,\tau)\right|^2 dsd\tau \le \int_{t_1}^{t_2}\int_{I_j}|\boldsymbol{V}_n(s,\tau)|^2\,dsd\tau \le 2D(\boldsymbol{\theta}_0). \tag{A2}$$

Hence we obtain

$$\|\theta_n^j(\cdot,t_2)-\theta_n^j(\cdot,t_1)\|_{L^2(I_j)} \le \sqrt{2D(\boldsymbol{\theta}_0)}(t_2-t_1)^{\frac{1}{2}}. \tag{A3}$$

We now turn to the proof of (3.14). First of all observe that by (A1) and embedding theory we have that

$$\sup_{t\in[0,T]}\|\theta_n^j(\cdot,t)\|_{L^\infty(I_j)} \le C, \qquad \text{with } C = C(L_j,T,D(\boldsymbol{\theta}_0),\|\theta_0^j\|_{L^2(I)},p). \tag{A4}$$

Moreover, again by (A1), for any $t \in [0,T]$ we have

$$|\theta_n^j(s_2,t)-\theta_n^j(s_1,t)| \le \int_{s_1}^{s_2}\left|\partial_s\theta_n^j(s,t)\right|ds \le C|s_2-s_1|^{1/q}, \tag{A5}$$

with $C = C(D(\boldsymbol{\theta}_0),p)$. Fix $0 \le t_1 \le t_2 \le T$ arbitrarily and set

$$\Gamma(\cdot) := \theta_n^j(\cdot,t_2)-\theta_n^j(\cdot,t_1) \in W^{1,p}(I_j).$$

By an interpolation inequality (see for instance [1, Thm. 5.9]) we find $\|\Gamma\|_{L^\infty} \le C\|\Gamma\|_{L^q}^{1/2}\|\Gamma\|_{W^{1,p}}^{1/2}$, and using (A1) we get $\|\Gamma\|_{L^\infty} \le C\|\Gamma\|_{L^q}^{1/2}$. In particular, for $p \ge 2$ we have $\|\Gamma\|_{L^\infty} \le C(L_j)\|\Gamma\|_{L^2}^{1/2}$, so that by (A3) we infer

$$\|\Gamma\|_{L^\infty} \le C|t_2-t_1|^{1/4}, \tag{A6}$$

with $C = C(T,L_j,D(\boldsymbol{\theta}_0),\|\theta_0^j\|_{L^2(I)},p)$. For $p \in (1,2)$, that is $q > 2$, another interpolation inequality gives $\|\Gamma\|_{L^q} \le \|\Gamma\|_{L^2}^{\theta}\|\Gamma\|_{L^\infty}^{1-\theta}$, with $\theta = 2/q$. Recalling that $\Gamma \in L^\infty$, we then obtain

$$\|\Gamma\|_{L^\infty} \le C\|\Gamma\|_{L^q}^{\frac{1}{2}} \le C\|\Gamma\|_{L^2}^{\frac{\theta}{2}}\|\Gamma\|_{L^\infty}^{\frac{1-\theta}{2}} \le C\|\Gamma\|_{L^2}^{\frac{\theta}{2}} = C\|\Gamma\|_{L^2}^{\frac{1}{q}} \le C(t_2-t_1)^{\frac{p-1}{2p}}. \tag{A7}$$

From (A6) and (A7) it follows that

$$\|\theta_n^j(t_2)-\theta_n^j(t_1)\|_{C^0(I_j)} \le C|t_2-t_1|^\alpha. \tag{A8}$$

From the above inequality and (A5) we then get

$$|\theta_n^j(s_2,t_2)-\theta_n^j(s_1,t_1)| \le C(|t_2-t_1|^\alpha+|s_2-s_1|^{1/q}) \le C(|t_2-t_1|^\alpha+|s_2-s_1|^\alpha) \tag{A9}$$

for any $(t_i,s_i) \in [0,T]\times[0,L_j]$, $i=1,2$. Application of the Arzelà-Ascoli Theorem yields (3.14). In particular $\theta^j(\cdot,t) \in W^{1,p}(I_j)$ for all times $t \in [0,T]$. Moreover, setting $t_1 = 0$ in (A8), we have that

$$\|\theta^j(t)-\theta_0^j\|_{C^0(I_j)} \to 0 \quad \text{as} \quad t \downarrow 0.$$

From (A2) we also infer that there exists $V^j \in L^2(0,T;L^2(I_j))$ such that

$$V_n^j = \partial_t\theta_n^j \rightharpoonup V^j \quad \text{in} \quad L^2(0,T;L^2(I_j)). \tag{A10}$$

Moreover, for any $v \in C_0^\infty((0,T)\times I_j)$, we have

$$\int_0^T \int_{I_j} \theta_n^j v_t \, dsdt = -\int_0^T \int_{I_j} V_n^j v \, dsdt \to -\int_0^T \int_{I_j} V^j v \, dsdt$$

as $n \to \infty$, and using the fact that $\theta_n^j \to \theta^j$ uniformly, we obtain

$$\int_0^T \int_{I_j} \theta_n^j v_t \, dsdt \to \int_0^T \int_{I_j} \theta^j v_t \, dsdt,$$

from which we infer that θ^j admits weak derivative $\theta_t^j = V^j$, $\theta^j \in H^1(0,T;L^2(I_j))$ and (3.13) holds. Finally, it follows from (A2) that (3.10) also holds. □

Proof of Lemma 3.7. This is a straight-forward adaptation of [31, Lemma 3.12]. Let $j \in \{1,2,3\}$. We show the proof only for $\bar{\theta}_n^j$, since analogous arguments holds for $\underline{\theta}_n^j$. Recalling (A1), we see that $\bar{\theta}_n^j \in L^\infty(0,T;W^{1,p}(I_j))$. In particular, equations (A4), (A5) hold with θ_n^j replaced by $\bar{\theta}_n^j$.

Fix now $t \in (\,0,T\,]$ arbitrarily. Then there exists a family of intervals $\{(\,(i_n-1)\tau_n, i_n\tau_n\,]\}_{n\in\mathbb{N}}$ such that $t \in (\,(i_n-1)\tau_n, i_n\tau_n\,]$. From (A8) we infer that

$$\begin{aligned}\|\bar{\theta}_n^j(t) - \theta_n^j(t)\|_{C^0(I_j)} &= \|\theta_{i_n,n}^j - \theta_n^j(t)\|_{C^0(I_j)} = \|\theta_n^j(i_n\tau_n) - \theta_n^j(t)\|_{C^0(I_j)} \\ &\le C|i_n\tau_n - t|^\alpha \le C\tau_n^\alpha \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Since $\theta_n^j \to \theta^j$ in $C^0([0,T]\times I_j)$ by Lemma 3.5, and t was arbitrarily chosen, we infer that $\bar{\theta}_n^j \to \theta^j$ in $C^0([0,T]\times I_j)$.

We turn to the proof of (3.16). Recalling again (A1), we also see that $\partial_s\bar{\theta}_n^j \in L^p(0,T;L^p(I_j))$ and $\|\partial_s\bar{\theta}_n^j\|_{L^p(0,T;L^p(I_j))} \le C$, for all $n \in \mathbb{N}$. Since $L^p(0,T;L^p(I_j))$ is a reflexive Banach space there exists $v^j \in L^p(0,T;L^p(I_j))$ such that $\partial_s\bar{\theta}_n^j \rightharpoonup v^j$. This implies that

$$\int_0^T \int_{I_j} \partial_s\bar{\theta}_n^j \cdot \varphi \, dsdt \to \int_0^T \int_{I_j} v^j \cdot \varphi \, dsdt$$

for any $\varphi \in L^q(0,T;L^q(I_j))$, with $q = p/(p-1)$. On the other hand, if $\varphi \in L^\infty(0,T;C_0^\infty(I_j))$ we infer that

$$\int_0^T \int_{I_j} \partial_s\bar{\theta}_n^j \cdot \varphi \, dsdt = -\int_0^T \int_{I_j} \bar{\theta}_n^j \cdot \partial_s\varphi \, dsdt \to -\int_0^T \int_{I_j} \theta^j \cdot \partial_s\varphi \, dsdt,$$

where we have used that $\bar{\theta}_n^j \to \theta^j$. Hence we obtain that $v^j = \partial_s\theta^j$, and the claim follows. □

Proof of Lemma 3.8. This is a straight-forward adaptation of [31, Lemma 3.13]. Notice first that, from (3.6), (3.5), and Theorem 3.4 it follows that

$$\||\partial_s\theta_{i,n}^j|^{p-2}\partial_s\theta_{i,n}^j\|_{H^1(I_j)} \le C(1 + \sum_{r=1}^3 \|V_{i,n}^r\|_{L^2(I_j)}) \tag{A11}$$

for $j = 1,2,3$ and for all $i = 1,\dots,n$, where $C = C(L_j, p, D(\boldsymbol{\theta}_0))$. Recalling Theorem 3.4, for all $j = 1,2,3$ we get

$$\int_0^T \||\partial_s\bar{\theta}_n^j|^{p-2}\partial_s\bar{\theta}_n^j\|_{H^1(I_j)}^2 \, dt \le C\int_0^T (1 + \|\boldsymbol{V}_n\|_{L^2(I_j)}^2) \, dt \le C. \tag{A12}$$

Thus, recalling also (3.7), we find $w^j \in L^2(0,T;H^1_0(I_j))$ such that

$$|\partial_s \bar\theta^j_n|^{p-2}\partial_s\bar\theta^j_n \rightharpoonup w^j \quad \text{in} \quad L^2(0,T;H^1(I_j)) \quad \text{as} \quad n\to\infty, \tag{A13}$$

and $\|w^j\|_{L^2(0,T;H^1(I_j))} \le C$. In particular, this implies that

$$\int_0^T\int_{I_j} |\partial_s\bar\theta^j_n|^{p-2}\partial_s\bar\theta^j_n\cdot\varphi\,dsdt \to \int_0^T\int_{I_j} w^j\cdot\varphi\,dsdt, \tag{A14}$$

for all $\varphi\in L^2(0,T;L^2(I_j))$. On the other hand, letting $q:=p/(p-1)$, from (A11) it follows that

$$\||(\theta^j_{i,n})_s|^{p-2}(\theta^j_{i,n})_s\|_{L^q(I_j)} \le C\||(\theta^j_{i,n})_s|^{p-2}(\theta^j_{i,n})_s\|_{H^1(I_j)} \le C(1+\sum_{r=1}^3\|V^r_{i,n}\|_{L^2(I_j)}),$$

so that, by Theorem 3.4 we also get

$$\int_0^T \||\partial_s\bar\theta^j_n|^{p-2}\partial_s\bar\theta^j_n\|^2_{L^q(I_j)}\,dt \le C\int_0^T(1+\|\boldsymbol{V}_n\|^2_{L^2(I_j)})\,dt \le C.$$

The space $L^2(0,T;L^q(I_j))$ is reflexive with dual space given by $(L^2(0,T;L^q(I_j)))^* = L^2(0,T;L^p(I_j))$. Hence there exists $\tilde\xi^j\in L^2(0,T;L^q(I_j))$ such that

$$\int_0^T\int_{I_j} |\partial_s\bar\theta^j_n|^{p-2}\partial_s\bar\theta^j_n\cdot\varphi\,dsdt \to \int_0^T\int_{I_j}\tilde\xi^j\cdot\varphi\,dsdt \quad \forall\varphi\in L^2(0,T;L^p(I_j)). \tag{A15}$$

Together with (A14) and Lemma 3.7, we infer that $w^j=\tilde\xi^j$.

Next, we set

$$F(\psi) := \frac{1}{p}\,\|\psi_s\|^p_{L^p(0,T;L^p(I_j))}.$$

Using the convexity of the map $y\to\frac{1}{p}|y|^p$, we see that

$$F(\psi)-F(\bar\theta^j_n) \ge \int_0^T\int_{I_j}|\partial_s\bar\theta^j_n|^{p-2}\partial_s\bar\theta^j_n\cdot(\psi-\bar\theta^j_n)_s\,dsdt \quad \text{for any } \psi\in L^\infty(0,T;W^{1,p}(I_j)). \tag{A16}$$

Recalling (3.16) and letting $n\to\infty$ in (A16), we have

$$F(\psi)-F(\theta^j) \ge \int_0^T\int_{I_j} w^j\cdot(\psi-\theta^j)_s\,dsdt, \tag{A17}$$

where we have used integration by parts (recall (3.7) and $w^j\in L^2(0,T;H^1_0(I_j))$) and (3.15). Letting now $\psi=\theta^j+\varepsilon\varphi$ in (A17) for some $\varphi\in L^\infty(0,T;W^{1,p}(I_j))$ and $\varepsilon>0$, we obtain

$$\frac{F(\theta^j+\varepsilon\varphi)-F(\theta^j)}{\varepsilon} \ge \int_0^T\int_{I_j} w^j\cdot\varphi_s\,dsdt. \tag{A18}$$

On the other hand, letting $\psi=w^j-\varepsilon\varphi$ in (A17), we also have

$$\frac{F(\theta^j)-F(\theta^j-\varepsilon\varphi)}{\varepsilon} \le \int_0^T\int_{I_j} w^j\cdot\varphi_s\,dsdt. \tag{A19}$$

Letting $\varepsilon \downarrow 0$, from (A18) and (A19) we then get

$$\int_0^T \int_{I_j} |\partial_s \theta^j|^{p-2} \theta^j_s \cdot \varphi_s \, ds dt = \int_0^T \int_{I_j} w^j \cdot \varphi_s \, ds dt = \int_0^T \int_{I_j} \tilde{\xi}^j \cdot \varphi_s \, ds dt, \tag{A20}$$

for all $\varphi \in L^\infty(0,T;W^{1,p}(I_j))$. Together with (A15) this gives the thesis. □

Lemma A.1. *Suppose $\theta = \theta(s,t) \in C^{0,\alpha}([0,T]\times\bar{I})$ for some $\alpha \in (0,1]$. Then osc $\theta \in C^\alpha([0,T])$.*

Proof. By definition we have osc $\theta(t) = \max_{s\in\bar{I}} \theta(s,t) - \min_{s\in\bar{I}} \theta(s,t) =: \theta(\bar{s},t) - \theta(\underline{s},t)$. Using this notation it follows

$$\begin{aligned} \text{osc } \theta(t_1) - \text{osc } \theta(t_2) &= [\theta(\bar{s}_1,t_1) - \theta(\underline{s}_1,t_1)] - [\theta(\bar{s}_2,t_2) - \theta(\underline{s}_2,t_2)] \\ &\leq \theta(\bar{s}_1,t_1) - \theta(\bar{s}_1,t_2) + \theta(\underline{s}_1,t_2) - \theta(\underline{s}_1,t_1) \leq C|t_1-t_2|^\alpha, \end{aligned}$$

where we have used $\theta(\bar{s}_2,t_2) \geq \theta(\bar{s}_1,t_2)$ and $\theta(\underline{s}_2,t_2) \leq \theta(\underline{s}_1,t_2)$. The claim follows. □

References

[1] Adams, R. A., and Fournier, J. J. F. *Sobolev spaces*, second ed., vol. 140 of *Pure and Applied Mathematics*. Elsevier/Academic Press, Amsterdam, 2003.

[2] Almgren, F., Taylor, J. E., and Wang, L. Curvature-driven flows: a variational approach. *SIAM J. Control Optim. 31*, 2 (1993), 387–438.

[3] Badal, R. Curve-shortening flow of open, elastic curves in $\mathbb{R}^2$ with repelling endpoints: A minimizing movement approach. *Preprint* (2019).

[4] Bellettini, G., and Kholmatov, S. Minimizing movements for mean curvature flow of partitions. *Preprint* (2017).

[5] Bronsard, L., and Reitich, F. On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation. *Arch. Rational Mech. Anal. 124*, 4 (1993), 355–379.

[6] Dall'Acqua, A., Lin, C.-C., and Pozzi, P. Evolution of open elastic curves in $\mathbb{R}^n$ subject to fixed length and natural boundary conditions. *Analysis (Berlin) 34*, 2 (2014), 209–222.

[7] Dall'Acqua, A., Lin, C.-C., and Pozzi, P. A gradient flow for open elastic curves with fixed length and clamped ends. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17*, 3 (2017), 1031–1066.

[8] Dall'Acqua, A., Lin, C.-C., and Pozzi, P. Flow of elastic networks: long-time existence result. *Preprint* (2018).

[9] Dall'Acqua, A., Novaga, M., and Pluda, A. Minimal elastic networks. *Indiana Univ. Math. J* (to appear).

[10] Dall'Acqua, A., and Pozzi, P. A Willmore-Helfrich L^2-flow of curves with natural boundary conditions. *Comm. Anal. Geom. 22*, 4 (2014), 617–669.

[11] Dall'Acqua, A., Pozzi, P., and Spener, A. The Ł ojasiewicz-Simon gradient inequality for open elastic curves. *J. Differential Equations 261*, 3 (2016), 2168–2209.

[12] Dziuk, G., Kuwert, E., and Schätzle, R. Evolution of elastic curves in $\mathbb{R}^n$: existence and computation. *SIAM J. Math. Anal. 33*, 5 (2002), 1228–1245 (electronic).

[13] Fonseca, I., Fusco, N., Leoni, G., and Morini, M. Motion of elastic thin films by anisotropic surface diffusion with curvature regularization. *Arch. Ration. Mech. Anal. 205*, 2 (2012), 425–466.

[14] Garcke, H., Menzel, J., and Pluda, A. Long time existence of solutions to an elastic flow of networks. *Preprint* (2019).

[15] Garcke, H., Menzel, J., and Pluda, A. Willmore flow of planar networks. *J. Differential Equations 266*, 4 (2019), 2019–2051.

[16] Koiso, N. On the motion of a curve towards elastica. In *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, vol. 1 of *Sémin. Congr.* Soc. Math. France, Paris, 1996, pp. 403–436.

[17] Langer, J., and Singer, D. A. Curve straightening and a minimax argument for closed elastic curves. *Topology 24*, 1 (1985), 75–88.

[18] Lin, C.-C. L^2-flow of elastic curves with clamped boundary conditions. *J. Differential Equations 252*, 12 (2012), 6414–6428.

[19] Lin, C.-C., Lue, Y.-K., and Schwetlick, H. R. The second-order L^2-flow of inextensible elastic curves with hinged ends in the plane. *J. Elasticity 119*, 1-2 (2015), 263–291.

[20] Luckhaus, S., and Sturzenhecker, T. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations 3*, 2 (1995), 253–271.

[21] Magni, A., Mantegazza, C., and Novaga, M. Motion by curvature of planar networks, II. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 15* (2016), 117–144.

[22] Mantegazza, C., Novaga, M., and Pluda, A. Motion by curvature of networks with two triple junctions. *Geom. Flows 2* (2017), 18–48.

[23] Mantegazza, C., Novaga, M., Pluda, A., and Schulze, F. Evolution of networks with multiple junctions. *Preprint* (2016).

[24] Mantegazza, C., Novaga, M., and Tortorelli, V. M. Motion by curvature of planar networks. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3*, 2 (2004), 235–324.

[25] Novaga, M., and Okabe, S. Regularity of the obstacle problem for the parabolic biharmonic equation. *Math. Ann. 363*, 3-4 (2015), 1147–1186.

[26] Novaga, M., and Okabe, S. Convergence to equilibrium of gradient flows defined on planar curves. *J. Reine Angew. Math. 733* (2017), 87–119.

[27] Öelz, D. Convergence of the penalty method applied to a constrained curve straightening flow. *Commun. Math. Sci. 12*, 4 (2014), 601–621.

[28] Öelz, D. B. On the curve straightening flow of inextensible, open, planar curves. *SeMA J.*, 54 (2011), 5–24.

[29] Okabe, S. The motion of elastic planar closed curves under the area-preserving condition. *Indiana Univ. Math. J. 56*, 4 (2007), 1871–1912.

[30] Okabe, S. The dynamics of elastic closed curves under uniform high pressure. *Calc. Var. 33*, 4 (2008), 493–521.

[31] Okabe, S., Pozzi, P., and Wheeler, G. A gradient flow for the p-elastic energy defined on closed planar curves. *Preprint* (2018).

[32] Polden, A. Curves and surfaces of least total curvature and fourth-order flows. *PhD Thesis, Universität Tübingen* (1996).

[33] Wen, Y. L^2 flow of curve straightening in the plane. *Duke Math. J. 70*, 3 (1993), 683–698.

[34] Wen, Y. Curve straightening flow deforms closed plane curves with nonzero rotation number to circles. *J. Differential Equations 120*, 1 (1995), 89–107.

[35] Wheeler, G. Global analysis of the generalised Helfrich flow of closed curves immersed in $\mathbb{R}^n$. *Trans. Amer. Math. Soc. 367*, 4 (2015), 2263–2300.