Bump solutions
for the mesoscopic Allen-Cahn equation
in periodic media

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Abstract

Given a double-well potential F, a $\mathbb{Z}^n$-periodic function H, small and with zero average, and $\varepsilon > 0$, we find a large R, a small δ and a function H_ε which is ε-close to H for which the following two problems have solutions:

1. Find a set $E_{\varepsilon,R}$ whose boundary is uniformly close to ∂B_R and has mean curvature equal to $-H_\varepsilon$ at any point,
2. Find $u = u_{\varepsilon,R,\delta}$ solving
$$-\delta\,\Delta u + \frac{F'(u)}{\delta} + \frac{c_0}{2} H_\varepsilon = 0,$$
such that $u_{\varepsilon,R,\delta}$ goes from a δ-neighborhood of $+1$ in B_R to a δ-neighborhood of -1 outside B_R.

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1 Introduction and main result

In this paper we deal with entire solutions of the elliptic equation

$$-\Delta u + F'(u) + H(x) = 0, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where the smooth function F is a double-well potential.
More precisely, we shall assume that

- $F(t) \geq 0$ for any $t \in \mathbb{R}$,
- $F(t) = 0$ if and only if $t = \pm 1$,
- there exist positive constants s_0 and c such that $F'(-1-s) \leq -c$ and $F'(1+s) \geq c$ for any $s \geq s_0$,
- $F(-1+s) = F(1+s)$ and $F''(1+s) > 0$ for any $s \in [-s_0, s_0]$.

The function $H \in C^2(\mathbb{R}^n)$ in (1.1) will be a small periodic perturbation of the operator. To this extent, we suppose that

- $\|H\|_{L^\infty(\mathbb{R}^n)}$ is suitably small,

- H is $\mathbb{Z}^n$-periodic, with zero average on $[0,1]^n$, that is

$$\begin{aligned} H(x+k) &= H(x) \qquad \forall x \in \mathbb{R}^n \text{ and } k \in \mathbb{Z}^n \\ \text{and } \int_{[0,1]^n} H(x)\,dx &= 0\,. \end{aligned} \tag{1.2}$$

Notice that (1.1) is the Euler-Lagrange equation of the functional

$$\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{2} + F(u) + H(x)u \; dx. \tag{1.3}$$

The functional in (1.3) has been considered in [DLN06, NV07, NV09a] as a mesoscopic model for phase transitions (see also [DY06, DYC08] for the analysis of the gradient flow of (1.3)).
When $H = 0$, (1.1) is called the Ginzburg-Landau or Allen-Cahn equation, which is a popular model for superconductors and superfluids. The term H may be seen as a small defect which favors locally one of the phases: condition (1.2) then says that such defect is "neutral" on large scales, in the sense that both the phases are equally treated.
In [NV07], minimizers of (1.3) have been dealt with. We say that $u \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ is a minimizer if

$$\begin{aligned} &\int_U \frac{|\nabla u|^2}{2} + F(u) + H(x)u \; dx \\ &\qquad \le \int_U \frac{|\nabla (u+\psi)|^2}{2} + F(u+\psi) + H(x)(u+\psi) \; dx \end{aligned} \tag{1.4}$$

for any $\psi \in C^\infty_0(U)$ and any bounded domain U (minimizers of this type are often called "local", or "class A", minimizers). As usual in the calculus of variation framework, the word minimizer for (1.4) refers to the fact that the energy is increased by compact perturbations, even if the energy (1.3) in the whole of $\mathbb{R}^n$ may well be infinite.
In particular, the following result has been proved in [NV07].

Theorem 1.1. *Let F and H be as above. Then, there exist two $\mathbb{Z}^n$-periodic minimizers $U^\pm$ of (1.3), with $U^+ = U^- + 2$. The uniform distance of the minimizers $U^\pm$ from ± 1, respectively, can be estimated in terms of the norm $\|H\|_{L^\infty(\mathbb{T}^n)}$.*
Moreover, given $\omega \in S^{n-1}$, there exists a minimizer u_ω of (1.3) which connects U^+ and U^- far from $\omega^\perp$, and such that

$$\{x \in \mathbb{R}^n : \langle \omega, x\rangle \le 0\} \subseteq \{x \in \mathbb{R}^n : u_\omega(x) > 0\} \subseteq \{x \in \mathbb{R}^n : \langle \omega, x\rangle \le C\}\,. \tag{1.5}$$

for a constant $C > 0$ independent of ω.

We also recall the following result on minimal surfaces in periodic media, which has been proved in [CdlL01] (see, in particular, Section 11.1 there).

Theorem 1.2. *Let H as above. Then, one can find a uniform constant $C > 0$ such that for all $\omega \in S^{n-1}$ there exists a local minimizer (i.e. a minimizer up to compact perturbations) E_ω of the functional*

$$P(E) + \int_E H(x)\,dx\,, \tag{1.6}$$

such that

$$\{x \in \mathbb{R}^n : \langle \omega, x\rangle \le 0\} \subseteq E_\omega \subseteq \{x \in \mathbb{R}^n : \langle \omega, x\rangle \le C\}\,. \tag{1.7}$$

The analogy between (1.5) and (1.7) is evident. We refer to [SZ97, NV09b] for further related results.
Given a function F as above, we let $\gamma : \mathbb{R} \to \mathbb{R}$ be the unique solution of

$$-\gamma'' + F'(\gamma) = 0, \tag{1.8}$$

connecting ± 1 at $\pm\infty$, respectively, and such that $\gamma(0) = 0$.
We also let

$$c_0 := \int_{\mathbb{R}} \big(\gamma'(x)\big)^2 \, dx \, . \tag{1.9}$$

The main result of this paper is the following:

Theorem 1.3. *Let F and H be as above, and assume that $H \not\equiv 0$. Then, for any $\varepsilon > 0$ there exist constants $R_0, \delta_0, C > 0$ and a function $H_\varepsilon \in C^\infty(\mathbb{T}^n)$, with*

$$\begin{aligned} \|H_\varepsilon\|_{L^\infty(\mathbb{T}^n)} &\leq \|H\|_{L^\infty(\mathbb{T}^n)} && (1.10)\\ \|H_\varepsilon - H\|_{L^1(\mathbb{T}^n)} &\leq 4\,\varepsilon\,\|H\|_{L^\infty(\mathbb{T}^n)}, && (1.11)\\ \int_{\mathbb{T}^n} H_\varepsilon(x)\,dx &= 0, && (1.12) \end{aligned}$$

such that the following holds:

1. *for any $R > R_0$ there exists a set $E_{\varepsilon,R}$ with smooth compact boundary such that*

$$B_R \subset E_{\varepsilon,R} \subset B_{R+C}\,, \tag{1.13}$$

and the mean curvature of $\partial E_{\varepsilon,R}$ agrees with $-H_\varepsilon$ at any point;

2. *for any $R > R_0$ and $\delta \in (0, \delta_0]$ there exists a function $u_{\varepsilon,R,\delta}$ of*

$$-\delta\,\Delta u + \frac{F'(u)}{\delta} + \frac{c_0}{2} H_\varepsilon = 0, \qquad x \in \mathbb{R}^n, \tag{1.14}$$

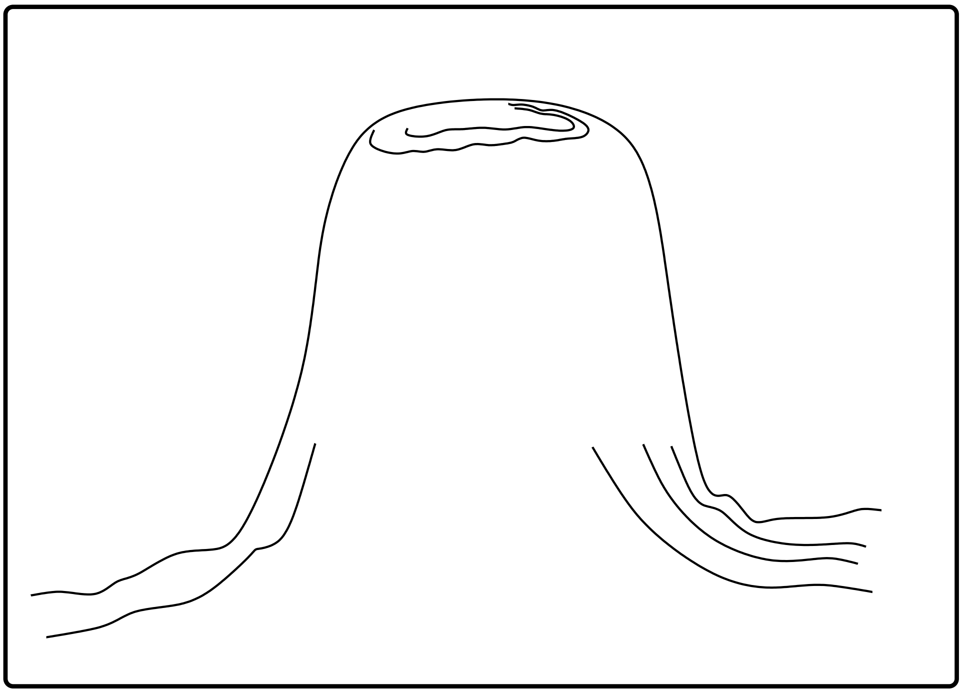
such that

$$\begin{aligned} &|u_{\varepsilon,R,\delta}(x) - 1| \leq C\delta \text{ on } B_R,\\ &\text{and } |u_{\varepsilon,R,\delta}(x) + 1| \leq C\delta \text{ on } \mathbb{R}^n \setminus B_{R+C}. \end{aligned} \tag{1.15}$$

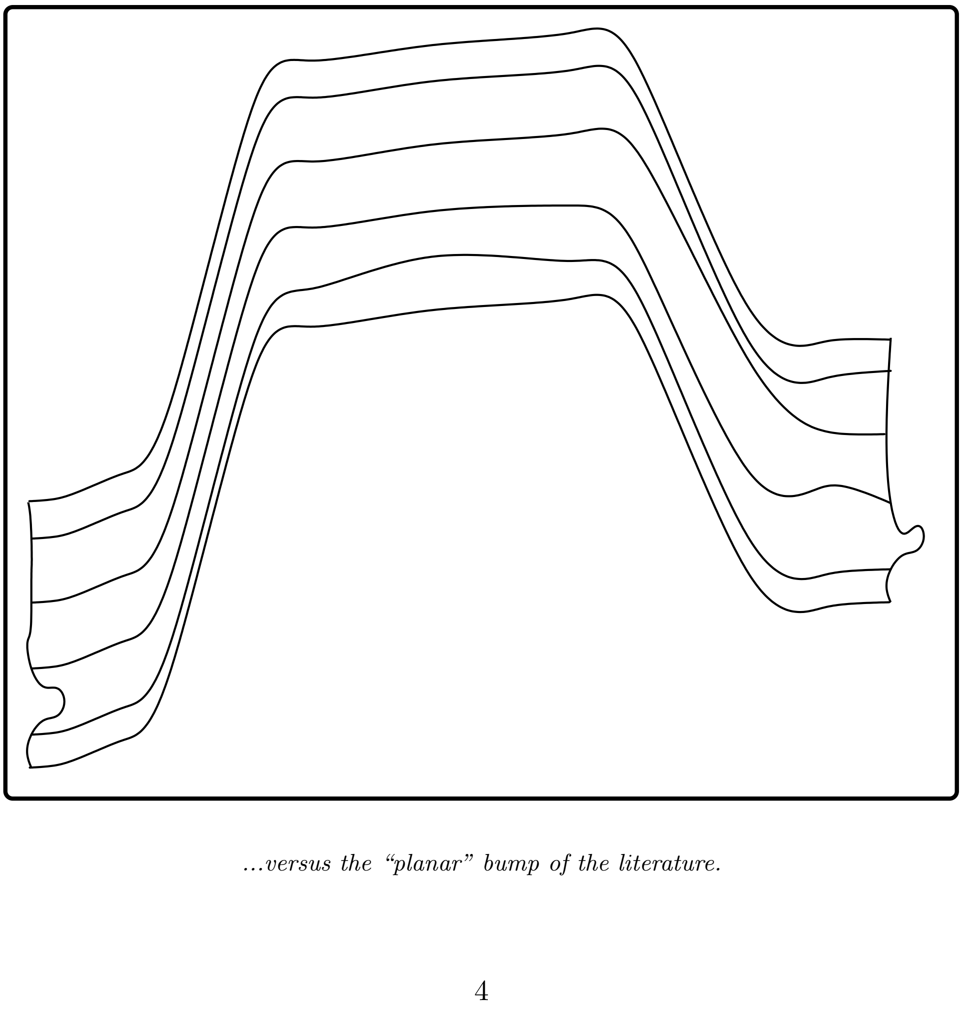
The factor $c_0/2$ in (1.14) is, of course, just a normalization constant: roughly speaking, it is needed to make (1.14) approach the prescribed mean curvature problem equal to $-H_\varepsilon$, and not a constant multiple of it.
We observe that Theorem 1.3 does not hold, in general, if we choose $H_\varepsilon := H$. However, it would be interesting to know:

- whether an analogous result holds if we replace the L^1 norm in (1.11) with a stronger one (e.g., the L^∞ norm),
- under which conditions on H it would be possible to choose $H_\varepsilon := H$ in Theorem 1.3,
- whether or not the results in Theorem 1.3 hold for $\delta = 1$.

Remark 1.1. The geometry of the solution $u_{\varepsilon,R,\delta}$ found in Theorem 1.3 is very different from the bump solutions usually obtained in the literature (see [RS01, RS04, RS08] for the Allen-Cahn case and [NV09a] for the mesoscopic case). Indeed the bump constructed in the previous literature was a somewhat "planar" oscillation from almost -1 to almost $+1$ and return. In Theorem 1.3 such an oscillation is not somewhat "planar", but somewhat "spherical". That is, the set $\{u_{\varepsilon,R,\delta} = 0\}$ is somewhat close to a sphere (compare (1.13) and (1.15)).
We refer to [AJM02, B05, dlLlV07] for other related results.



The "spherical" bump of Theorem 1.3...



...versus the "planar" bump of the literature.

With essentially the same proof one can get the following extension of Theorem 1.3, showing the existence of "multibump" solutions of (2.12) and (1.14). For this, it is enough to modify the barriers by appropriately repeating their bumps (notice that $v_R^{\pm}$ are constants outside a ball).
To more precisely state the result analogous to Theorem 1.3 for multibump solutions, we introduce some notation. Given R, $C > 0$, we let $\mathcal{F}_{R,C}$ be the family of subsets of $\mathbb{R}^n$ defined as follows: $E \in \mathcal{F}_{R,C}$ if and only if there exist a sequence of points $x_i \in \mathbb{R}^n$ and a sequence of numbers $R_i \geq R$, with $i \in I \subseteq \mathbb{N}$, such that $|x_i^j - x_k^j| \geq R_i + R_k + C$ for all $i, k \in I$ and for all $j \in \{1, \dots, n\}$, and

$$E = \bigcup_{i \in I} B_{R_i}(x_i).$$

We also define

$$E \oplus B_C := \bigcup_{x \in E} B_C(x).$$

Then, in analogy with Theorem 1.3, we have the following result of multibump type:

Theorem 1.4. *Let F and H be as above.*
Then, for any $\varepsilon > 0$ there exist constants $R_0, \delta_0, C > 0$ and a function $H_\varepsilon \in C^\infty(\mathbb{T}^n)$ satisfying (1.10)–(1.12) *such that for any $E \in \mathcal{F}_{R_0,C}$:*

1. *there exists a smooth set E_ε such that*

$$E \subset E_\varepsilon \subset E \oplus B_C\,,$$

and the mean curvature of ∂E_ε agrees with $-H_\varepsilon$ at any point;

2. *for any $\delta < \delta_0$ there exists a solution $u_{\varepsilon,\delta}$ of* (1.14) *such that*

$$\begin{aligned} &|u_{\varepsilon,\delta}(x) - 1| \leq C\delta \text{ on } E, \\ &\text{and } |u_{\varepsilon,\delta}(x) + 1| \leq C\delta \text{ on } \mathbb{R}^n \setminus (E \oplus B_C). \end{aligned}$$

Once more, the geometry of the multibump obtained here is quite different from the multibumps of the previous literature, since the excursions obtained in Theorem 1.4 are somewhat "spherical" instead of somewhat "planar" (recall Remark 1.1).

2 Proof of Theorem 1.3

Step 1. We prove the first statement of Theorem 1.3. Since H does not vanish identically and it has zero average,

$$-\inf_{\mathbb{T}^n} H > 0.$$

Thus, we may fix $\hat{R} > 0$ large enough such that

$$\frac{n}{\hat{R}} \leq -\frac{1}{2} \inf_{\mathbb{T}^n} H. \tag{2.1}$$

Let

$$U := B_{\hat{R}+c\varepsilon} \setminus B_{\hat{R}-c\varepsilon}. \tag{2.2}$$

We denote by $\pi : \mathbb{R}^n \to \mathbb{T}^n$ the natural projection, we let $U_\varepsilon^{(1)} := \pi(U)$, and we choose the constant c in such a way that $|U_\varepsilon^{(1)}| = \varepsilon/4$. Of course, "$|\cdot|$" is here the Lebesgue measure on $\mathbb{T}^n$.

We let $U_\varepsilon^{(2)} \subset \mathbb{T}^n$ be an open set such that $\overline{U_\varepsilon^{(1)}} \subset U_\varepsilon^{(2)}$ and $|U_\varepsilon^{(2)}| = \varepsilon/2$, and we take a partition of unity $\psi_\varepsilon \in C^\infty(\mathbb{T}^n, [0,1])$, with $\psi_\varepsilon \equiv 1$ on $U_\varepsilon^{(1)}$ and $\psi_\varepsilon \equiv 0$ outside $U_\varepsilon^{(2)}$.
Let also ρ_ε be a convolution kernel such that $\|H - H * \rho_\varepsilon\|_{L^1(\mathbb{T}^n)} \le \varepsilon \|H\|_{L^\infty(\mathbb{T}^n)}$.
Finally, we let

$$\begin{aligned} K_\varepsilon^\star(x) &:= \begin{cases} -\dfrac{n}{\hat{R}} & \text{if } x \in U_\varepsilon^{(1)} \\ -\psi_\varepsilon(x)\,\dfrac{n}{\hat{R}} + \big(1-\psi_\varepsilon(x)\big)(1-\varepsilon)(H*\rho_\varepsilon)(x) & \text{if } x \in U_\varepsilon^{(2)} \setminus U_\varepsilon^{(1)} \\ (1-\varepsilon)(H*\rho_\varepsilon)(x) & \text{if } x \in \mathbb{T}^n \setminus U_\varepsilon^{(2)} \end{cases} \\ \alpha_\varepsilon &:= \int_{\mathbb{T}^n} K_\varepsilon^\star(y)\,dy \\ H_\varepsilon(x) &:= K_\varepsilon^\star(x) - \alpha_\varepsilon, \end{aligned} \tag{2.3}$$

and we extend H_ε by periodicity to the whole of $\mathbb{R}^n$. We claim that

$$|\alpha_\varepsilon| \le \varepsilon \|H\|_{L^\infty(\mathbb{T}^n)}. \tag{2.4}$$

In order to prove (2.4), we first observe that

$$\|H * \rho_\varepsilon\|_{L^\infty(\mathbb{T}^n)} \le \|H\|_{L^\infty(\mathbb{T}^n)}. \tag{2.5}$$

Also, in $U_\varepsilon^{(1)}$,

$$|K_\varepsilon^\star| = \frac{n}{\hat{R}} \le \frac{\|H\|_{L^\infty(\mathbb{T}^n)}}{2}, \tag{2.6}$$

due to (2.1).
Thus, we deduce from (2.5) and (2.6) that

$$\Big| \int_{U_\varepsilon^{(1)}} K_\varepsilon^\star - H * \rho_\varepsilon \Big| \le \frac{3\,\|H\|_{L^\infty(\mathbb{T}^n)}}{2}\,|U_\varepsilon^{(1)}| = \frac{3\varepsilon\,\|H\|_{L^\infty(\mathbb{T}^n)}}{8}. \tag{2.7}$$

Moreover, in $U_\varepsilon^{(2)} \setminus U_\varepsilon^{(1)}$, we have that

$$K_\varepsilon^\star - H * \rho_\varepsilon = -\psi_\varepsilon \left(\frac{n}{\hat{R}} - (H*\rho_\varepsilon) \right) - \varepsilon\big(1-\psi_\varepsilon\big)(H*\rho_\varepsilon).$$

Therefore, making use of (2.1) and (2.5) once more, we see that, in $U_\varepsilon^{(2)} \setminus U_\varepsilon^{(1)}$,

$$\begin{aligned} |K_\varepsilon^\star - H*\rho_\varepsilon| &\le \psi_\varepsilon \left(\frac{n}{\hat{R}} + |H*\rho_\varepsilon| \right) + \varepsilon\big(1-\psi_\varepsilon\big)|H*\rho_\varepsilon| \\ &\le \psi_\varepsilon \left(\frac{\|H\|_{L^\infty(\mathbb{T}^n)}}{2} + \|H\|_{L^\infty(\mathbb{T}^n)} \right) + \varepsilon\big(1-\psi_\varepsilon\big)\|H\|_{L^\infty(\mathbb{T}^n)} \\ &\le \left(\frac{\|H\|_{L^\infty(\mathbb{T}^n)}}{2} + \|H\|_{L^\infty(\mathbb{T}^n)} \right) + \varepsilon\|H\|_{L^\infty(\mathbb{T}^n)} \\ &= \left(\frac{3}{2} + \varepsilon \right) \|H\|_{L^\infty(\mathbb{T}^n)}. \end{aligned}$$

Consequently,

$$\Big| \int_{U_\varepsilon^{(2)} \setminus U_\varepsilon^{(1)}} K_\varepsilon^\star - H*\rho_\varepsilon \Big| \le \frac{3\,\|H\|_{L^\infty(\mathbb{T}^n)}}{2}\,\big|U_\varepsilon^{(2)} \setminus U_\varepsilon^{(1)}\big| = \left(\frac{3}{8} + \varepsilon \right) \varepsilon\,\|H\|_{L^\infty(\mathbb{T}^n)}. \tag{2.8}$$

Also, from (1.2), we have that

$$\int_{\mathbb{T}^n} H * \rho_\varepsilon \, dx \;=\; 0. \tag{2.9}$$

Thus, since

$$K_\varepsilon^\star - H * \rho_\varepsilon = -\varepsilon H * \rho_\varepsilon \text{ in } \mathbb{T}^n \setminus U_\varepsilon^{(2)},$$

we obtain from (2.9) and (2.5) that

$$\begin{aligned} \Big| \int_{\mathbb{T}^n \setminus U_\varepsilon^{(2)}} K_\varepsilon^\star - H * \rho_\varepsilon \Big| &= \varepsilon \Big| \int_{\mathbb{T}^n \setminus U_\varepsilon^{(2)}} H * \rho_\varepsilon \Big| = \varepsilon \Big| \int_{U_\varepsilon^{(2)}} H * \rho_\varepsilon \Big| \\ &\le \varepsilon \, \|H * \rho_\varepsilon\|_{L^\infty(\mathbb{T}^n)} \, |U_\varepsilon^{(2)}| \le \frac{\varepsilon^2 \, \|H\|_{L^\infty(\mathbb{T}^n)}}{2}. \end{aligned} \tag{2.10}$$

By collecting the results of (2.7), (2.8) and (2.10), we obtain that

$$\Big| \int_{\mathbb{T}^n} K_\varepsilon^\star - H * \rho_\varepsilon \Big| \le \varepsilon \|H\|_{L^\infty(\mathbb{T}^n)},$$

and so, using (2.9) once more,

$$\Big| \int_{\mathbb{T}^n} K_\varepsilon^\star \Big| \le \varepsilon \|H\|_{L^\infty(\mathbb{T}^n)}.$$

This proves (2.4).

As a consequence, we see that condition (1.10) follows immediately from (2.1) and (2.4), condition (1.12) is automatically satisfied, and condition (1.11) comes from (2.3) and (2.4).

Notice that, since

$$H_\varepsilon \equiv -n/\hat{R} - \alpha_\varepsilon \text{ on } U \tag{2.11}$$

from (2.4) we get that the mean curvature of ∂B_R, that is $(n-1)/R$, is strictly less than $-H_\varepsilon$, for all $R \in (\hat{R} - c\varepsilon, \hat{R} + c\varepsilon)$, so that ∂B_R is a strict subsolution of the geometric equation

$$\kappa + H_\varepsilon = 0, \tag{2.12}$$

where κ denotes the mean curvature of ∂B_R. We recall that (2.12) is the Euler-Lagrange equation of the functional

$$P(E) + \int_E H_\varepsilon(x)\, dx\,. \tag{2.13}$$

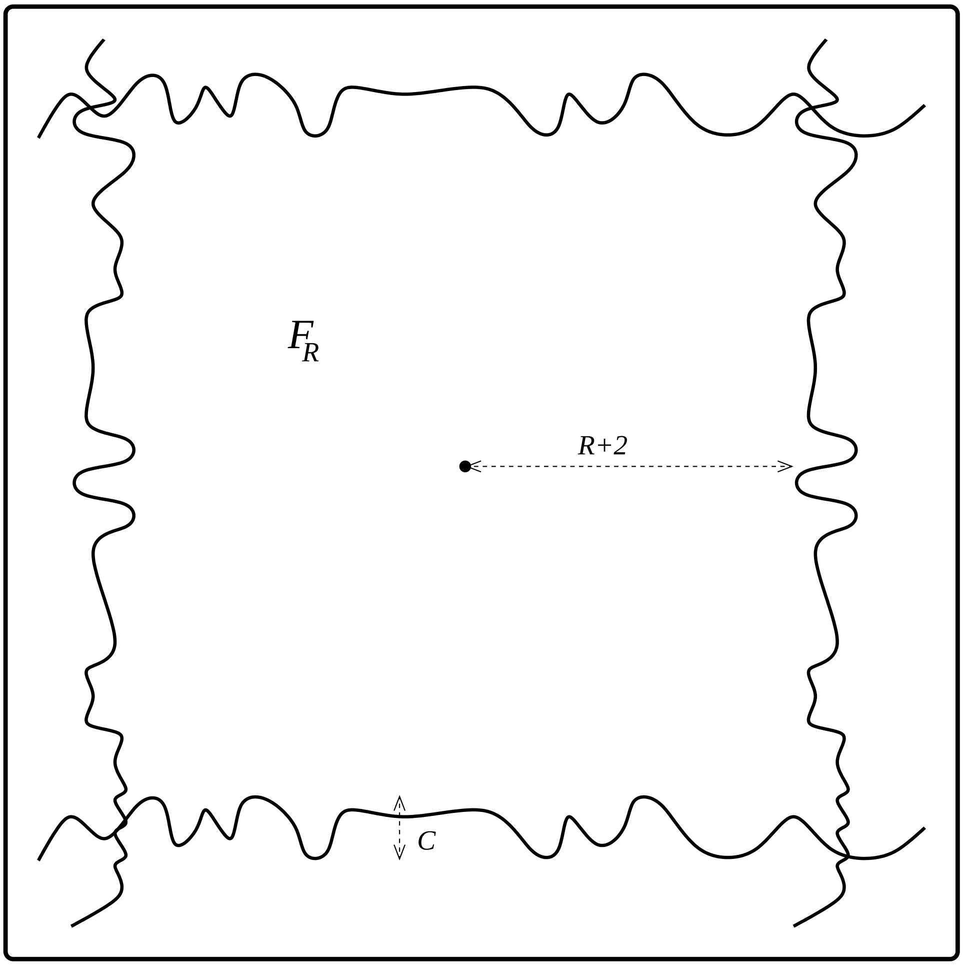
Fix now $R \ge \hat{R}$ and let

$$\begin{aligned} E_R \;&:=\; \bigcup_{z \in \mathbb{Z}^n : \, z + B_{\hat{R}} \subset B_{R+1}} \big(z + B_{\hat{R}}\big) \\ F_R \;&:=\; \bigcap_{\omega \in S^{n-1} \cap \mathbb{Z}^n} [R+2]\, \omega + E_\omega\,, \end{aligned}$$

where $[x]$ denotes the integer part of x, and the set E_ω is as in Theorem 1.2 (applied here with $H := H_\varepsilon$).

Notice also that

$$S^{n-1} \cap \mathbb{Z}^n = \Big\{ (\pm 1, 0, 0, \dots, 0),\; (0, \pm 1, 0, \dots, 0),\; (0, 0, \pm 1, \dots, 0),\; (0, 0, 0, \dots, \pm 1) \Big\}.$$



The set F_R.

By construction,

$$E_R \text{ is a subsolution and } F_R \text{ is a supersolution of (2.12).} \tag{2.14}$$

Also,

$$B_R \subset E_R \subset B_{R+1} \subset F_R \subset B_{R+C+2} \,.$$

where C is as in Theorem 1.2.
Then, the set $E_{\varepsilon,R}$ claimed in the statement of Theorem 1.3 can be obtained by minimizing the functional (2.13), with the additional constraint $E_R \subseteq E \subseteq F_R$. Notice that by (2.14) and the strong maximum principle (see, e.g., [CdlL01]) we have

$$\partial E_{\varepsilon,R} \cap (\partial E_R \cup \partial F_R) \,=\, \emptyset \,,$$

so that the mean curvature of $\partial E_{\varepsilon,R}$ agrees with $-H_\varepsilon$, as required.
Step 2. We now prove the second statement of Theorem 1.3. Let $\eta : \mathbb{R} \to \mathbb{R}$ be the solution of

$$-\eta'' + F''(\gamma)\eta = -\gamma' + \frac{c_0}{2} \,, \tag{2.15}$$

such that $\eta(0) = 0$.

Note that, since the right-hand side of (2.15) is orthogonal to γ', the function η is uniquely defined under mild growth conditions at infinity and

$$\lim_{x\to\pm\infty} \eta(x) = \eta_\infty := \frac{c_0}{2\,F''(1)}\,.$$

We refer to Section 6.1 of [Pa97] for further details on the construction of η.
We now fix $\delta \in (0,1)$, to be taken suitably small in the sequel (possibly in dependence of ε too). Since γ and η approach their limit values exponentially fast, together with their derivatives, we have that

$$\sum_{j=0}^{2} \left|D^j(|\gamma(x)|-1)\right| + \left|D^j(\eta(x)-\eta_\infty)\right| \le \delta^4 \qquad \text{if } |x| \ge K|\log\delta|. \tag{2.16}$$

Here above, $K>0$ is a suitably large structural constant, depending only on the potential F.
We now follow some ideas of [Pa97] in order to construct useful barriers by means of γ and η.
We let γ_δ, $\eta_\delta \in C^1(\mathbb{R})$ be such that

$$\begin{array}{ll} \gamma_\delta(x) = \gamma(x) & \text{for } x \in (-K|\log\delta|, K|\log\delta|) \\ \eta_\delta(x) = \eta(x) & \text{for } x \in (-K|\log\delta|, K|\log\delta|), \end{array} \tag{2.17}$$

$$\gamma_\delta(x) = \begin{cases} 1 & \text{for } x \ge 2K|\log\delta| \\ -1 & \text{for } x \le -2K|\log\delta|. \end{cases}$$

and

$$\eta_\delta(x) = \eta_\infty \qquad \text{for } x \ge 2K|\log\delta|.$$

Due to (2.16), we may construct γ_δ and η_δ in such a way that

$$\sum_{j=0}^{2} \left|D^j(|\gamma_\delta(x)|-1)\right| + \left|D^j(\eta_\delta(x)-\eta_\infty)\right| \le \delta^3 \qquad \text{if } |x| \ge K|\log\delta|. \tag{2.18}$$

We set

$$v_{\varepsilon,\delta}(x) := \gamma_\delta\left(\frac{\hat{R}-|x|}{\delta}\right) - \delta\,\eta_\delta\left(\frac{\hat{R}-|x|}{\delta}\right) H_\varepsilon(x) - c_1\delta^2\,,$$

Here above, $c_1 > 0$ is a constant, to be taken suitably large with respect to other structural constants.
Notice that

$$\begin{array}{l} v_{\varepsilon,\delta}(x) = 1 - \delta\eta_\infty H_\varepsilon(x) - c_1\delta^2 \text{ when } |x| \le \hat{R} - 2K\delta|\log\delta|, \text{ and} \\ v_{\varepsilon,\delta}(x) = -1 - \delta\eta_\infty H_\varepsilon(x) - c_1\delta^2 \text{ when } |x| \ge \hat{R} - 2K\delta|\log\delta|. \end{array} \tag{2.19}$$

We claim that

$$v_{\varepsilon,\delta} \text{ is a strict subsolution of (1.14)} \tag{2.20}$$

provided c_1 is sufficiently large and $\delta \in (0,\delta_0]$, with δ_0 sufficiently small with respect to ε and to the other structural constants.

To prove (2.20), we use polar coordinates to compute the Laplacian and we obtain

$$\begin{aligned}
&-\delta\,\Delta v_{\varepsilon,\delta} + \frac{F'(v_{\varepsilon,\delta})}{\delta} + \frac{c_0}{2}H_\varepsilon \\
&\leq -\frac{\gamma_\delta{}''}{\delta}(\star) + \frac{n-1}{|x|}\gamma_\delta{}'(\star) + \big(\eta_\delta{}''(\star) - \frac{\delta(n-1)}{|x|}\eta_\delta{}'(\star)\big)H_\varepsilon + \delta|\eta_\delta{}'(\star)||\nabla H_\varepsilon| \\
&\quad + \delta^2\eta_\delta(\star)|\Delta H_\varepsilon| + \frac{F'(\gamma_\delta(\star) - \delta\eta_\delta(\star)H_\varepsilon - c_1\delta^2)}{\delta} + F''(1)\eta_\infty H_\varepsilon \qquad (2.21)\\
&\leq -\frac{\gamma_\delta{}''}{\delta}(\star) + \frac{n-1}{|x|}\gamma_\delta{}'(\star) + \big(\eta_\delta{}''(\star) - \frac{\delta(n-1)}{|x|}\eta_\delta{}'(\star)\big)H_\varepsilon \\
&\quad + \frac{F'(\gamma_\delta(\star) - \delta\eta_\delta(\star)H_\varepsilon - c_1\delta^2)}{\delta} + F''(1)\eta_\infty H_\varepsilon + \mathrm{const}\delta.
\end{aligned}$$

Here and in the sequel, we use "$\star$" as a short hand notation for "$(\hat{R}-|x|)/\delta$" and "const" to denote suitable quantities, possibly depending on ε, F and H, but independent of δ.
Thus, we distinguish now the case in which $|\hat{R}-|x||/\delta \geq K|\log\delta|$ from the one in which $|\hat{R}-|x||/\delta \leq K|\log\delta|$.
When $|\hat{R}-|x||/\delta \geq K|\log\delta|$, we use that

$$|F'(\pm 1 + s) - F''(1)s| \leq \mathrm{const}s^2$$

for small s and so, recalling (2.18),

$$F'(\gamma_\delta(\star) - \delta\eta_\delta(\star)H_\varepsilon - c_1\delta^2) \leq F''(1)\big(\gamma_\delta(\star) \pm 1 - \delta\eta_\delta(\star)H_\varepsilon - c_1\delta^2\big) + \mathrm{const}\delta^2$$

for $|\hat{R}-|x||/\delta \geq K|\log\delta|$.
Consequently, when $|\hat{R}-|x||/\delta \geq K|\log\delta|$, (2.21) gives that

$$\begin{aligned}
&-\delta\,\Delta v_{\varepsilon,\delta} + \frac{F'(v_{\varepsilon,\delta})}{\delta} + \frac{c_0}{2}H_\varepsilon \\
\leq\;& -\frac{\gamma_\delta{}''}{\delta}(\star) + \frac{n-1}{|x|}\gamma_\delta{}'(\star) + \big(\eta_\delta{}''(\star) - \frac{\delta(n-1)}{|x|}\eta_\delta{}'(\star)\big)H_\varepsilon \\
&+\frac{F''(1)\big(\gamma_\delta(\star) \pm 1 - \delta\eta_\delta(\star)H_\varepsilon - c_1\delta^2\big)}{\delta} + F''(1)\eta_\infty H_\varepsilon + \mathrm{const}\delta \\
\leq\;& -c_1F''(1)\delta + \mathrm{const}\delta \;<\; 0,
\end{aligned}$$

where (2.18) was used once more.
This proves (2.20) when $|\hat{R}-|x||/\delta \geq K|\log\delta|$.
On the other hand, when $|\hat{R}-|x||/\delta \leq K|\log\delta|$, we make use of (1.8), (2.15), (2.17) and (2.21) to deduce that

$$\begin{aligned}
&-\delta\,\Delta v_{\varepsilon,\delta} + \frac{F'(v_{\varepsilon,\delta})}{\delta} + \frac{c_0}{2}H_\varepsilon \\
\leq\;& -\frac{\gamma''}{\delta}(\star) + \frac{n-1}{|x|}\gamma'(\star) + \big(\eta''(\star) - \frac{\delta(n-1)}{|x|}\eta'(\star)\big)H_\varepsilon \\
&+\frac{F'(\gamma(\star) - \delta\eta(\star)H_\varepsilon - c_1\delta^2)}{\delta} + F''(1)\eta_\infty H_\varepsilon \\
=\;& -\frac{F'(\gamma(\star))}{\delta} + \frac{n-1}{|x|}\gamma'(\star) + \big(F''(\gamma(\star))\eta(\star) + \gamma'(\star) - F''(1)\eta_\infty\big)H_\varepsilon \\
&-\frac{\delta(n-1)}{|x|}\eta'(\star)H_\varepsilon + \frac{F'(\gamma(\star) - \delta\eta(\star)H_\varepsilon - c_1\delta^2)}{\delta} + F''(1)\eta_\infty H_\varepsilon \\
\leq\;& \gamma'(\star)\left[\frac{n-1}{|x|} + H_\varepsilon\right] - c_1F''(\gamma)\delta + \mathrm{const}\delta.
\end{aligned}$$

Also, if δ is small enough with respect to ε we have that

$$\big\{|\hat{R}-|x||/\delta \leq K|\log\delta|\big\} \subset \big\{|x| \in (\hat{R}-c\varepsilon, \hat{R}+c\varepsilon)\big\}$$

and so, from (2.4), (2.2) and (2.11), we conclude that

$$\frac{n-1}{|x|} + H_\varepsilon \leq -\frac{1}{8\hat{R}}$$

for ε small enough; we recall that $\hat{R}$ is a structural constant, given by (2.1).
Therefore,

$$-\delta\,\Delta v_{\varepsilon,\delta} + \frac{F'(v_{\varepsilon,\delta})}{\delta} + \frac{c_0}{2}H_\varepsilon \leq -\frac{\gamma'(\star)}{8\hat{R}} - c_1 F''(\gamma(\star))\delta + \mathrm{const}\,\delta. \tag{2.22}$$

Now, if $\gamma'(\star) \geq \sqrt{\delta}$, we have

$$-\frac{\gamma'(\star)}{8\hat{R}} - c_1 F''(\gamma(\star))\delta + \mathrm{const}\,\delta \leq -\frac{\sqrt{\delta}}{8\hat{R}} + \mathrm{const}(c_1+1)\,\delta < 0 \tag{2.23}$$

for small δ.
On the other hand, if $\gamma'(\star) \leq \sqrt{\delta}$, that is $|\gamma'(\star)| \leq \sqrt{\delta}$, we have that $F''(\gamma(\star)) \geq F''(1)/2$, and so

$$-\frac{\gamma'(\star)}{8\hat{R}} - c_1 F''(\gamma(\star))\delta + \mathrm{const}\,\delta \leq -\frac{c_1 F''(1)\delta}{2} + \mathrm{const}\,\delta < 0 \tag{2.24}$$

as long as c_1 is suitably large.
From (2.22), (2.23) and (2.24), we conclude that

$$-\delta\,\Delta v_{\varepsilon,\delta} + \frac{F'(v_{\varepsilon,\delta})}{\delta} + \frac{c_0}{2}H_\varepsilon < 0$$

when $|\hat{R}-|x||/\delta \leq K|\log\delta|$.
This proves (2.20) also when $|\hat{R}-|x||/\delta \leq K|\log\delta|$.
Step 3. Let $\omega \in S^{n-1}$ be a rational direction such that the set E_ω given by Theorem 1.2 is a nondegenerate minimizer (see [Pa97, Definition 4.1]). Then, by [Pa97, Theorem 8.1] there exists a supersolution $u^+_{\omega,\delta}$ of (1.14) such that

- the Hausdorff distance between $\{u_{\omega,\delta} > 0\}$ and the set E_ω is of order $\delta^2|\log\delta|^2$;
- there exists absolute constants $\xi_\infty, c_2, N > 0$ (independent of ω and δ) such that

$$\begin{aligned} u^+_{\omega,\delta}(x) &= \ 1 - \delta\eta_\infty H_\varepsilon(x) + \delta^2\xi_\infty\left(H_\varepsilon(x)\right)^2 + c_2\delta^3|\log\delta|^2 \quad \text{if } \langle\omega, x\rangle \geq N \\ u^+_{\omega,\delta}(x) &= -1 - \delta\eta_\infty H_\varepsilon(x) - \delta^2\xi_\infty\left(H_\varepsilon(x)\right)^2 + c_2\delta^3|\log\delta|^2 \quad \text{if } \langle\omega, x\rangle \leq -N. \end{aligned} \tag{2.25}$$

Notice that from (2.25) it follows that there exists $\widetilde{N} \in \mathbb{N}$ such that

$$u^+_{\omega,\delta}\left(x - N\,\langle\omega, x\rangle\right) > v_{\varepsilon,\delta}(x) \tag{2.26}$$

for all $N \geq \widetilde{N}$ and any $x \in \mathbb{R}^n$.
Let

$$\Sigma := \bigcup_{\omega\in S^{n-1}\cap\mathbb{Z}^n} \pi\left(\partial E_\omega\right).$$

We can slightly perturb H_ε in a neighborhood of Σ, without changing its value on Σ itself, in such a way that

– conditions (1.10)–(1.12) still hold;

– the sphere ∂B_R is a strict subsolution of (2.12) for all $R \in (\hat{R} - c\varepsilon, \hat{R} + c\varepsilon)$;

– E_ω is a nondegenerate minimizer of (1.6) for all $\omega \in S^{n-1} \cap \mathbb{Z}^n$.

In particular, we can use [Pa97, Theorem 8.1] and obtain (2.25), and so (2.26), for any $\omega \in S^{n-1} \cap \mathbb{Z}^n$.
Fix now $R \geq \hat{R}$ and let

$$\begin{aligned} v_R^-(x) &:= \max_{z \in \mathbb{Z}^n :\, z + B_{\hat{R}} \subset B_{R+1}} v_{\varepsilon,\delta}(x - z) \\ v_R^+(x) &:= \min_{\omega \in S^{n-1} \cap \mathbb{Z}^n} u^+_{\omega,\delta}\left(x - [R + 2 + \widetilde{N}] \langle \omega, x \rangle \right). \end{aligned}$$

We observe that $v_R^-(x) < v_R^+(x)$ for all $x \in \mathbb{R}^n$, thanks to (2.26).
The function $u_{\varepsilon,R,\delta}$ solving (1.14) and satisfying (1.15) that was claimed in Theorem 1.3 can now be obtained by choosing $u_{\varepsilon,R,\delta}$ as a minimizer of (1.3), under the additional constraint $v_R^- \leq u \leq v_R^+$.
As above, by strong maximum principle we have

$$v_R^-(x) < u_{\varepsilon,R,\delta}(x) < v_R^+(x)$$

for all $x \in \mathbb{R}^n$, so that $u_{\varepsilon,R,\delta}$ is a solution of (1.14).
This completes the proof of Theorem 1.3. □

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