Closed curves of prescribed curvature and a pinning effect

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Abstract

We prove that for any $H : \mathbb{R}^2 \to \mathbb{R}$ which is $\mathbb{Z}^2$-periodic, there exists H_ε, which is smooth, ε-close to H in L^1, with L^∞-norm controlled by the one of H, and with the same average of H, for which there exists a smooth closed curve γ_ε whose curvature is H_ε. A pinning phenomenon for curvature driven flow with a periodic forcing term then follows. Namely, curves in fine periodic media may be moved only by small amounts, of the order of the period.

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''Si fa la trigonometria,
nei finestrini corrispondenti agli occhi alessandrini,
di lei che guarda fissa
un suo sussulto fuso nel vetro,
che le ricorda tanto un suo sussulto.''
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(P. Panella & L. Battisti, *La metro eccetera*)

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1 Introduction

In this paper, curves in the plane with prescribed curvature are dealt with.
We show that, for a "generic" H, periodic, possibly with small L^∞-size, and with prescribed (possibly zero) average, there exists a closed, convex curve whose curvature at any points agrees with H. The genericity is in the L^1-sense.
We then apply this result to show a pinning phenomenon in an evolutionary problem driven by the curvature.
In further detail, our geometric result is the following:

Theorem 1.1. *For any $H \in L^\infty(\mathbb{T}^2)$, with $H \not\equiv 0$, and for any $\varepsilon > 0$ there exists $H_\varepsilon \in C^\infty(\mathbb{T}^2)$, with*

$$\|H_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \le \|H\|_{L^\infty(\mathbb{T}^2)}, \tag{1.1}$$

$$\|H_\varepsilon - H\|_{L^1(\mathbb{T}^2)} \le \varepsilon \, \|H\|_{L^\infty(\mathbb{T}^2)}, \tag{1.2}$$

and

$$\int_{\mathbb{T}^2} H_\varepsilon(x)\,dx = \int_{\mathbb{T}^2} H(x)\,dx, \tag{1.3}$$

such that there exists a set E_ε, with smooth compact boundary, whose curvature agrees with H_ε at any point of ∂E_ε. Moreover, we can choose E_ε such that either E_ε or $\mathbb{R}^2 \setminus E_\varepsilon$ is a convex set (with the convention that the curvature of a convex set is positive).

We observe that Theorem 1.1 does not hold, in general, if we choose $H_\varepsilon := H$. However, it would be interesting to know:

- whether a result analogous to Theorem 1.1 holds if we replace the L^1 norm in (1.2) with a stronger one (e.g., the L^∞ norm),
- whether a result analogous to Theorem 1.1 holds in higher dimension,
- under which conditions on H it would be possible to choose $H_\varepsilon := H$ in Theorem 1.1, possibly studying concrete cases (such as a chessboard like H),
- whether the random setting (instead of the periodic one) exhibits similar phenomena,
- whether a PDE analogue holds (for instance, whether there exists a mesoscopic phase transition [NV07] in the plane whose interface is a closed curve).

As a consequence of Theorem 1.1, we have a pinning phenomenon for the curvature flow.
Namely, given $\delta > 0$, for an open interval $I \subseteq \mathbb{R}$ and a function $H : \mathbb{T}^2 \to \mathbb{R}$, we say that a family of closed, smoothly embedded curves $\{\Gamma_t\}_{t\in I}$, with $\Gamma_t = \partial E_t$, moves by δ-periodic H-curvature if

$$v(x,t) = \left(\frac{H(x/\delta)}{\delta} - \kappa(x)\right)\nu(x) \tag{1.4}$$

for any $x \in \Gamma_t$ and any $t \in I$.
Here above v, κ and ν denote, respectively, the normal velocity, the curvature and the exterior unit normal of E_t at $x \in \Gamma_t$. Notice that when $H := 0$, equation (1.4) boils down to the usual curvature flow [GH86].
We denote by $d_{\mathcal{H}}(A, B)$ the Hausdorff distance between two sets A, $B \subseteq \mathbb{R}^2$. With this notation, we have that solutions of (1.4) are, for a "typical" H, confined in a δ-neighborhood of their initial data, according to the following result:

Theorem 1.2. *Let $H \in L^\infty(\mathbb{T}^2)$ be such that both $H^+ \not\equiv 0$ and $H^- \not\equiv 0$, where $H^\pm$ denote respectively the positive and the negative part of H. Then, for any $\varepsilon > 0$ there exist $H_\varepsilon \in C^\infty(\mathbb{T}^2)$, satisfying (1.1), (1.2) and (1.3), and $C_\varepsilon > 0$ such that any $\{\Gamma_t\}_{t\in I}$, $\Gamma_t = \partial E_t$, which moves by δ-periodic H_ε-curvature satisfies*

$$\sup_{s,t\in I} d_{\mathcal{H}}(\Gamma_s, \Gamma_t) \le C_\varepsilon \delta. \tag{1.5}$$

Related pinning effects in different frameworks have been also studied by [DY06]. The pinning effect of Theorem 1.2 should be compared with the limit of the functionals

$$E \mapsto \mathrm{Per}(E) + \frac{1}{\delta}\int_E H(x/\delta)\,dx, \tag{1.6}$$

which has been investigated in [DLN06] (as usual, in (1.6), we denoted by Per the perimeter of a Caccioppoli set), where it is shown that the functionals in (1.6) converge, in the sense of Γ-convergence, to an anisotropic perimeter, with anisotropy depending on H. Since equation (1.4) corresponds to the gradient flow of (1.6), one may expect that the solutions of (1.4) converge, as $\delta \to 0$, to a solution of the gradient flow of the limit functional, that is to an anisotropic curvature flow. However, the result of Theorem 1.2 indicates that this is generically *not* the case, and the solutions of (1.4) do not move in the limit, due to the effect of the strong forcing term.

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1.1, by making use of an auxiliary result, namely Proposition 2.1, which is proved in Section 3. The proof of Theorem 1.2 is given in Section 4.

2 Proof of Theorem 1.1

The main step towards the proof of Theorem 1.1 consists in the following

Proposition 2.1. *Let $K \in C^\infty(\mathbb{R}^2)$, with $K(x) \geq 0$ for any $x \in \mathbb{R}^2$. Suppose that there exist r' and $r > 0$ in such a way that $r' \in [r, 1/4]$ and $c > 0$ for which*

$$K(x) \geq c \text{ for any } x \in \bigcup_{j \in \mathbb{Z}^2} B_r(j)$$

and

$$K(x) = 0 \text{ for any } x \text{ outside } \bigcup_{j \in \mathbb{Z}^2} B_{r'}(j).$$

Then, there exists a C^∞ closed, convex curve γ whose curvature at any points is equal to K.

We postpone the proof of Proposition 2.1 to Section 3 and we show now that Proposition 2.1 implies Theorem 1.1.
For this, we fix a small $\varepsilon > 0$ and we take H as in the statement of Theorem 1.1.
We consider a standard mollifier ρ_ε and we define the mollification of H as

$$\tilde{K}_\varepsilon := (1 - \sqrt{\varepsilon})(H * \rho_\varepsilon).$$

Note that $\tilde{K}_\varepsilon \in C^\infty(\mathbb{T}^2)$. Since H is not identically zero, we have that there exist $c_\varepsilon > 0$, $r_\varepsilon > 0$, and $x_o \in \mathbb{R}^2$ such that $\tilde{K}_\varepsilon(x) \geq c_\varepsilon$ or $\tilde{K}_\varepsilon(x) \leq -c_\varepsilon$, for any $x \in B_{3r_\varepsilon}(x_o)$. For simplicity, we assume that $\tilde{K}_\varepsilon \geq c_\varepsilon$ on $B_{3r_\varepsilon}(x_o)$, since the other case can be trated analogously.
Up to change of coordinates, we may suppose $x_o = 0$. Then, by periodicity,

$$\tilde{K}_\varepsilon(x) \geq c_\varepsilon \text{ for any } x \in \bigcup_{j \in \mathbb{Z}^2} B_{3r_\varepsilon}(j). \tag{2.1}$$

We take a bump function $\tau_\varepsilon \in C^\infty(\mathbb{T}^2, [0,1])$ such that

$$\tau_\varepsilon(x) = 1 \text{ for any } x \in \bigcup_{j \in \mathbb{Z}^2} B_{r_\varepsilon}(j)$$

and

$$\tau_\varepsilon(x) = 0 \text{ for any } x \text{ outside } \bigcup_{j \in \mathbb{Z}^2} B_{3r_\varepsilon}(j).$$

We set

$$K_\varepsilon := \tau_\varepsilon \tilde{K}_\varepsilon.$$

Then, by (2.1),

$$K_\varepsilon(x) \geq c_\varepsilon \text{ for any } x \in \bigcup_{j \in \mathbb{Z}^2} B_{r_\varepsilon}(j),$$

and $K_\varepsilon \geq 0$ on $\mathbb{R}^2$.

Thus, in both the cases considered above, we have found $K_\varepsilon \in C^\infty(\mathbb{R}^2)$ such that $K_\varepsilon \geq 0$ on $\mathbb{R}^2$,

$$\|K_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq \|\tilde{K}_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq (1-\varepsilon)\|H\|_{L^\infty(\mathbb{R}^2)} = (1-\varepsilon)\|H\|_{L^\infty(\mathbb{T}^2)}, \tag{2.2}$$

$$K_\varepsilon(x) = 0 \text{ for any } x \text{ outside } \bigcup_{j\in\mathbb{Z}^2} B_{3r_\varepsilon}(j) \tag{2.3}$$

and

$$K_\varepsilon(x) \geq c_\varepsilon \text{ for any } x \in \bigcup_{j\in\mathbb{Z}^2} B_{r_\varepsilon}(j),$$

for suitably small c_ε, $r_\varepsilon > 0$.
We can thus apply Proposition 2.1 and obtain a C^∞ curve $\gamma_\varepsilon = \partial E_\varepsilon$, with E_ε compact convex set, such that

$$\text{the curvature of } \gamma_\varepsilon \text{ is equal to } K_\varepsilon \text{ at any point.} \tag{2.4}$$

We denote by

$$\pi : \mathbb{R}^2 \to \mathbb{T}^2$$

the natural projection.
Notice that $\pi(\gamma_\varepsilon)$ is a closed set of zero Lebesgue measure in $\mathbb{T}^2$ and so we can find a ball β_ε, with Lebesgue measure $b_\varepsilon \in (0,1)$, and open sets $U_\varepsilon^{(1)} \subset U_\varepsilon^{(2)} \subset \mathbb{T}^2$ such that $\pi(\gamma_\varepsilon) \subset U_\varepsilon^{(1)}$, $U_\varepsilon^{(2)} \cap \beta_\varepsilon = \emptyset$ and

$$\text{the Lebesgue measure of } U_\varepsilon^{(2)} \text{ is less than } \varepsilon^2 b_\varepsilon. \tag{2.5}$$

We consider a bump function $\psi_\varepsilon \in C^\infty(\mathbb{T}^2, [0,1])$ such that $\psi_\varepsilon(x) = 1$ for any $x \in U_\varepsilon^{(1)}$ and $\psi_\varepsilon(x) = 0$ for any x outside $U_\varepsilon^{(2)}$.
Hence, we take $\alpha_\varepsilon \in C^\infty(\mathbb{T}^2, [0,+\infty))$ to be a bump function such that $\alpha_\varepsilon(x) = 0$ for any x outside β_ε and

$$\int_{\beta_\varepsilon} \alpha_\varepsilon(x)\, dx = 1.$$

By definition of b_ε, we can also suppose that

$$\|\alpha_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \leq \frac{\text{const}}{b_\varepsilon}. \tag{2.6}$$

We consider now $K_\varepsilon^\star \in C^\infty(\mathbb{T}^2)$ in such a way that

$$\|K_\varepsilon^\star\|_{L^\infty(\mathbb{T}^2)} \leq \|H\|_{L^\infty(\mathbb{T}^2)} \tag{2.7}$$

and

$$\|H - K_\varepsilon^\star\|_{L^\infty(\mathbb{T}^2)} \leq \varepsilon^2 b_\varepsilon \|H\|_{L^\infty(\mathbb{T}^2)}. \tag{2.8}$$

Let also

$$\begin{aligned} \ell_\varepsilon := &\int_{\mathbb{T}^2\setminus U_\varepsilon^{(2)}} \big(H(x) - K_\varepsilon^\star(x)\big)\, dx + \int_{U_\varepsilon^{(2)}} H(x)\, dx \\ &- \int_{U_\varepsilon^{(2)}\setminus U_\varepsilon^{(1)}} \psi_\varepsilon(x) K_\varepsilon(x)\, dx - \int_{U_\varepsilon^{(2)}\setminus U_\varepsilon^{(1)}} \big(1-\psi_\varepsilon(x)\big) K_\varepsilon^\star(x)\, dx - \int_{U_\varepsilon^{(1)}} K_\varepsilon(x)\, dx. \end{aligned} \tag{2.9}$$

We define

$$H_\varepsilon(x) := \begin{cases} K_\varepsilon(x) & \text{if } x \in U_\varepsilon^{(1)}, \\ \psi_\varepsilon(x)\, K_\varepsilon(x) + \big(1-\psi_\varepsilon(x)\big) K_\varepsilon^\star(x) & \text{if } x \in U_\varepsilon^{(2)} \setminus U_\varepsilon^{(1)}, \\ \ell_\varepsilon \alpha_\varepsilon(x) + K_\varepsilon^\star(x) & \text{if } x \in \mathbb{T}^2 \setminus U_\varepsilon^{(2)}. \end{cases}$$

Note that the curvature of γ_ε agrees with H_ε, due to (2.4), since the support of $\pi(\gamma_\varepsilon)$ lies in $U_\varepsilon^{(1)}$. Therefore, γ_ε satisfies the claim of Theorem 1.1.
We now prove that H_ε also satisfies the claim of Theorem 1.1.
For this, we use (2.2), (2.5) and (2.8) to get that

$$|\ell_\varepsilon| \leq 6\varepsilon^2 b_\varepsilon \|H\|_{L^\infty(\mathbb{R}^2)}. \tag{2.10}$$

As a consequence, from (2.2), (2.6) and (2.7), we obtain (1.1).
Also, by (2.2), (2.5), (2.8) and (2.10), we have

$$\begin{aligned}\int_{\mathbb{T}^2} |H_\varepsilon(x) - H(x)|\,dx &\leq \int_{U_\varepsilon^{(2)}} |K_\varepsilon(x) - H(x)| + |K_\varepsilon^\star(x) - H(x)\,dx \\ &\quad +|\ell_\varepsilon| \int_{\mathbb{T}^2 - U_\varepsilon^{(2)}} \alpha_\varepsilon(x)\,dx \\ &\quad + \int_{\mathbb{T}^2 - U_\varepsilon^{(2)}} |K_\varepsilon^\star(x) - H(x)\,dx \\ &\leq 9\varepsilon^2 \|H\|_{L^\infty(\mathbb{T}^2)}.\end{aligned}$$

This proves (1.2).
Finally, (2.9) gives (1.3) and H_ε is $C^\infty(\mathbb{T}^2)$ by construction.
Notice that, if we have instead $\tilde{K}_\varepsilon \leq -c_\varepsilon$ on $B_{3r_\varepsilon}(x_o)$, we can reason as above replacing the function H with $-H$. The only difference is that in this case we obtain a curve $\gamma_\varepsilon = \partial E_\varepsilon$, still satisfying (2.4), where E_ε is unbounded and $\mathbb{R}^2 \setminus E_\varepsilon$ is a compact convex set.
This completes the proof of Theorem 1.1 when Proposition 2.1 is in force.

3 Proof of Proposition 2.1

First of all, we fix $\alpha > 0$, to be taken conveniently small in what follows, and we construct a closed convex polygon $\mathcal{P}_\alpha$ whose vertex are in $\mathbb{Z}^2$ and such that the angles between its edges are in $[\pi - \alpha, \pi)$.
For this scope, we fix a small $a > 0$ and a point $P_1 \in \mathbb{Z}^2$. We take a half-line λ_1 with rational slope through P_1 whose angle with respect to the horizontal axis is in $[a, 2a]$. Say, for definiteness, that the angles we consider are taken to be oriented anticlockwisely.
Due to the rationality of the slope of λ_1, there exists $P_2 \in \mathbb{Z}^2 \cap \lambda_1$. We then take a half-line λ_2 with rational slope through P_2 whose angle with respect to λ_1 is in $[a, 2a]$.
We then iterate this procedure (see Figure 1) and we find a half-line λ_n with rational slope through P_n whose angle with respect to λ_{n-1} is in $[a, 2a]$.

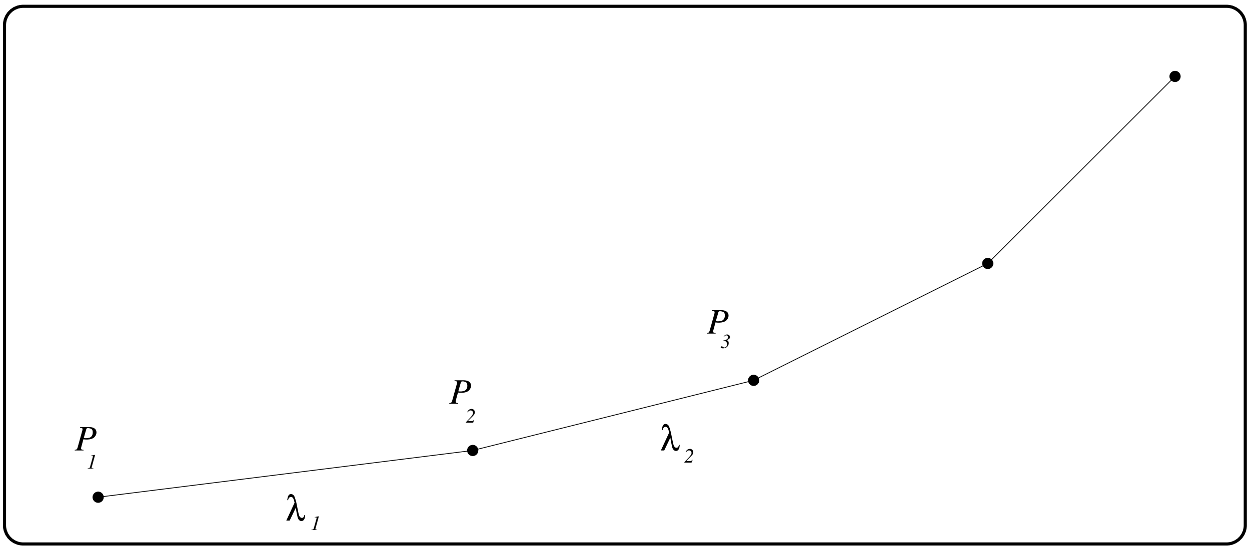

Figure 1

We denote by β_n the angle between λ_n and the horizontal axis. By construction,

$$\beta_n \in [\beta_{n-1} + a, \beta_{n-1} + 2a] \tag{3.1}$$

and therefore we can take m to be the first angle for which $\beta_m \geq (\pi/2) - 3a$.
We observe that, from (3.1), we have

$$(\pi/2) - 3a \geq \beta_{m-1} \geq \beta_m - 2a$$

hence (see Figure 2)

$$\beta_m \in \big[(\pi/2) - 3a, (\pi/2)\big).$$

In particular, the angle between λ_m and the vertical axis is in $(0, 3a]$.

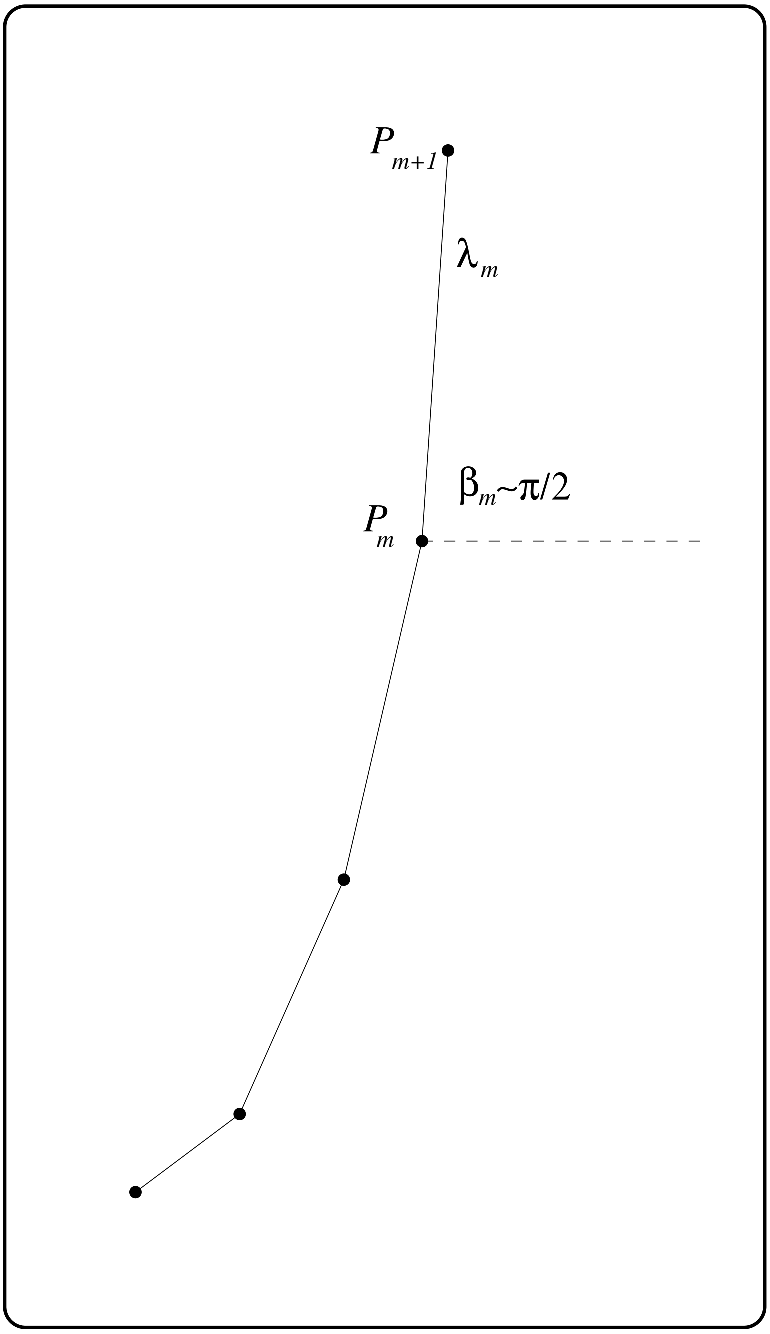

Figure 2

The polygon $\mathcal{P}_\alpha$ is then obtained by the segments $P_1 P_2 \ldots P_{m+1}$ by even reflections along the horizontal and vertical axes.
The reflections make $\mathcal{P}_\alpha$ closed. Since $P_n \in \mathbb{Z}^2$ for any n, the vertices of $\mathcal{P}_\alpha$ are in $\mathbb{Z}^2$. Also, if a is chosen suitably small, the angles of $\mathcal{P}_\alpha$ are close to π but less then π (thus, in particular, $\mathcal{P}_\alpha$ is convex).
We now take c and $r > 0$ as in the statement of Proposition 2.1 and we construct a closed $C^{1,1}$ curve Γ which consists in:

- pieces of segments outside

$$\mathcal{B}_r := \bigcup_{j\in\mathbb{Z}^2} B_{r/2}(j)$$

- arcs of circumferences with curvature less then $c/2$ in $\mathcal{B}_r$.

The curve Γ is constructed by modifying $\mathcal{P}_\alpha$. Indeed, we take Γ to agree with $\mathcal{P}_\alpha$ outside $\mathcal{B}_r$. Then, if P is a vertex of $\mathcal{P}_\alpha$, we call Q and R to be the two points in $\partial B_{r/2}(P)\cap\mathcal{P}_\alpha$ and we take Γ in $B_{r/2}(P)$ to be the arc of circumference passing through Q and R and tangent to $\mathcal{P}_\alpha$ from inside (see Figure 3).

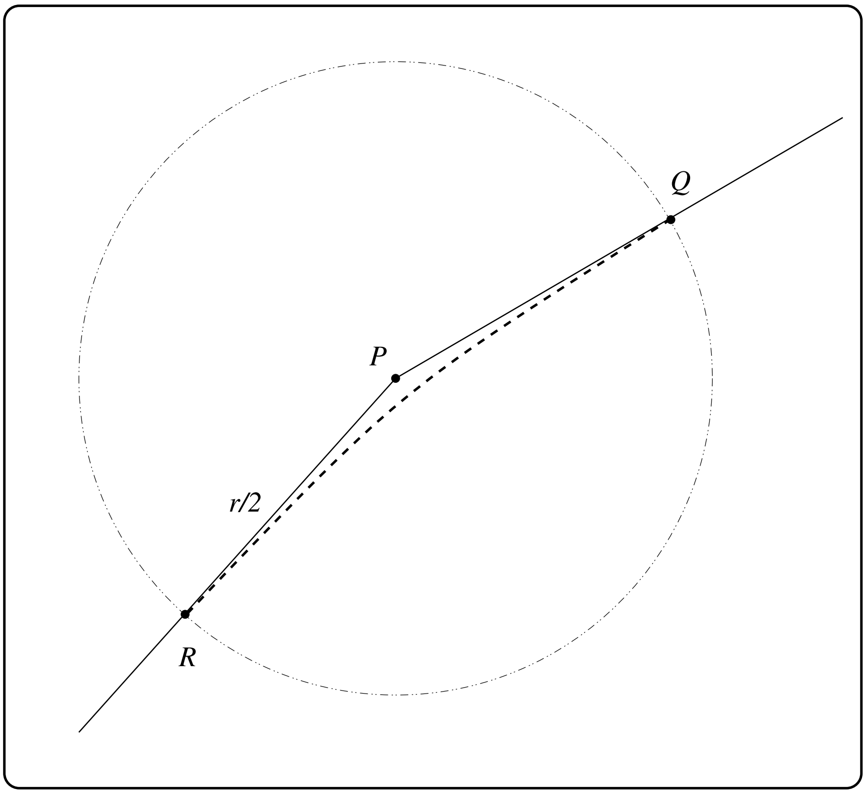

Figure 3

If we call 2θ the angle of $\mathcal{P}_\alpha$ in P, the radius ρ of such circumference satisfies

$$\rho = \frac{r}{2}\tan\theta,$$

due to standard trigonometry (see Figure 4).

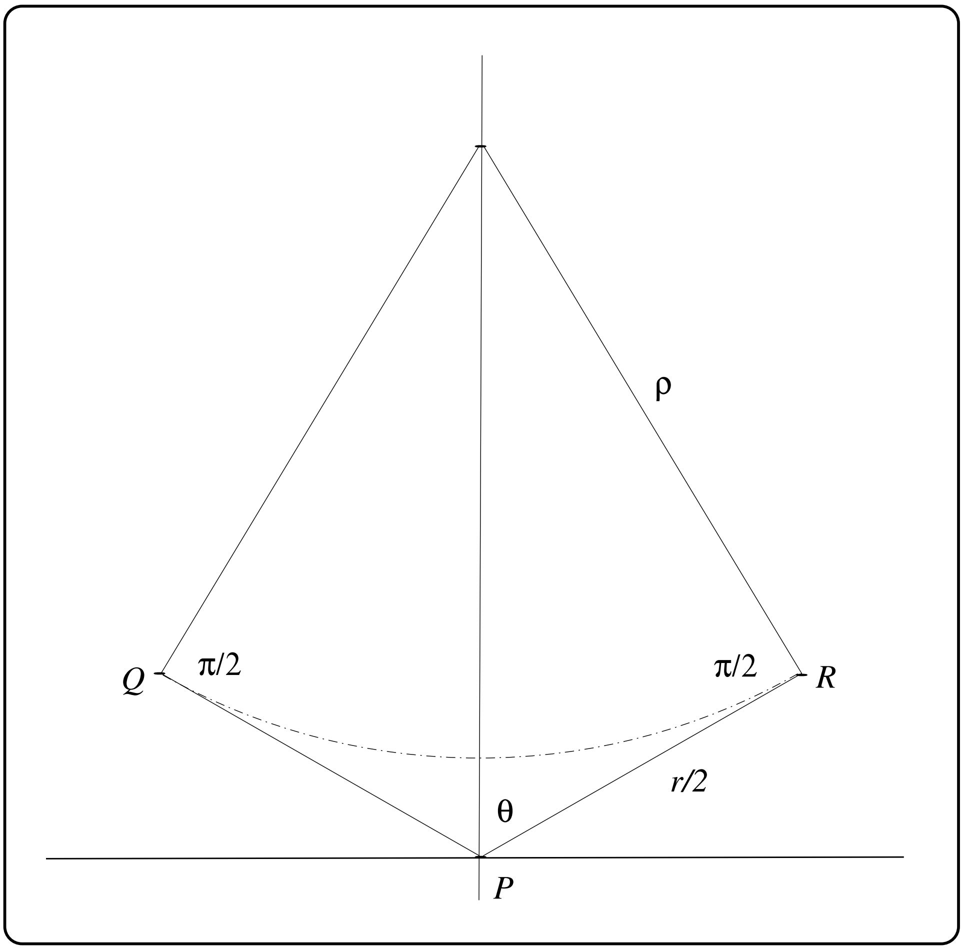

Figure 4

Accordingly, the curvature of Γ inside $B_{r/2}(P)$ is of the order of $1/(r\tan\theta)$. Since we know that $\theta \in \big[(\pi-\alpha)/2, \pi/2\big)$, such curvature is smaller than $c/2$, provided that α is small enough (possibly in dependence of r and c).

This ends the construction of the curve Γ satisfying the desired properties.

We define $E_\star$ to be the bounded set for which $\partial E_\star = \Gamma$.

Let also $R_\star$ to be a square, with horizontal/vertical edges, in such a way that

$$\partial R_\star \cap \bigcup_{j\in\mathbb{Z}^2} B_{r'}(j) = \emptyset. \tag{3.2}$$

By (3.2) and our hypotheses on K, we have that

$$K \text{ is zero near } \partial R_\star\ . \tag{3.3}$$

We look at the following functional. Given any bounded Caccioppoli set $F\subset\mathbb{R}^2$ (see [Giu84] for the definition and the basic properties of such an F), we define

$$\mathcal{I}(F) := \mathrm{Per}(\partial F) - \int_F K(x)\,dx.$$

By standard compactness arguments (see, for instance, [Giu84] or page 1425 in [CdlL01]), the functional $\mathcal{I}$ attains its minimum under the constraint that

$$E_\star \subseteq F \subseteq R_\star.$$

Let $F_\star$ be one of such minima. We have that the curvature of $\gamma := \partial F_\star$ is equal to K at any point in which γ does not touch $\partial E_\star \cup \partial R_\star$ (see, for instance, Section 11.1 in [CdlL01]).
Then, the proof of Proposition 2.1 will be finished once we show that

$$\gamma \cap (\partial E_\star \cup \partial R_\star) = \emptyset. \tag{3.4}$$

To prove (3.4), we first study the neighborhood of $\partial E_\star$. We observe that

$$\text{the curvature of } \gamma \text{ is bigger than, or equal to, } K \text{ in the vicinity of } \partial E_\star. \tag{3.5}$$

Indeed, if we take a small perturbation F_ϵ of $F_\star$, supported in the vicinity of $\partial E_\star$, for which $F_\star \subseteq F_\epsilon$, we know that

$$\mathcal{I}(F_\epsilon) \geq \mathcal{I}(F_\star). \tag{3.6}$$

We take ν to be the external normal of F and we write F_ϵ as a normal deformation (see page 119 in [Giu84]), that is

$$F_\epsilon = \{x + \eta\nu(x)\zeta(x),\ x \in \partial F_\star,\ \eta \in [0, \epsilon]\},$$

for some smooth compactly supported function ζ and $\epsilon > 0$.
Then, if $\pi_{\partial F_\star}$ is the natural projection onto $\partial F_\star$, we have

$$\int_{F_\epsilon \setminus F_\star} K(x)\, dx = \int_{F_\epsilon \setminus F_\star} K(\pi_{\partial F_\star} x)\, dx + o(\epsilon) = \epsilon \int_{\partial F_\star} K(y)\, \zeta(y)\, d\mathcal{H}^{n-1}(y) + o(\epsilon), \tag{3.7}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.
Also (see formula (10.12) in [Giu84]),

$$\operatorname{Per}(\partial F_\epsilon) - \operatorname{Per}(\partial F_\star) = \epsilon \int_{\partial F_\star} \mathcal{C}(y)\, \zeta(y)\, d\mathcal{H}^{n-1}(y) + o(\epsilon), \tag{3.8}$$

where $\mathcal{C}$ denotes the curvature (in fact, here, the only curvature) of $\partial F_\star$.
Thus, by (3.6), (3.7) and (3.8),

$$\begin{aligned} 0 &\leq \frac{\mathcal{I}(F_\epsilon) - \mathcal{I}(F_\star)}{\epsilon} \\ &= \int_{\partial F_\star} \mathcal{C}(y)\, \zeta(y)\, d\mathcal{H}^{n-1}(y) - \int_{\partial F_\star} K(y)\, \zeta(y)\, d\mathcal{H}^{n-1}(y) + o(1) \end{aligned}$$

thence $\mathcal{C} \geq K$ on $\partial F_\star$, which proves (3.5).
We now make an elementary observation of strong comparison principle type. Namely, for $\delta > 0$,

$$\begin{aligned} &\text{if } u \in C^2((0,\delta)) \cap C^1([0,\delta)) \text{ with } u(t) \geq 0 \text{ for any } t \in [0,\delta),\ u(0) = 0 \text{ and} \\ &\operatorname{div}\left(\frac{u'(t)}{\sqrt{1 + (u'(t))^2}}\right) \leq 0 \text{ for any } t \in (0,\delta), \\ &\text{then } u(t) = 0 \text{ for any } t \in [0,\delta). \end{aligned} \tag{3.9}$$

To prove (3.9) we just write the equation as

$$\frac{u''}{(1 + (u')^2)^{3/2}} \leq 0$$

and therefore, since $u'(0) = u(0) = 0$,

$$0 \leq u(t) = \int_0^t \int_0^\tau u''(s)\, ds\, d\tau \leq 0,$$

for any $t \in [0, \delta)$, proving (3.9).
Now, we have that

$$\gamma \text{ cannot touch } \partial E_\star \text{ in the interior of any } B_r(j), \text{ for } j \in \mathbb{Z}^2. \tag{3.10}$$

Indeed, thanks to (3.5), the osculating circle of γ has curvature bigger than, or equal to, c in $B_r(j)$. Since the curvature of the osculating circle of $\partial E_\star$ in the interior of $B_{r/2}(j)$ is at most $c/2$, we see that (3.10) holds true.
Moreover,

$$\gamma \text{ cannot touch } \partial E_\star \text{ in the closure of } \mathbb{R}^2 \setminus \bigcup_{j\in\mathbb{Z}^2} B_r(j). \tag{3.11}$$

Indeed, if such a touching point $P_\star$ existed, since $\partial E_\star$ contains a segment passing through $P_\star$, we would obtain from (3.9) that γ and $\partial E_\star$ agree as long as $\partial E_\star$ is flat, that is up to $\partial B_{r/2}(j_\star)$, for some $j_\star \in \mathbb{Z}^2$. But this would be in contradiction with (3.10) and it thus proves (3.11).
Therefore, from (3.10) and (3.11), we have that

$$\gamma \cap \partial E_\star = \emptyset. \tag{3.12}$$

Furthermore, γ cannot touch $\partial R_\star$ at its corner, since cutting the corner would decrease the perimeter and leave unchanged the term $\int_F K(x)\, dx$, thanks to (3.3), thus decreasing $\mathcal{I}$.
In particular, γ cannot be equal to $\partial R_\star$. Also, γ cannot touch $\partial R_\star$ at the other points as well, since it must be a straight line in the vicinity of $\partial R_\star$, due to (3.3).
These observations and (3.12) imply (3.4) and so they complete the proof of Proposition 2.1.

4 Proof of Theorem 1.2

For all $\varepsilon > 0$, we let $\gamma_\varepsilon^\pm = \partial E_\varepsilon^\pm$ be the smooth curves given by Theorem 1.1, which correspond to the forcing term $\pm H$ respectively.
Thanks to our assumptions on the function H, we may assume that the sets $E_\varepsilon^\pm$ are both compact and convex. Therefore, we can find a square with integer vertices containing $\gamma_\varepsilon^\pm$, and we denote by C_ε the sidelength of such square. Thus, we consider a tiling of $\mathbb{R}^2$ made by squares of sides C_ε each containing an integer translation of $E_\varepsilon^\pm$ (see Figure 5).

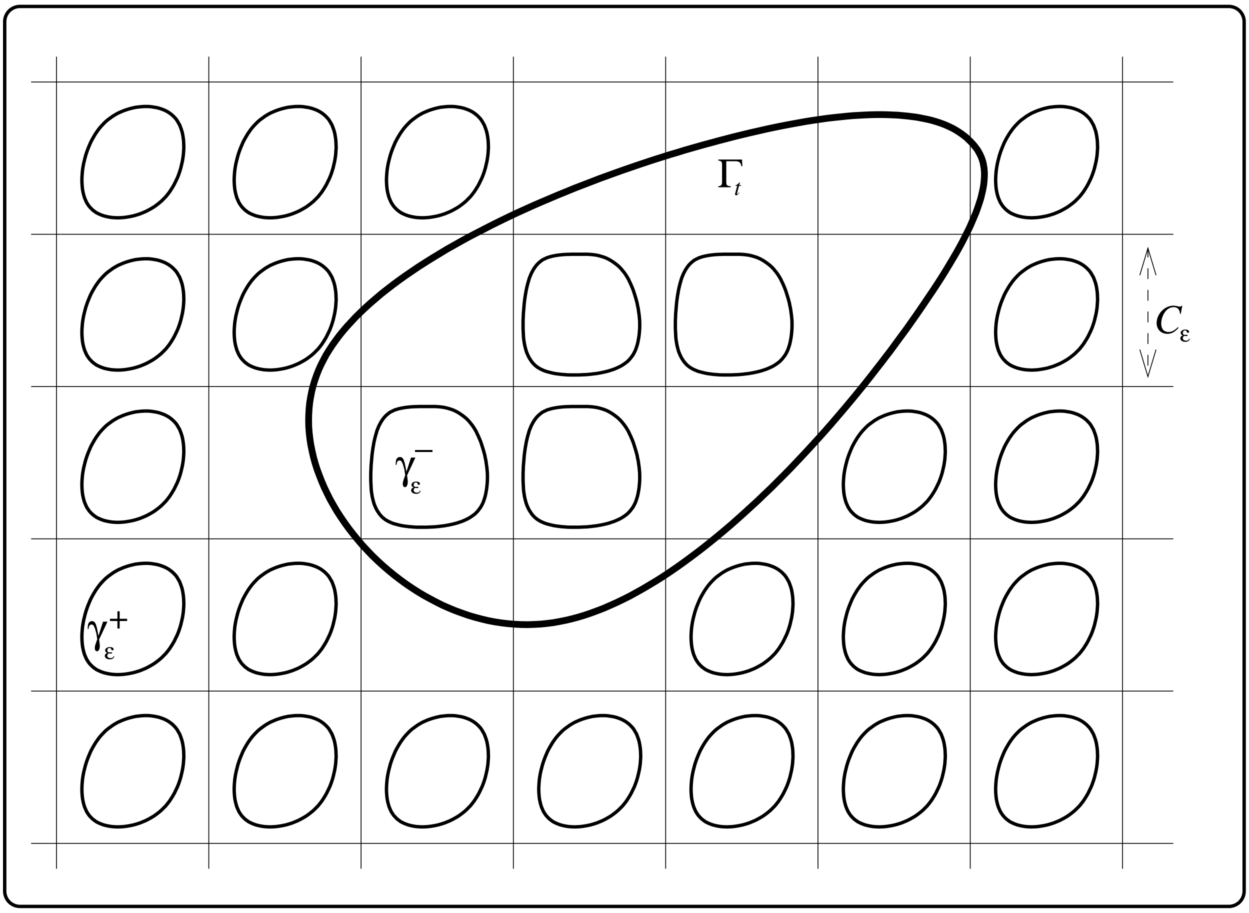

Figure 5

In dealing with the proof of Theorem 1.2, up to a dilation of factor $1/\delta$, we may and do assume that $\delta := 1$ in (1.4). Thus, we take any $\{\Gamma_t\}_{t\in I}$, with $\Gamma_t = \partial E_t$, that moves by 1-periodic H_ε-curvature and we show that

$$\sup_{s,t\in I} d_{\mathcal{H}}(\Gamma_s,\Gamma_t) \leq \operatorname{const} C_\varepsilon. \tag{4.1}$$

Dilating back by a factor δ the estimate in (4.1), we then obtain (1.5).
To prove (4.1), we observe that all the integer translations of E_ε^+ and of $\mathbb{R}^2 \setminus E_\varepsilon^-$ (which is an unbounded set) are stationary solutions of (1.4), with $\delta := 1$. Consequently, by comparison principle (see, for instance, page 18 in [Eck04]), Γ_t cannot travel neither through the translations $z + \gamma_\varepsilon^+$ such that $(z+E_\varepsilon^+) \subset E_t$, $z \in \mathbb{Z}^2$, nor through the translations $z+\gamma_\varepsilon^-$ such that $E_t \subset (z+\mathbb{R}^2 \setminus E_\varepsilon^-)$. Such confinement proves (4.1) and thus completes the proof of Theorem 1.2.

Acknowledgments

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